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Variational Iteration Method for Solving Riccati Matrix Differential Equations

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Abstract

Riccati matrix differential equation has long been known to be so difficult to solve analytically and/or numerically. In this connection, most of the recent studies are concerned with the derivation of the necessary conditions that ensure the existence of the solution. Therefore, in this paper, He's Variational iteration method is used to derive the general form of the iterative approximate sequence of solutions and then proved the convergence of the obtained sequence of approximate solutions to the exact solution. This proof is based on using the mathematical induction to derive a general formula for the upper bound proved to be converging to zero under certain conditions.

Keywords: *Matrix Riccati differential equation, Variational iteration method, Iterative methods, He's iterative method, Matrix differential equations.*

1. Introduction

Mathematical modeling of real life problems, especially in control problems leads to differential equations, integral equations, system of differential and algebraic equations. While the solution of such obtained models may be so difficult to be evaluated analytically, therefore numerical and approximate methods seem to be necessary to be used to evaluate the solution of such problems.

Among the most popular and simplest accurate methods, the iterative methods is the variational iteration method (VIM), which is proposed by Ji-Huan He in 1999 as a special approach of the homotopy analysis method. The VIM has been shown to solve functional equations efficiently. In related literature, this method has been referred to as a modification of the general Lagrange multiplier method [18].

Several researchers compared the VIM with other numerical and approximate methods, where it is shown by all that this method may give more accurate results and faster than the other methods. In addition, in this method, the convergent to the exact solution is considered into a count and proved in the research work [9].

The main feature of the VIM is that the solution of the problem under consideration with linearization assumption is used as an initial approximate solution to more accurate and precise approximate solution which is obtained through iterative.

The VIM has been used by other researchers to solve various problems such as He in 1999 used the VIM to give approximate solution for some well known non-linear problems, [11]. He in 2007 used this method to solve autonomous systems of ordinary differential equations, [10]. In 2006 the VIM was applied for solving nonlinear integro-differential equation of functional order by Kurulay M. [13, 17]. David K. and Hadi R. G. in 2010 studied the convergence of the VIM for the solution of the telegraph problem, [3]. Ghorbani A. and Saberi J. proved the convergence of the VIM of nonlinear oscillators with an illustrative example [6]. Nguyen T. et al in 2010 propose a new method to solve Riccati matrix differential equation, which is based on the differential Lyapunov equation to calculate numerically the solution of Riccati matrix differential equation [19]. The metod of Nguyen T., et al., [19] is shown to be robust and numerically efficient. While the application of the proposed method by Nguyen T. et al. had been applied [20] to the singular perturbed linear quadratic optimal control problem, in which for this objective, the author's compose the full-order optimal linear –quadratic control problem into an reduced order subproblems. A similar transformation idea can be found in the works of Glizer

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and Dmitriev [7, 8], in which a series of expansion method is proposed. Biazar J., et al., in 2011 studied the numerical solution of certain functional integral equations by the VIM [2]. Geng F. used the modified VIM in 2009 to solving scalar Riccati differential equation [4]. Lu J. solved a nonlinear system of second order boundary value problems using VIM in 2007 [16] while Hemeda A.A. used the VIM to solve the wave partial differential equation [12]. The VIM has also been used to solve more advance problems movable boundaries by Ghomanjani F. and Ghaderi S. in 2012 [5].

In addition, in recent years, there is a wide occurrence of Riccati matrix differential equation as a control model in which the analytical and theoretical results concerning this matrix equation has been established, but still its solution seems to be difficult to be evaluated.

In the current study, the VIM will be used to solve Riccati matrix differential equation, where the convergence of the sequence of iterated approximate solutions have been proved to be converge to the exact solution of the problem, which is assumed to exist and be unique depending on the results of the other researchers [15].

2. The Ricatti Equation

The study of scalar and Riccati matrix and/or differential equations has a great importance, which dates from the early days of modern mathematical analysis, since such equations represent one of the obtained types of nonlinear equations, especially in mathematical physics. Also, within recent years, there is a wide occurrence of Riccati matrix differential equations notably in variational theory and allied areas of optimal control, invariant embedding and dynamic programming [21].

An atypical algebraic Riccati equation is similar to one of the following [14, 22]. The first is a quadratic matrix equation for the unknown $n \times n$ matrix X of the form:

$$A^T X + XA + XBX - C = 0 \tag{1}$$

Where A, B and C are $n \times n$ complex matrices with B and C hermitian, and T refers to the matrix transpose. Equation (1) is called the continuous time Riccati equation.

