Indonesian Journal of Electrical Engineering and Computer Science
Vol. 7, No. 3, September 2017, pp. 755 ~ 760
DOI: 10.11591/ijeecs.v7.i3.pp755-760

Analysis of Memory Effects in Digital Filters with Overflow Arithmetic

Mani Kant Kumar*, Haranath Kar
Department of Electronics and Communication Engineering, Motilal Nehru National Institute of Technology Allahabad, Allahabad 211004, India
*Corresponding author, e-mail: manikantmnnit@gmail.com

Abstract

This paper deals with the problem of undesired memory effects in nonlinear digital filters owing to the influence of past excitations on future outputs. The nonlinearities under consideration cover the usual types of overflow arithmetic employed in practice. Based on the Hankel norm performance, a new criterion is proposed to ensure the reduction of undesired memory effects in digital filters with overflow arithmetic. In absence of external input, the nonexistence of overflow oscillations is also confirmed by the proposed criterion. A numerical example together with simulation result showing the effectiveness of the criterion is given.

Keywords: *digital filter, finite wordlength effect, hankel norm performance, lyapunov method, memory effect*

Copyright © 2017 Institute of Advanced Engineering and Science. All rights reserved.

1. Introduction

Digital filters are linear time-invariant systems that perform mathematical operations on discrete-time signals to enhance or reduce certain features of such signals. They found wide applications in many fields such as speech and image processing, communications, noise suppression, control systems, spectrum and vibration analyses, etc. [1]. During the implementation of recursive digital filters on a fixed-point digital signal processor, the generation of nonlinearities due to quantization and overflow is a common phenomenon [2]. Due to the presence of such nonlinearities, the implemented filter may exhibit undesired zero-input oscillations [2-3]. The problem of stability of digital filters with finite wordlength nonlinearities has been studied extensively [2-14]. Several stability results to reduce the effects of nonlinear phenomenon have appeared recently [10-16].

If an excitation of finite duration is applied to a digital filter, ringing may occur after the withdrawal of the excitation [16]. The stored energy within the digital filter due to past excitations may produce undesired memory effects or ringing. Ringing may lead to noise, undesired oscillations, echoes etc. and degrade the performance of the system [16-17]. The Hankel norm performance can be used to measure the undesired memory effects of past excitations on future outputs [16-22].

The analysis of undesired memory effects in digital filters with finite wordlength nonlinearities is an important problem. The criteria in [3-15] cannot be used to measure the memory effects in digital filters. Recently, the Hankel norm performance criterion for the reduction of memory effects of digital filters with saturation arithmetic has been established in [16]. However, this problem has not been fully investigated in the literature so far.

This paper deals with the problem of memory effect analysis for digital filters with overflow nonlinearities. The paper is organised as follows. In Section 2, a description of the system under consideration is presented and the problem is formulated. Section 3 proposes a new linear matrix inequality (LMI)-based criterion for the Hankel norm performance of digital filters with overflow arithmetic. Unlike [16], the proposed criterion can be used to affirm the reduction of undesired memory effects in digital filters employing any kind of overflow arithmetic used in practice (e.g., saturation, zeroing, two’s complement and triangular). A numerical example is given in Section 4 to illustrate the applicability of the presented criterion. Finally, the paper is concluded in Section 5.

Received May 19, 2017; Revised July 29, 2017; Accepted August 13, 2017

2. System Description and Problem Formulation

Consider a fixed-point state-space digital filter given by:

$$\boldsymbol{x}(r+1)=\boldsymbol{f}(\boldsymbol{y}(r))+\boldsymbol{u}(r), \tag{1a}$$

$$\boldsymbol{y}(r)=\boldsymbol{A}\boldsymbol{x}(r)+\boldsymbol{u}(r), \tag{1b}$$

$$\boldsymbol{y}(r)=\boldsymbol{A}\boldsymbol{x}(r)+\boldsymbol{u}(r), \tag{1c}$$

