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$M^{[x]}$/G/1 Multistage Queue with Stand-by Server during Main Server's Interruptions

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ABSTRACT

This paper investigates a multistage batch arrival queue with different server interruptions and a second server replaces the main server during the interruptions. The different server interruptions are assumed to be: multiple vacation, extended vacation, breakdown with delay time and server under two phases of repair. Customers are assumed to arrive in batches according to Poisson process and a single server provides service to the customers. When the main server is inactive due to the interruptions, stand-by server provide service to the arrivals. In addition, customers may renege during server breakdown or during server vacation due to impatience. Transient solution and the corresponding steady state solution is derived using supplementary variable technique.

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1. INTRODUCTION

Many authors have studied queueing models by importing many aspects on them such as server vacation, breakdown, repair, reneging, balking etc.

A batch arrival queueing model with different vacation policies is studied by numerous authors including R. Vimaladevi [1], G.Ayyappan & S.Shyamala [2] and Ke.J.C, Huang.H.I and Chu.Y.K [3]. G. Ayyappan and S. Shyamala [4] obtained transient state and steady state of batch arrival queue subject to random breakdowns and Bernoulli schedule server vacation with second optional repair. Khalaf [5] studied the single stage queueing model with single original and extended vacation having breakdown with delay time and the system equipped with a stand-by-server works during the main server stops. Monita Baruah, K.C.Madan and Tillal Eldabi [6] studied a two stage batch arrival queue with reneging during vacation and breakdown periods. Batch arrival retrial queue with multi optional repair is discussed by D.Sumitha and K.Udaya Chandrika [7]. Multistage Batch arrival queue subject to different vacation policy with two phases of repair is discussed in C.Yuvarani and C.Vijayalakshmi [8].

In this work, we consider M[x]/G/1 queue with 'N' stages of services under different vacation policy and extended vacation subject to system breakdown with delay time and two phase of repairs. In addition, we assume that the customers may renege during breakdown or vacation period due to impatience.

This paper is organized as follows. The assumptions of our model are given in section 2. Definitions and Equations governing the system are given in section 3. The time dependent solutions have been obtained in section 4 and corresponding steady state results have been derived explicitly in section 5. Mean queue size and mean waiting time are computed in section 6. Some particular cases are discussed in section 7 and the numerical results are given in section 8.

2. MODEL ASSUMPTIONS

We assume the following to describe the queueing model of our study.

a) The mean arrival rate of customers in batches is λ and they are served one by one on a first come - first served basis. The first order probability that a batch of *i* customers arrives at the system during a short interval of time $(t; t + dt]$ is $\lambda c_i dt$ $(i \geq 1)$, where $0 \leq c_i \leq 1$ and $\sum_{i=1}^{\infty} c_i = 1$.

b) A single server provides 'N' stages of services for each customer, with the service times having general distribution. Let $B_i(x)$ and $b_i(x)$ (i =1, 2, 3,…,N) be the distribution and the density function of i stage service respectively. Let $\mu_i(x)dx$ be the conditional probability density of service completion during the interval (x; x + dx], given that the elapsed time is x, so that

$$\mu_i(x) = \frac{b_i(x)}{1-B_i(x)}$$

and therefore, $b_i(t) = \mu_i(t) e^{-\int_0^t \mu_i(x)dx}$, i =1, 2, 3,…,N

c) After service completion of a customer, the server may remain in the system to serve the next customer with probability β_0 or he may proceed on j^{th} vacation scheme with probability β_j $(1 \leq j \leq M)$ and $\sum_{j=0}^{M} \beta_j = 1$. The server's vacation time follows a general (arbitrary) distribution with distribution function $V_j(x)$ and density function $v_j(x)$. Let $\gamma_j(x)dx$ be the conditional probability of a completion of a vacation during the interval (x; x + dx] given that the elapsed vacation time is x, so that

$$\gamma_j(x) = \frac{v_j(x)}{1-V_j(x)}$$

and therefore, $v_j(t) = \gamma_j(t) e^{-\int_0^t \gamma_j(x)dx}$, j =1, 2, 3,…,M