The second equation has the functional form for the unknown $n \times n$ matrix X.

$$X = A^T XA - (C + B^T XA)^T (R + B^T XB)^{-1} (C + B^T XA) + Q \tag{2}$$

Where R and Q are $m \times m$ and $n \times n$ matrices respectively and A, B, C are complex matrices with respect to sizes $n \times n$, $n \times m$ and $m \times n$ respectively. In addition, both equations arise in systems theory, differential equations and filter design in control theory [14].

More advanced and interesting form of Riccati matrix equation is the nonlinear Riccati matrix differential equation of the form [21]:

$$\dot{X} + XA + A^T X - XBX + C = 0 \tag{3}$$

Where A, B and C are constants $n \times n$ matrices such that $B = B^T$ and $C = C^T$. The existence conditions of a unique solution are imposed and satisfied on equation (3).

Solution of equation (3) may be achieved analytically, which is so complicated and difficult to evaluate in some cases and therefore numerical and/or approximate methods seems to be necessary to find the solution of equation (3). In the next section, we will present and use one of the He's iteration methods to solve equation (3), namely we will use the VIM.

3. The Variational Iteration Method

To illustrate the basic ideas of the VIM, consider the following nonlinear equation given in abstract form as an operator equation:

$$Au(t) = g(t), \ t \in [a, b] \tag{4}$$

Where A is a nonlinear operator, $g(t)$ is a given function, $a,b \in \mathbb{R}$ and u is the unknown function. If it is assumed that the operator A may be decomposed into two operators, namely; L and N that will be rewritten equation (4) as follows:

$$Lu(t)+Nu(t)=g(t) \tag{5}$$

Where L is a linear operator, N is a nonlinear operator and g is any given function, which is called the nonhomogeneous term.

The analysis of the VIM is to rewrite equation (5) as follows:

$$Lu(t)+Nu(t)-g(t)=0 \tag{6}$$

And if it is supposed that u_n is the n^{th} approximate solution of equation (6), then it follows that:

$$Lu_n(t)+Nu_n(t)-g(t)\neq 0 \tag{7}$$

And therefore, a correction functional for equation (5) may be given by:

$$u_{n+1}(t)=u_n(t)+\int_{t_0}^{t}\lambda(s)\left\{Lu_n(s)+N\tilde{u}_n(s)-g(s)\right\}ds \tag{8}$$

Where λ is the general Lagrange multiplier that can be identified optimally via variational theory, the subscript n denotes the n^{th} iterative approximate solution of u, and $\tilde{u}_n$ is considered as a restricted variation, i.e., $\delta\tilde{u}_n=0$, [1].

To solve equation (8) by the VIM, the Lagrange multiplier λ should be determined and evaluated first that will be identified optimally through integration by parts. Then, the successive approximations $u_n(t)$, $n = 0, 1, \ldots$; of the solution $u(t)$ was obtained upon applying the obtained Lagrange multiplier λ.

Now, using selective function $u_0(t)$ as an initial guess for $u(t)$, the zeroth approximation u_0 may be selected by any function that just satisfies at least the initial and boundary conditions with λ predetermined, then several approximations $u_n(t)$, $n = 0, 1, \ldots$; follows immediately and consequently the exact solution may be arrived since:

$$u(t)=\lim_{n\to\infty}u_n(t) \tag{9}$$

In other words, the correction functional for equation (8) will give a sequence iterated approximations, and therefore the exact solution is obtained as the limit of the resulting successive approximations [25].

4. Approximate Solution of Riccati Matrix Differential Equations Using VIM

It is obvious now that the main aspects of the VIM requires first the determinative of the Lagrange multiplier $\lambda(t)$, which may be found by the method of integration by parts, i.e. , for equation (8) that can be used:

$$\int_{t_0}^{t}\lambda(t)u_n'(t)dt=\lambda(t)u_n(t)-\int_{t_0}^{t}\lambda'(t)u_n(t)dt$$

$$\int_{t_0}^{t} \lambda(t)u_n''(t)dt = \lambda(t)u_n'(t) - \lambda'(t)u_n(t) + \int_{t_0}^{t} \lambda''(t)u_n(t)dt$$

And so on. Therefore, after having determined $\lambda(t)$, the sequence of approximate solutions $u_{n+1}, n = 0,1,...$, of the solution will be obtained immediately upon any selective function *u* as an initial guess. As it is found previously in literatures (see for example [23,24]), the general form of $\lambda(t)$ for the n^{th} order ODE:

$$u^{(n)}(t) + f(t, u(t), u'(t), ..., u^{(n-1)}(t)) = 0 \tag{10}$$

Is proved by induction to be:

$$\lambda(t) = (-1)^n \frac{1}{(n-1)!}(t-x)^{n-1} \tag{11}$$

Where it is clear that for the first order ODE $\lambda(t) = -1$, for the second order ODE $\lambda(t) = -1$ and so on.