where $\boldsymbol{x}(r)\in R^n$ is the state vector, $\boldsymbol{u}(r)\in R^n$ denotes the external input or interference, $\boldsymbol{f}(\boldsymbol{y}(r))=[f_1(y_1(r))\quad f_2(y_2(r))\quad\ldots\quad f_n(y_n(r))]^T$ stands for a nonlinear function vector, $\boldsymbol{A}\in R^{n\times n}$ is the coefficient matrix, $\boldsymbol{H}\in R^{m\times n}$ is a known constant matrix and superscript T denotes the transpose. It is assumed that $\boldsymbol{u}(r)$ acts as an excitation over the time period $0\le r<T$ for the system with zero initial condition and the excitation disappears if $r\ge T.$ The vector $\boldsymbol{z}(r)=[z_1(r)\quad z_2(r)\quad\ldots\quad z_m(r)]^T$ is a linear combination vector of the states representing ring of the system for $r\ge T.$ It is assumed that the nonlinearities under consideration satisfy.

$$|f_i(y_i(r))|\le|y_i(r)|,\qquad i=1,2,\ldots,n, \tag{2}$$

which include all kinds of overflow arithmetic employed in practice.

Given a scalar level $\gamma>0$ and a time $T>0$ the purpose of this paper is to establish a new condition such that the digital filter (1) and (2) satisfies.

$$\sum_{r=T}^{\infty}\boldsymbol{z}^T(r)\boldsymbol{z}(r)<\gamma^2\sum_{r=0}^{T-1}\boldsymbol{u}^T(r)\boldsymbol{u}(r), \tag{3}$$

under zero initial condition. If (3) is satisfied, then the digital filter has the Hankel norm performance γ. The Hankel norm performance γ can be interpreted as a measure of the effects of past excitations on the future outputs.

3. Proposed Criterion

The proposed criterion may be stated as follows.

Theorem 1. Given a scalar $\gamma>0$ suppose there exist two positive definite diagonal matrices ***D*** and ***C*** satisfying

$$\boldsymbol{Q}=\begin{bmatrix}\boldsymbol{D}-\boldsymbol{A}^T\boldsymbol{C}\boldsymbol{A} & \boldsymbol{0} & -\boldsymbol{A}^T\boldsymbol{C}\\ \boldsymbol{0} & \boldsymbol{C}-\boldsymbol{D} & -\boldsymbol{D}\\ -\boldsymbol{C}\boldsymbol{A} & -\boldsymbol{D} & \gamma^2\boldsymbol{I}-\boldsymbol{C}-\boldsymbol{D}\end{bmatrix}>\boldsymbol{0}, \tag{4a}$$

$$\boldsymbol{R}=\boldsymbol{D}-\boldsymbol{A}^T\boldsymbol{D}\boldsymbol{A}-\boldsymbol{H}^T\boldsymbol{H}>\boldsymbol{0}, \tag{4b}$$

where '>' signifies that the matrix is positive definite. Then, the digital filter described by (1) and (2) has the Hankel norm performance γ

Proof. Consider a quadratic Lyapunov function

$$V(\boldsymbol{x}(r))=\boldsymbol{x}^T(r)\boldsymbol{D}\boldsymbol{x}(r). \tag{5}$$

Along the trajectory of (1a), one has

$$\begin{aligned}\Delta V(\boldsymbol{x}(r))&=V(\boldsymbol{x}(r+1))-V(\boldsymbol{x}(r))\\ &=[\boldsymbol{f}(\boldsymbol{y}(r))+\boldsymbol{u}(r)]^T\boldsymbol{D}[\boldsymbol{f}(\boldsymbol{y}(r))+\boldsymbol{u}(r)]-\boldsymbol{x}^T(r)\boldsymbol{D}\boldsymbol{x}(r).\end{aligned} \tag{6}$$