d) Once the original vacation gets over, the server has an option of taking an extended vacation with probability p or he may rejoin the system immediately with probability 1-p. The server's extended vacation time follows a general (arbitrary) distribution with distribution function $W_j(x)$ and density function $w_j(x)$. Let $\theta_j(x)dx$ be the conditional probability of a completion of a extended vacation during the interval (x; x + dx] given that the elapsed extended vacation time is x, so that

$$\theta_j(x) = \frac{w_j(t)}{1-W_j(t)}$$

and therefore, $w_j(t) = \theta_j(t) e^{-\int_0^t \theta_j(x)dx}$, j =1, 2, 3,…,M

e) Reneging is assumed to follow exponential distribution with parameter η. Thus ηdt is the probability that a customer can renege during a short interval of time $(t; t + dt]$.

f) Once the server breakdown, the repair do not start immediately. There is a delay time to start the repairs. The delay time follows general distribution with distribution and density function $H(x)$ and $h(x)$ respectively. Let $\phi(x)dx$ be the conditional probability of a completion of a delay time during the interval (x; x + dx] given that the elapsed delay time is x, so that

$$\phi(x) = \frac{h(x)}{1-H(x)} \quad \text{and therefore,} \quad h(t) = \phi(t) e^{-\int_0^t \phi(x)dx}$$

g) The server breakdown is assumed to occur according to a poisson stream with mean breakdown rate α > 0.

h) When the server breakdown the repair process may start any time. First the server sent for first essential repair (FER). After the completion of FER, the server may opt for the second optional repair (SOR) with probability r or may join the system with probability 1-r to render the service to the customers.

i) Both the repair process FER and SOR follows a general (arbitrary) distribution with distribution function $U_j(x)$ and density function $u_j(x)$ for j =1,2 respectively. Let $\kappa_1(x)dx$ and $\kappa_2(x)dx$ be the conditional probability of a completion of a FER and SOR during the interval (x; x + dx] given that the elapsed repair time is x, so that

$$\kappa_1(x) = \frac{u_1(x)}{1-U_1(x)} \quad \text{and therefore,} \quad u_1(t) = \kappa_1(t)e^{-\int_0^t \kappa_1(x)dx}$$

$$\kappa_2(x) = \frac{u_2(x)}{1-U_2(x)} \quad \text{and therefore,} \quad u_2(t) = \kappa_2(t)e^{-\int_0^t \kappa_2(x)dx}$$

The stand-by server provide service to the customers when the main server is on vacation, extended vacation, waiting for repair to start or under repair until the main server returns. The stand-by service time is assumed to follow exponential distribution with parameter δ.

3. DEFINITIONS AND EQUATIONS GOVERNING THE SYSTEM

We define

(i) $P_n^{(i)}(x,t)$ denotes the probability that there are 'n' (n ≥ 0) customers in the queue excluding the one in service at time t and the server is active providing i^{th} stage (i =1, 2, 3,…,N) of service with the elapsed service time for this customer is x and its corresponding probability irrespective of value of x is denoted by $P_n^{(i)}(t)$.

(ii) $V_n^{(j)}(x,t)$ denotes the probability that at time t, the server is on j^{th} vacation (j =1, 2, 3,…,M) and there are 'n' (n ≥ 1) customers waiting in the queue for service and its corresponding probability irrespective of value of x is denoted by $V_n^{(j)}(t)$.

(iii) $R_n^{(1)}(x,t)$ denotes the probability that at time t, the server is inactive due to breakdown and the system is under FER while there are 'n' (n ≥ 0) customers in the queue and its corresponding probability irrespective of value of x is denoted by $R_n^{(1)}(t)$.

(iv) $R_n^{(2)}(t)$ denotes the probability that at time t, the server is inactive due to breakdown and the system is under SOR while there are 'n' (n ≥ 0) customers in the queue and its corresponding probability irrespective of value of x is denoted by $R_n^{(2)}(t)$.

(v) Q(t) = Probability that at time t, there are no customers in the queue and the server is idle but available in the system.