The generalization of the correction functional (8) with its corresponding Lagrange multiplier for systems of n-operator equations may be considered similarly by using system of n-correction functionals with related n-Lagrange multiplier.

Hence, in connection with Ricaati matrix differential equation (3), the correction functional may be considered for a system of n-equations of the first order ODE's as follows:

Equation (3) may be written in the form of equation (10) as:

$$\dot{X}(t) + F(t, X(t) = 0 \tag{12}$$

Where,

$$F(t, X(t)) = AX(t) + X(t)A^T - X(t)BX(t) + C$$

And hence equation (3) (or equivalently equation 12) may be written in the form of the nonlinear equation (4), where the related nonlinear operator in the lefthand side of equation (4) is defined by:

$$A\cdot = \frac{d}{dt}\cdot + \cdot A + A^T \cdot - \cdot B\cdot \tag{13}$$

And the nonhomogenuous function in the righthand side is given by:

$$g(t) = -C \tag{14}$$

Therefore, in connection with the nonlinear operator (13) that may be decomposed into two operators, namely; a linear operator L and a nonlinear operator N, which are defined as:

$$L\cdot = \frac{d}{dt}\cdot + \cdot A + A^T \cdot \tag{15}$$

$$N\cdot = - \cdot B\cdot \tag{16}$$

Hence, the statement of the variational iteration formula (8) related to the Riccati matrix differential equation:

$$LX(t)+NX(t)=g(t) \tag{17}$$

Where L, N and *g* are defined by equations (14), (15) and (16) respectively. The derivation of the approximated formula is presented and it is easily verified that the correction functional takes the form as in the next theorem.

Theorem (1):

The correction functional of the Ricaati matrix differential equation (3) has the form:

$$X_{n+1}(t)=X_n(t)-\int_{t_0}^{t}[\dot{X}_n(s)+X_n(s)A+A^T X_n(s)-X(s)BX_n(s)+C]ds \tag{18}$$

For all n=0,1,…

Proof:

Rewrite equation (3) in the form of equation (12) in matrix form. Then using the resulting Lagrange multiplier for equations (10) and (11) implies that the *nxn* values of the Lagrange multiplier $\lambda(t)$ to be $\lambda(t)=-1.$

In order to ensure the convergence of the sequence of approximate solutions obtained by using equation (18), which may be proved to be converge to the exact solution $X(s)$. Before introducing the proof, the nonlinear operator M is defined by:

$$MX=-XBX$$

Which is assumed to satisfy Lipschitz condition with constant L, i.e., satisfy:

$$\|X_1BX_1-X_2BX_2\|\le L\|X_1-X_2\| \tag{19}$$

Where $\|\cdot\|$ is an appropriate matrix norm.

Theorem (2):

Let $X_1,...,X_n\in(C^1[0,b],\|\cdot\|_\infty)$, *n* = 0,1,…; be the exact and approximate solutions of the Riccati matrix differential equation (1). If $E_n(t)=X_n(t)-X(t)$ and $MX=-XBX$ satisfies Lipschitz condition with constant L, then the sequence of approximate solutions $\{X_n(t)\}$, *n* = 0,1,…; converges to the exact solution $X(t)$.

Proof

The approximate solution of Riccati matrix equation (1) using the VIM is given by (18) and since X is the exact solution, then it satisfy:

$$X=X-\int_{t_0}^{t}[\dot{X}+XA+A^TX-XBX+C]ds \tag{20}$$

Now, substract equation (20) from equation (18) to get:

$$X_{n+1}-X=X_n-X-\int_{t_0}^{t}[\dot{X}_n+X_nA+X_nA^T-X_nBX_n+C-\dot{X}-XA-XA^T+XBX-C]ds$$

Since $E_n = X_n - X$, then:

$$E_{n+1}(t) = E_n(t) - \int_{t_0}^{t} [\dot{E}_n(s) + E_n(s)A + A^T E_n(s) - (X_n(s)BX_n(s) - X(s)BX(s))]ds$$
$$= E_n(t) - \int_{t_0}^{t} \dot{E}_n(s)ds - \int_{t_0}^{t} E_n(s)Ads - \int_{t_0}^{t} A^T E_n(s)ds + \int_{t_0}^{t} (X_n(s)BX_n(s) - X(s)BX(s))ds \quad (21)$$

And upon using the method of integration by parts for the first integral of equation (21), to get:

$$E_{n+1}(t) = E_n(t) - E_n(t) + E_n(0) - \int_{t_0}^{t} E_n(s)Ads - \int_{t_0}^{t} A^T E_n(s)ds +$$
$$\int_{t_0}^{t} (X_n(s)BX_n(s) - X(s)BX(s))]ds \quad (22)$$