Equation (6) can be rearranged as

$$\Delta V(\boldsymbol{x}(r)) = -\begin{bmatrix}\boldsymbol{x}^T(r) & \boldsymbol{f}^T(\boldsymbol{y}(r)) & \boldsymbol{u}^T(r)\end{bmatrix}\boldsymbol{Q}\begin{bmatrix}\boldsymbol{x}(r)\\ \boldsymbol{f}(\boldsymbol{y}(r))\\ \boldsymbol{u}(r)\end{bmatrix} + \gamma^2\boldsymbol{u}^T(r)\,\boldsymbol{u}(r) - \beta_1, \tag{7}$$

where

$$\beta_1 = [\boldsymbol{A}\boldsymbol{x}(r)+\boldsymbol{u}(r)]^T\boldsymbol{C}\,[\boldsymbol{A}\boldsymbol{x}(r)+\boldsymbol{u}(r)] - \boldsymbol{f}^T(\boldsymbol{y}(r))\boldsymbol{C}\,\boldsymbol{f}(\boldsymbol{y}(r)). \tag{8}$$

Since is a positive definite diagonal matrix, given by (8) is nonnegative in view of (2).
From (7), it is clear that

$$\Delta V(\boldsymbol{x}(r)) < \gamma^2\boldsymbol{u}^T(r)\,\boldsymbol{u}(r), \tag{9}$$

if $\boldsymbol{Q} > \boldsymbol{0}$. Using (9), we obtain

$$V(\boldsymbol{x}(T)) = \sum_{r=0}^{T-1}\Delta V(\boldsymbol{x}(r)) < \gamma^2\sum_{r=0}^{T-1}\boldsymbol{u}^T(r)\boldsymbol{u}(r), \tag{10}$$

under the zero initial condition.
For all $r \ge T$, $\boldsymbol{u}(r) = \boldsymbol{0}$. Therefore, it follows from (6) that

$$\Delta V(\boldsymbol{x}(r)) = \boldsymbol{f}^T(\boldsymbol{y}(r))\boldsymbol{D}\,\boldsymbol{f}(\boldsymbol{y}(r)) - \boldsymbol{x}^T(r)\boldsymbol{D}\,\boldsymbol{x}(r), \tag{11}$$

for all $r \ge T$
Equation (11) can be expressed as

$$\Delta V(\boldsymbol{x}(r)) = -\boldsymbol{x}^T(r)\,\boldsymbol{R}\,\boldsymbol{x}(r) - \boldsymbol{z}^T(r)\,\boldsymbol{z}(r) - \beta_2, \tag{12}$$

where

$$\beta_2 = \boldsymbol{x}^T(r)\boldsymbol{A}^T\boldsymbol{D}\boldsymbol{A}\,\boldsymbol{x}(r) - \boldsymbol{f}^T(\boldsymbol{y}(r))\,\boldsymbol{D}\,\boldsymbol{f}(\boldsymbol{y}(r)). \tag{13}$$

Note that, owing to (2), β_2 given by (13) is nonnegative. From (12), it is clear that

$$\Delta V(\boldsymbol{x}(r)) < -\boldsymbol{z}^T(r)\,\boldsymbol{z}(r), \tag{14}$$

if (4b) holds true.
From (14), it follows that

$$V(\boldsymbol{x}(\infty)) - V(\boldsymbol{x}(T)) = \sum_{r=T}^{\infty}\Delta V(\boldsymbol{x}(r)) < -\sum_{r=T}^{\infty}\boldsymbol{z}^T(r)\,\boldsymbol{z}(r). \tag{15}$$

Using (10), (15) and the relation $V(\boldsymbol{x}(\infty)) \ge 0$, one obtains (3). This completes the proof.

Remark 1. In view of (9), $\Delta V(\boldsymbol{x}(r)) < 0$ if $\boldsymbol{u}(r) = \boldsymbol{0}$. Therefore, the zero-input digital filter is globally asymptotically stable under the conditions stated in Theorem 1.

Remark 2. The approach in [16] focuses on saturation overflow arithmetic. By contrast, the present approach is applicable not only to saturation, but also to two's complement, zeroing, triangular overflow nonlinearities.

Remark 3. The conditions given by (4) are LMIs. Therefore, they can be easily tested by using MATLAB LMI toolbox [23, 24].