(vi) $E_n^{(j)}(x,t)$ denotes the probability that at time t, the server is on j^{th} extended vacation (j =1, 2, 3,…,M) and there are 'n' (n ≥ 1) customers waiting in the queue for service and its corresponding probability irrespective of value of x is denoted by $E_n^{(j)}(t)$.

(vii) $D_n(x,t)$ denotes the probability that there are 'n' (n ≥ 0) customers waiting in the queue for service at time t, and the server is waiting for repair to start with elapsed delay time x and its corresponding probability irrespective of value of x is denoted by $D_n(t)$.

The queueing model is then, governed by the following set of differential-difference equations:

$$\frac{\partial}{\partial x}P_n^{(i)}(x,t) + \frac{\partial}{\partial t}P_n^{(i)}(x,t) + [\lambda + \mu_i(x) + \alpha]P_n^{(i)}(x,t) = \lambda\sum_{k=1}^{n} c_k P_{n-k}^{(i)}(x,t)$$

i =1, 2, 3,…,N , n ≥ 1

$$\frac{\partial}{\partial x}P_0^{(i)}(x,t) + \frac{\partial}{\partial t}P_0^{(i)}(x,t) + [\lambda + \mu_i(x) + \alpha]P_0^{(i)}(x,t) = 0 \qquad i=1, 2, 3,\ldots,N$$

$$\frac{\partial}{\partial x}V_n^{(j)}(x,t)+\frac{\partial}{\partial t}V_n^{(j)}(x,t)+\left[\lambda+\gamma_j(x)+\delta+\eta\right]V_n^{(j)}(x,t)=(\delta+\eta)V_{n+1}^{(j)}(x,t)+\lambda\sum_{k=1}^{n}c_kV_{n-k}^{(j)}(x,t)$$

j =1, 2, 3,…,M , n ≥ 1

$$\frac{\partial}{\partial x}V_0^{(j)}(x,t)+\frac{\partial}{\partial t}V_0^{(j)}(x,t)+\left[\lambda+\gamma_j(x)+\delta\right]V_0^{(j)}(x,t)=(\delta+\eta)V_1^{(j)}(x,t)$$

j =1, 2, 3,…,M

$$\frac{\partial}{\partial x}E_n^{(j)}(x,t)+\frac{\partial}{\partial t}E_n^{(j)}(x,t)+\left[\lambda+\theta_j(x)+\delta+\eta\right]E_n^{(j)}(x,t)=(\delta+\eta)E_{n+1}^{(j)}(x,t)+\lambda\sum_{k=1}^{n}c_kE_{n-k}^{(j)}(x,t)$$

j =1, 2, 3,…,M , n ≥ 1

$$\frac{\partial}{\partial x}E_0^{(j)}(x,t)+\frac{\partial}{\partial t}E_0^{(j)}(x,t)+\left[\lambda+\theta_j(x)+\delta\right]E_0^{(j)}(x,t)=(\delta+\eta)E_1^{(j)}(x,t)$$

j =1, 2, 3,…,M

$$\frac{\partial}{\partial x}R_n^{(1)}(x,t)+\frac{\partial}{\partial t}R_n^{(1)}(x,t)+(\lambda+\kappa_1(x)+\delta+\eta)R_n^{(1)}(x,t)=\lambda\sum_{k=1}^{n}c_kR_{n-k}^{(1)}(x,t)+(\delta+\eta)R_{n+1}^{(1)}(x,t)$$

$$\frac{\partial}{\partial x}R_0^{(1)}(x,t)+\frac{\partial}{\partial t}R_0^{(1)}(x,t)+(\lambda+\kappa_1(x)+\delta)R_0^{(1)}(x,t)=(\delta+\eta)R_1^{(1)}(x,t)$$

$$\frac{\partial}{\partial x}R_n^{(2)}(x,t)+\frac{\partial}{\partial t}R_n^{(2)}(x,t)+(\lambda+\kappa_2(x)+\delta+\eta)R_n^{(2)}(x,t)=\lambda\sum_{k=1}^{n}c_kR_{n-k}^{(2)}(x,t)+(\delta+\eta)R_{n+1}^{(2)}(x,t)+r\kappa_1(x)R_n^{(1)}(x,t),\quad n\ge 1$$

$$\frac{\partial}{\partial x}R_0^{(2)}(x,t)+\frac{\partial}{\partial t}R_0^{(2)}(x,t)+(\lambda+\kappa_2(x)+\delta)R_0^{(2)}(x,t)=(\delta+\eta)R_1^{(2)}(x,t)+r\kappa_1(x)R_0^{(1)}(x,t)$$

$$\frac{\partial}{\partial x}D_n(x,t)+\frac{\partial}{\partial t}D_n(x,t)+\left[\lambda+\phi(x)+\delta+\eta\right]D_n(x,t)=(\delta+\eta)D_{n+1}(x,t)+\lambda\sum_{k=1}^{n}c_kD_{ni-k}(x,t)$$

$$\frac{\partial}{\partial x}D_0(x,t)+\frac{\partial}{\partial t}D_0(x,t)+\left[\lambda+\phi(x)+\delta\right]D_0(x,t)=(\delta+\eta)D_1(x,t)$$

$$\frac{d}{dt}Q(t)=\beta_0\int_0^\infty P_0^{(N)}(x,t)\,\mu_N(x)dx+(1-p)\sum_{j=1}^{M}\int_0^\infty V_0^{(j)}(x,t)\gamma_j(x)\,dx-\lambda Q(t)+(1-r)\int_0^\infty\kappa_1(x)R_0^{(1)}(x,t)dx+\int_0^\infty\kappa_2(x)R_0^{(2)}(x,t)dx$$

The above equations are to be solved subject to the following boundary conditions:

$$P_n^{(1)}(0,t)=\beta_0\int_0^\infty P_{n+1}^{(N)}(x,t)\,\mu_N(x)dx+(1-p)\sum_{j=1}^{M}\int_0^\infty V_{n+1}^{(j)}(x,t)\gamma_j(x)\,dx+\lambda c_{n+1}Q(t)+\sum_{j=1}^{M}\int_0^\infty E_{n+1}^{(j)}(x,t)\theta_j(x)\,dx+(1-r)\int_0^\infty\kappa_1(x)R_{n+1}^{(1)}(x,t)dx+\int_0^\infty\kappa_2(x)R_{n+1}^{(2)}(x,t)dx$$

$$P_n^{(i)}(0,t) = \int_0^\infty \mu_{i-1}(x)\, P_n^{(i-1)}(x,t)\,dx \qquad i = 2,3,\ldots,N$$

$$V_n^{(j)}(0,t) = \beta_j \int_0^\infty \mu_N(x)\, P_{n+1}^{(N)}(x,t)\,dx \qquad j = 1,2,\ldots,M$$

$$E_n^{(j)}(0,t) = p\int_0^\infty V_n^{(j)}(x,t)\,\gamma_j(x)\,dx \qquad j = 1,2,\ldots,M$$

$$R_n^{(1)}(0,t) = \int_0^\infty D_n(x,t)\,\phi(x)\,dx$$

$$D_n(0,t) = \alpha\sum_{i=1}^{N}\int_0^\infty P_{n-1}^{(i)}(x,t)dx = \alpha\sum_{i=1}^{N} P_{n-1}^{(i)}(t)$$

We assume that initially there are no customers in the system and the server is idle. So the initial conditions are

$V_0^{(j)}(0) = V_n^{(j)}(0) = 0$, j =1, 2, 3,…,M and Q(0) = 1 , $D_0(0) = 0$ and