Remark 4. The H_∞ performance $\tilde{\gamma}$ given by [25]

$$\tilde{\gamma} = \sqrt{\frac{\sum_{r=0}^{\infty} \boldsymbol{z}^T(r)\boldsymbol{z}(r)}{\sum_{r=0}^{\infty} \boldsymbol{u}^T(r)\boldsymbol{u}(r)}}, \tag{16}$$

is not capable of measuring the memory effects of the digital filter precisely. Clearly, $\gamma \le \tilde{\gamma}$ in view of (3) and (16).

4. An Example

To illustrate the applicability of Theorem 1, consider a second-order digital filter represented by (1) and (2) with

$$\boldsymbol{A} = \begin{bmatrix} 0.5 & -0.3 \\ 0.1 & 0.4 \end{bmatrix}, \ \boldsymbol{H} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \ \gamma = 0.35, \tag{17a}$$

and the external input $\boldsymbol{u}(r)$ is given by

$$\boldsymbol{u}(r) = \begin{cases} 0.2\,[\cos(r) \quad \sin(r)]^T, & 0 \le r < 100, \\ \boldsymbol{0}, & r \ge 100. \end{cases} \tag{17b}$$

For this example, it is found that the conditions in Theorem 1 are satisfied for

$$\boldsymbol{D} = \begin{bmatrix} 0.0171 & 0 \\ 0 & 0.0218 \end{bmatrix}, \ \boldsymbol{C} = \begin{bmatrix} 0.0265 & 0 \\ 0 & 0.0390 \end{bmatrix}. \tag{18}$$

With $\boldsymbol{D}$ and $\boldsymbol{C}$ given by (18), the matrices $\boldsymbol{Q}$ and $\boldsymbol{R}$ in Theorem 1 become

$$\boldsymbol{Q} = \begin{bmatrix} 0.0100 & 0.0024 & 0 & 0 & -0.0133 & -0.0039 \\ 0.0024 & 0.0131 & 0 & 0 & 0.0080 & -0.0156 \\ 0 & 0 & 0.0095 & 0 & -0.0171 & 0 \\ 0 & 0 & 0 & 0.0172 & 0 & -0.0218 \\ -0.0133 & 0.0080 & -0.0171 & 0 & 0.0789 & 0 \\ -0.0039 & -0.0156 & 0 & -0.0218 & 0 & 0.0617 \end{bmatrix}, \tag{19}$$

$$\boldsymbol{R} = \begin{bmatrix} 0.0026 & 0.0017 \\ 0.0017 & 0.0067 \end{bmatrix}. \tag{20}$$

The matrices $\boldsymbol{Q}$ and $\boldsymbol{R}$ turn out to be positive definite. Therefore, Theorem 1 ensures the Hankel norm performance $\gamma = 0.35$ for this digital filter under the zero initial condition. It may be mentioned that the criteria in [3-15] are not suitable for the analysis of memory effects in digital filters.

Pertaining to the present system with zero initial condition, the effects of $\boldsymbol{u}(r)$ on $\boldsymbol{z}(r) = [z_1(r) \quad z_2(r)]^T$ are shown in Figures 1 and 2. The memory effect of the digital filter under consideration can be represented by $\boldsymbol{z}(r)$ for $r \ge 100$. For the present example, a routine computation shows that:

$$\gamma^* = \sqrt{\frac{\sum_{r=100}^{\infty} \mathbf{z}^T(r)\mathbf{z}(r)}{\sum_{r=0}^{99} \mathbf{u}^T(r)\mathbf{u}(r)}} = 0.0175. \quad (20)$$

Note that $\gamma^* < \gamma = 0.35$. This confirms the Hankel norm performance of the present system. Further, it can also be seen from Figures 1 and 2 that $\mathbf{z}(r) \to \mathbf{0}$ as $r \to \infty$ which, in turn, implies $\mathbf{x}(r) \to \mathbf{0}$ as $r \to \infty$. Thus, the present digital filter is globally asymptotically stable in absence of external input.