$P_n^{(i)}(0) = 0$ for n = 0, 1, 2, …, i =1, 2, 3,…,N

4. THE TIME-DEPENDENT SOLUTION

By using the supplementary variable technique, we obtain the following transient solution

$$\overline{P}^{(i)}(z,s) = \overline{B}_1(f_1(z))\overline{B}_2(f_1(z))\ldots\overline{B}_{i-1}(f_1(z))\left[1-\overline{B}_i(f_1(z))\right] X\ \frac{f_2(z)\left[(1-s\overline{Q}(s))+\lambda(C(z)-1)\overline{Q}(s)\right]}{DR}$$

i = 2, 3,4,…,N

$$\overline{V}^{(j)}(z,s) = \beta_j\overline{B}_1(f_1(z))\overline{B}_2(f_1(z))\ldots\overline{B}_N(f_1(z))\frac{f_1(z)\left[(1-s\overline{Q}(s))+\lambda(C(z)-1)\overline{Q}(s)\right]}{DR} . \frac{\left[1-\overline{V}_j(f_2(z))\right]}{f_2(z)} \qquad j = 1,2,\ldots,M$$

$$\overline{E}^{(j)}(z,s) = p\beta_j\overline{B}_1(f_1(z))\overline{B}_2(f_1(z))\ldots\overline{B}_N(f_1(z))\frac{f_1(z)\left[(1-s\overline{Q}(s))+\lambda(C(z)-1)\overline{Q}(s)\right]}{DR} X\ \frac{\overline{V}_j(f_2(z))\left[1-\overline{W}_j(f_2(z))\right]}{f_2(z)} \qquad j = 1,2,\ldots,M$$

$$\overline{R}^{(1)}(z,s) = \frac{\alpha z\overline{H}(f_2(z))\left[(1-s\overline{Q}(s))+\lambda(C(z)-1)\overline{Q}(s)\right]\left[1-\overline{B}_1(f_1(z))\overline{B}_2(f_1(z))\ldots\overline{B}_N(f_1(z))\right]}{DR} X\ \frac{\left[1-\overline{U}_1(f_2(z))\right]}{f_2(z)}$$

$$\overline{R}^{(2)}(z,s) = \frac{rz\alpha\overline{H}(f_2(z))\left[(1-s\overline{Q}(s))+\lambda(C(z)-1)\overline{Q}(s)\right]\left[1-\overline{B}_1(f_1(z))\overline{B}_2(f_1(z))\ldots\overline{B}_N(f_1(z))\right]}{DR} X\ \frac{\overline{U}_1(f_2(z))\left[1-\overline{U}_2(f_2(z))\right]}{f_2(z)}$$

$$\overline{D}(z,s)=\frac{\alpha z\left[(1-s\overline{Q}(s))+\lambda(C(z)-1)\overline{Q}(s)\right]\left[1-\overline{B}_1(f_1(z))\overline{B}_2(f_1(z))\ldots\overline{B}_N(f_1(z))\right]}{DR}\;X\;\frac{\left[1-\overline{H}(f_2(z))\right]}{f_2(z)}$$

where DR is given by

$$DR=f_1(z)\left\{z-\overline{B}_1(f_1(z))\overline{B}_2(f_1(z))\ldots\overline{B}_N(f_1(z))\left[\begin{array}{l}\beta_0+(1-p)\sum_{j=1}^{M}\beta_j\overline{V}_j(f_2(z))+\\ p\sum_{j=1}^{M}\beta_j\overline{V}_j(f_2(z))\overline{W}_j(f_2(z))\end{array}\right]\right\}$$
$$-\alpha z\overline{H}(f_2(z))\overline{U}_1(f_2(z))\left[(1-r)+r\overline{U}_2(f_2(z))\right]\left[1-\overline{B}_1(f_1(z))\overline{B}_2(f_1(z))\ldots\overline{B}_N(f_1(z))\right]$$

$$f_1(z)=s+\lambda(1-C(z))+\alpha\,,\qquad f_2(z)=s+\lambda(1-C(z))+\delta+\eta-\frac{\delta}{z}-\frac{\eta}{z}$$

$\overline{B}_i(f_1(z)),\overline{V}_j(f_2(z)),\overline{W}_j(f_2(z)),\overline{H}(f_2(z)),\overline{U}_1(f_2(z)),\overline{U}_2(f_2(z))$ are the Laplace –Stieltjes transform of Service time $B_i(x)$, vacation time $V_j(x)$, extended vacation $W_j(x)$, delay time H(x), and repair time $U_1(x)$ and $U_2(x)$ respectively.

5. THE STEADY STATE RESULTS

By using the property

$$\lim_{s\to 0}s\bar{f}(s)=\lim_{t\to\infty}f(t)$$

We obtain the following steady state result

$$\overline{P}^{(i)}(z)=\frac{\lambda(C(z)-1)\overline{B}_1(w)\overline{B}_2(w)\ldots\overline{B}_{i-1}(w)\left[1-\overline{B}_i(w)\right]Q}{dr}\qquad i=1,2,\ldots,N$$

$$\overline{V}^{(j)}(z)=\beta_j w\overline{B}_1(w)\overline{B}_2(w)\ldots\overline{B}_N(w)\frac{\lambda(C(z)-1)Q}{dr}\frac{\left[1-\overline{V}_j(m)\right]}{m}\qquad j=1,2,\ldots,M$$

$$\overline{E}^{(j)}(z)=p\beta_j\overline{B}_1(w)\overline{B}_2(w)\ldots\overline{B}_N(w)\frac{w\lambda(C(z)-1)Q}{dr}\frac{\overline{V}_j(m)\left[1-\overline{W}_j(m)\right]}{m}\qquad j=1,2,\ldots,M$$

$$\overline{R}^{(1)}(z)=\frac{\alpha z\overline{H}(m)\lambda(C(z)-1)Q\left[1-\overline{B}_1(w)\overline{B}_2(w)\ldots\overline{B}_N(w)\right]}{dr}\frac{\left[1-\overline{U}_1(m)\right]}{m}$$

$$\overline{R}^{(2)}(z)=\frac{rz\alpha\overline{H}(m)\lambda(C(z)-1)\left[1-\overline{B}_1(w)\overline{B}_2(w)\ldots\overline{B}_N(w)\right]\overline{U}_1(m)}{dr}\frac{\left[1-\overline{U}_2(m)\right]}{m}$$

$$\overline{D}(z)=\frac{\alpha z\lambda(C(z)-1)Q\left[1-\overline{B}_1(w)\overline{B}_2(w)\ldots\overline{B}_N(w)\right]}{dr}\frac{\left[1-\overline{H}(m)\right]}{m}$$

where dr is given by

$$dr=w\left\{z-\overline{B}_1(w)\overline{B}_2(w)\ldots\overline{B}_N(w)\left[\beta_0+(1-p)\sum_{j=1}^{M}\beta_j\overline{V}_j(m)+p\sum_{j=1}^{M}\beta_j\overline{V}_j(m)\overline{W}_j(m)\right]\right\}$$
$$-\alpha z\overline{H}(m)\overline{U}_1(m)\left[(1-r)+r\overline{U}_2(m)\right]\left[1-\overline{B}_1(w)\overline{B}_2(w)\ldots\overline{B}_N(w)\right]$$

$$w = \lambda(1 - C(z)) + \alpha, \qquad m = \lambda(1 - C(z)) + \delta + \eta - \frac{\delta}{z} - \frac{\eta}{z}$$

Let $W_q(z)$ be the PGF of queue size irrespective of the state of the system. Then we have,

$$W_q(z) = \sum_{i=1}^{N} \overline{P}^{(i)}(z) + \sum_{j=1}^{M} \overline{V}^{(j)}(z) + \sum_{j=1}^{M} \overline{E}^{(j)}(z) + \overline{R}^{(1)}(z) + \overline{R}^{(2)}(z) + \overline{D}(z)$$

In order to obtain Q, using the normalization condition

$$W_q(1) + Q = 1$$

We see that for z=1, Wq(z) is indeterminate of the form 0 / 0. Therefore, we apply L'Hopital's rule and on simplifying we obtain the result (1), where C(1) = 1, C′(1) = E(I), is mean batch size of the arriving customers, $-\overline{V}_j'(0) = E(V_j)$, , $j = 1,2,3,\ldots,M$ is the mean of vacation time, $-\overline{W}_j'(0) = E(eV_j)$ is the mean of extended vacation time, $-\overline{H}'(0) = E(D)$, is the mean of delay time, $-\overline{U}_1(0) = E(R_1)$ is the mean of FER time , $-\overline{U}_2(0) = E(R_2)$ is the mean of SOR time and let