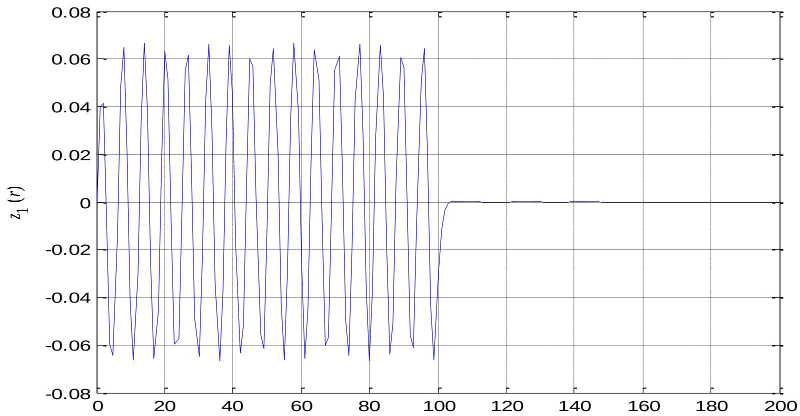

Figure 1. Plot of $z_1(r)$

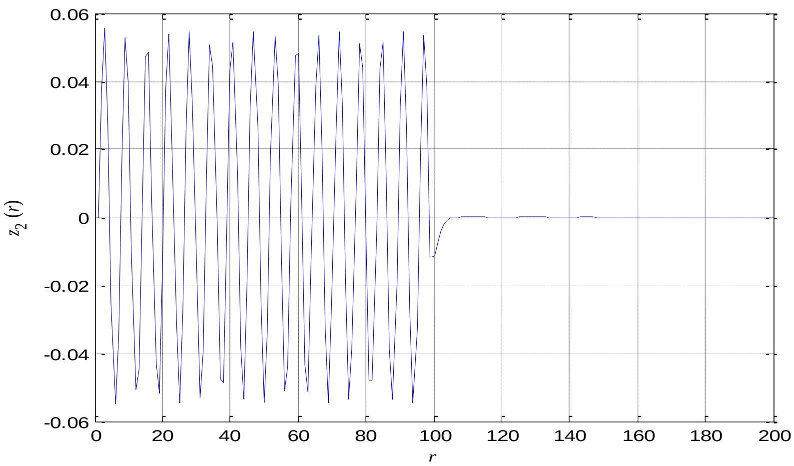

Figure 2. Plot of $z_2(r)$

5. Conclusion

A new LMI-based criterion (Theorem 1) for the Hankel norm performance of digital filters employing overflow arithmetic has been established. Unlike the approach in [16], the presented approach is applicable to saturation, zeroing, two's complement and triangular overflow nonlinearities. The criterion obtained in this paper is useful to check the reduction of undesired memory effects in digital filters to past inputs. Without external input, the nonexistence of overflow oscillations is also guaranteed by the presented criterion. An example showing the effectiveness of the criterion has been provided.