$$E(V) = \sum_{j=1}^{M} \beta_j E(V_j) + p\sum_{j=1}^{M} \beta_j E(eV_j), \qquad E(R) = E(R_1) + rE(R_2),$$

$$\overline{B}(\alpha) = \overline{B}_1(\alpha)\overline{B}_2(\alpha)\ldots\overline{B}_N(\alpha).$$

$$W_q(1) = \frac{\lambda E(I)Q\{1 + \alpha E(R) + \overline{B}(\alpha)[\alpha E(V) - \alpha E(R) - 1]\}}{-\lambda E(I) + \alpha m'(1)[E(D) + E(R)] + \overline{B}(\alpha)\{\alpha + \lambda E(I) + \alpha m'(1)[E(V) - E(R) - E(D)]\}}$$

6. THE MEAN QUEUE SIZE AND THE SYSTEM SIZE

Let L_q denote the mean number of customers in the queue under the steady state, then

$$L_q = \frac{d}{dz}[W_q(z)]_{z=1}$$

Since this gives 0/0 form we write $W_q(z) = \frac{N(z)}{D(z)}$ where *N(z) and D(z)* are the numerator and denominator of the RHS of equation (54) respectively. Then we use

$$L_q = \left[\frac{D'(1)N''(1) - D''(1)N'(1)}{2(D'(1))^2}\right]$$

$$N'(1) = \lambda E(I)Q\{1 + \alpha E(R) + \overline{B}(\alpha)[\alpha E(V) - \alpha E(R) - 1]\}$$

$$\begin{aligned} N''(1) = (1 - \overline{B}(\alpha))&\left\{\begin{matrix} \lambda E(I(I-1))[1 - \alpha E(R) - \alpha E(D)] + 2\alpha\lambda E(I)[E(R) + E(D)] \\ + 2m'(1)\alpha\lambda E(I)[E(R^2) - E(R)E(D) + rE(R_1)E(R_2)] \end{matrix}\right\} \\ &+ 2\lambda^2 E(I)^2 \overline{B}'(\alpha)\{\alpha E(V) - \alpha E(R) - \alpha E(D) - 1\} \\ &+ \overline{B}(\alpha)\{2\lambda^2 E(I)^2 E(V) + \alpha\lambda E(I(I-1))E(V) + \alpha m'(1)\lambda E(I)E(V^2)\} \end{aligned}$$

$$D'(1) = -\lambda E(I) + \alpha m'(1)[E(D) + E(R)] + \overline{B}(\alpha)\{\alpha + \lambda E(I) + \alpha m'(1)[E(V) - E(R) - E(D)]\}$$

$$\begin{aligned} D''(1) = -2\lambda E(I) + (1 - \overline{B}(\alpha))&\left\{\begin{matrix} -\lambda E(I(I-1)) + \alpha[E(R) + E(D)](2m'(1) + m''(1)) \\ + m'(1)^2 \alpha[E(R^2) - 2E(R)E(D) + E(D^2)] \end{matrix}\right\} \\ &- 2\lambda E(I)\overline{B}'(\alpha)\{\lambda E(I) + \alpha + m'(1)\alpha[E(V) - E(D) - E(R)]\} \\ &+ \overline{B}(\alpha)\{m'(1)E(V) + m''(1)E(V) + m'(1)^2 E(V^2)\} \end{aligned}$$

Where $E(V^2) = \sum_{j=1}^{M} \beta_j E(V_j^2) + p\sum_{j=1}^{M} \beta_j E(eV_j^2) + 2p\sum_{j=1}^{M} \beta_j E(V_j)E(eV_j)$,

$E(R^2) = E(R_1^{\ 2}) + rE(R_2^{\ 2})$ and $E(I(I-1))$ is the second factorial moment of the batch size of arriving customers and $\overline{B}'(\alpha) = \overline{B}_1'(\alpha)\overline{B}_2(\alpha)\ldots\overline{B}_N(\alpha) + \overline{B}_1(\alpha)\overline{B}_2'(\alpha)\ldots\overline{B}_N(\alpha) + \ldots + \overline{B}_1(\alpha)\overline{B}_2(\alpha)\ldots\overline{B}_N'(\alpha)$. Then, if we substitute the values $N'(1), N''(1), D'(1), D''(1)$ in L_q, We get the performance measure in the closed form.

7. PARTICULAR CASES

Our queueing system can be considered as a very general system and lot of systems already studied can be a particular case of our system.

Case (i) No Reneging

If we assume there is no reneging, single stage of service , single vacation policy and there is no second optional repair then $E(V) = \beta_1 E(V_1) + p\beta_1 E(eV_1)$, $E(R) = E(R_1)$, $\overline{B}(\alpha) = \overline{B}_1(\alpha)$,

$m = \lambda(1 - C(z)) + \delta - \frac{\delta}{z}$

also we get

$$W_q(1) = \frac{\lambda E(I)Q\{1 + \alpha E(R) + \overline{B}(\alpha)[\alpha E(V) - \alpha E(R) - 1]\}}{-\lambda E(I) + \alpha m'(1)[E(D) + E(R)] + \overline{B}(\alpha)\{\alpha + \lambda E(I) + \alpha m'(1)[E(V) - E(R) - E(D)]\}}$$

The above result agrees with the result of Khalaf.R.F [5]

Case (ii) No Delay, No standby Server, No Extended Vacation, No Optional Repair

If we assume there is no delay, standby server, extended vacation and second optional repair with two stages of service having single vacation policy then $E(V) = \beta_1 E(V_1)$, $E(R) = E(R_1)$,

$\overline{B}(\alpha) = \overline{B}_1(\alpha)\overline{B}_2(\alpha)$, $m = \lambda(1 - C(z)) + \eta - \frac{\eta}{z}$ also we get

$$W_q(1) = \frac{\lambda E(I)Q\{1 + \alpha E(R) + \overline{B}(\alpha)[\alpha E(V) - \alpha E(R) - 1]\}}{-\lambda E(I) + \alpha m'(1)[E(R)] + \overline{B}(\alpha)\{\alpha + \lambda E(I) + \alpha m'(1)[E(V) - E(R)]\}}$$

The above result agrees with the result given Monita Baruah et al.,[6]

8. NUMERICAL RESULT

In order to examine the validity of the results we approach numerical result. For that purpose we assume service time, vacation time, extended vacation time, delay time, FER and SOR are all follows exponential distribution and we choose the following arbitrary values : N =1 , M = 1 (single server and single vacation),

$$E(I) = 1;\ E(I(I-1)) = 0;\ \lambda = 2;\ \mu_1 = 7,\ \gamma_1 = 5;\ \kappa_1 = 4, \kappa_2 = 5, \beta_1 = 0.5;\ r = 0.5;\ p = 0.5;$$
$$\alpha = 2, \eta = 5, \delta = 0$$

Table 1. Computed Values of Various Queue Characteristics

θ	ϕ	Q	ρ	L_q	W_q
4	3	0.8585	0.1415	0.6288	0.3144
7		0.8684	0.1416	0.6019	0.3010
10		0.8724	0.1276	0.5931	0.2966
12		0.8739	0.1261	0.59	0.295
4	5	0.8314	0.1686	0.7024	0.3512
7		0.8434	0.1566	0.6819	0.3409
10		0.8482	0.1518	0.6757	0.3379
12		0.85	0.15	0.6737	0.3369

This table clearly shows that as the delay time and the extended vacation time increases, the server idle time increases and the utilization factor decreases.

9. CONCLUSION

We have studied $M^{[x]}/G/1$ queue with 'N' stages of services under different vacation policy and extended vacation subject to system breakdown with delay time and two phase of repairs. In addition we assume that the customers may renege during breakdown or vacation period due to impatience. The service time, vacation time, extended vacation time, delay time and repair are all follow general distribution. This work presents the closed form of the important performance measure. Many queueing system studied already are the special cases of this model.

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