References

[1] Rabiner LR, Gold B. Theory and Application of Digital Signal Processing. Prentice-Hall. 1975.
[2] Butterweck HJ, Ritzerfeld JHF, Werter MJ. Finite Wordlength in Digital Filters: A Review. Eindhoven University Technology, Eindhoven. 1988.
[3] Mills WL, Mullis CT, Roberts RA. Digital Filter Realizations without Overflow Oscillations. *IEEE Transactions on Acoustics, Speech, and Signal Processing.* 1978; 26(4): 334-338.
[4] Kar H. An Improved Version of Modified Liu-Michel's criterion for Global Asymptotic Stability of Fixed-point State-space Digital Filters using Saturation Arithmetic. *Digital Signal Processing.* 2010; 20(4): 977-981.
[5] Li X, Yu X, Wang C, Ling BW-K. Periodic Input Response of a Second-order Digital Filter with Two's Complement Arithmetic. *IEEE Transactions on Circuits and Systems II.* 2009; 56(3): 225-229.
[6] Shen T, Yuan Z. Stability of Fixed-point State-space Digital Filters Using Two’s Complement Arithmetic: Further Insight. *Automatica.* 2010; 46(12): 2109-2111.
[7] Kar H. A Note on the Improved LMI-based criterion for Global Asymptotic Stability of 2-D State-space Digital Filters Described by Roesser Model Using Two's Complement Overflow Arithmetic. *Digital Signal Processing.* 2013; 23(5): 1767-1772.
[8] Agarwal N, Kar H. New results on Saturation Overflow Stability of 2-D State-space Digital Filters Described by the Fornasini–Marchesini Second Model. *Signal Processing.* 2016; 128: 504-511.
[9] Agarwal N, Kar H. Criterion for non-existence of Limit Cycles in 2D State-space Digital Filters Described by the Fornasini-Marchesini Second Model with Finite Wordlength non-linearities. *IET Signal Processing.* 2016; 10(5): 449-456.
[10] Ahn CK, Kar H. Passivity and Finite-gain Performance for Two-dimensional Digital Filters: the FM LSS model case. *IEEE Transactions on Circuits and Systems II.* 2015; 62(9): 871-875.
[11] Ahn CK, Kar H. Expected Power Bound for Two-dimensional Digital Filters in the Fornasini-Marchesini Local State-space Model. *IEEE Signal Processing Letters.* 2015; 22(8): 1065-1069.
[12] Ahn CK, Shi P. Dissipativity Analysis for Fixed-point Interfered Digital Filters. *Signal Processing.* 2015; 109: 148-153.
[13] Ahn CK, Shi P. Generalized Dissipativity Analysis of Digital Filters with Finite-wordlength Arithmetic. *IEEE Transactions on Circuits and Systems II.* 2016; 63(4): 386-390.
[14] Kokil P, Arockiaraj SX. An improved criterion for induced l_∞ stability of fixed-point digital filters with saturation arithmetic. *Indonesian Journal of Electrical Engineering and Computer Science (IJEECS).* 2016; 4(1): 65-72.
[15] Arockiaraj SX, Kokil P, Kar H. Passivity Based Stability Condition for Interfered Digital Filters. *Indonesian Journal of Electrical Engineering and Computer Science (IJEECS).* 2017; 6(2): 431-437.
[16] Ahn CK, Shi P. Hankel Norm Performance of Digital Filters Associated with Saturation. *IEEE Transactions on Circuits and Systems II.* 2017; 64(6): 720-724.
[17] Dao MN, Noll D. Minimizing Memory Effects of a system. *Mathematics of Control, Signals, and Systems.* 2015; 27(1): 77-110.
[18] Glover K. All Optimal Hankel-norm Approximations of linear multivariable systems and their L, $^\infty$-error bounds. *International Journal of Control.* 1984; 39(6): 1115-1193.
[19] Zhou K, Doyle JC, Glover K. Robust and Optimal Control. New Jersey: Prentice hall. 1996.
[20] Gao H, Lam J, Wang C, Wang Q. Hankel Norm Approximation of Linear Systems with Time-varying Delay: continuous and discrete cases. *International Journal of Control.* 2004; 77(17): 1503-1520.
[21] Wu L, Shi P, Su X. Hankel-norm Model Approximation for LPV Systems with Parameter-varying Time Delays. *International Journal of Systems Science*. 2010; 41(10): 1173-1185.
[22] Su X, Wu L, Shi P, Chen CLP. Model Approximation for Fuzzy Switched Systems with Stochastic Perturbation. *IEEE Transactions on Fuzzy Systems.* 2015; 23(5): 1458-1473.
[23] Boyd S, Ghaoui LEI, Feron E, Balakrishnan V. Linear Matrix Inequalities in System and Control Theory. Philadelphia: SIAM. 1994.
[24] Gahinet P, Nemirovski A, Laub AJ, Chilali M. LMI Control Toolbox for use with MATLAB. Natick, MA; The Mathworks Inc. 1995.
[25] Gao H, Wang C. Delay-dependent robust H_∞ and L_2-L_∞ filtering for a class of uncertain nonlinear time-delay systems. *IEEE Transactions on Automatic Control.* 2003; 48(9): 1661-1666.