Hence taking the suprumum norm to the both sides of equation (22) yields to:

$$\|E_{n+1}(t)\| \le \int_{t_0}^{t} \|E_n(s)\|\|A\|\,ds + \int_{t_0}^{t} \|A^T\|\|E_n(s)\|\,ds + \int_{t_0}^{t} \|X_n(s)BX_n(s) - X(s)BX(s)\|\,ds$$
$$\le \int_{t_0}^{t} \|E_n(s)\|\|A\|\,ds + \int_{t_0}^{t} \|A^T\|\|E_n(s)\|\,ds + L\int_{t_0}^{t} \|E_n(s)\|ds$$
$$= (2\|A\| + L)\int_{t_0}^{t} \|E_n(s)\|ds$$

Therefore:

$$\|E_{n+1}(t)\| \le (2\|A\| + L)\int_{t_0}^{t} \|E_n(s)\|ds$$

Now, if n=0, then:

$$\|E_1(t)\| \le (2\|A\| + L)\int_{t_0}^{t} \|E_0(s)\|ds$$
$$\le (2\|A\| + L) \sup_{s\in[t_0,b]} |E_0(s)| \int_{t_0}^{t} ds$$
$$= (2\|A\| + L)t \sup_{s\in[t_0,b]} |E_0(s)|$$

If n=1, then:

$$\|E_2(t)\| \le (2\|A\| + L)\int_{t_0}^{t} \|E_1(s)\|\,ds$$

$$\leq (2\|A\|+L)\int_{t_0}^{t}[2\|A\|+L)s \sup_{s\in[t_0,b]}|E_0(s)|]ds$$

$$=(2\|A\|+L)^2 \sup_{s\in[t_0,b]}|E_0(s)|\int_{t_0}^{t} sds$$

$$\leq \frac{(2\|A\|+L)^2}{2} t^2 \sup_{s\in[t_0,b]}|E_0(s)|$$

If n=2, then:

$$\|E_3(t)\| \leq (2\|A\|+L)\int_{t_0}^{t}\|E_2(s)\|ds$$

$$\leq (2\|A\|+L)\frac{\int_{t_0}^{t}(2\|A\|+L)^2}{2} s^2 \sup_{s\in[t_0,b]}|E_0(s)|ds$$

$$\leq \frac{(2\|A\|+L)^3 t^3}{3!} \sup_{s\in[t_0,b]}|E_0(s)|$$

And so on in general for any natural number n.

$$\|E_n(s)\| \leq \frac{(2\|A\|+L)^n t^n}{n!} \sup_{s\in[t_0,b]}|E_0(s)|$$

$$\leq \frac{(2\|A\|+L)^n b^n}{n!} \sup_{s\in[t_0,b]}|E_n(s)|$$

Since $(2\|A\|+L)b<1$ is a constant and as $n\to\infty$ then $\frac{1}{n!}\to 0$ which implies $\|E_n(t)\|\to 0$ as $n\to\infty$, i.e., $X_n(t)\to X(s)$ as $n\to\infty$, which means that the sequence of approximate solutions using the variational iteration formula equation (18) converge to the exact solution of Riccati matrix differential equation .

5. Illustrative Examples

In this section, two illustrative examples will be considered, where it is notable that the matrix C may be considered as a matrix function of t which has no effect on the derivation of the Lagrange multipliers $\lambda(t)$ (see theorem (1)). The first example is for the one-dimensional Riccati differential equation while the second example is for 2×2 matrix Riccati differential equation. In each example a comparison have been made with the exact solution.

Example (1): Consider the first order Riccati differential equation:

$$y'(x)=1+x^2-y^2, \qquad x\in[0,1]$$

With initial condition $y(0)=1$, where comparison with equation (3), give:

$A=0, B=-1$ and $C=-1-x^2$

The exact solution for comparison purpose is given by:

$$y(x) = x + \frac{e^{-x^2}}{1+\int_{t_0}^{t} e^{-t^2} dt}$$

Hence, starting with the initial solution $y_0(x) = 1$, and by applying the variational iteration formula (18), we get the first three approximate solutions:

$$y_1(x) = y_0(x) - \int_{t_0}^{t} [y_0'(x) - 1 - s^2 + y_0{}^2(s)]ds$$

$$= 1 + \frac{x^3}{3}$$

$$y_2(x) = y_1(x) - \int_{t_0}^{t} [y_1'(x) - 1 - s^2 + y_1{}^2(s)]ds$$

$$= 1 + \frac{x^3}{3} - \frac{x^4(2x^3+21)}{126}$$

$$y_3(x) = y_2(x) - \int_{t_0}^{t} [y_2'(x) - 1 - s^2 + y_2{}^2(s)]ds$$

$$= \frac{x^3}{3} - \frac{x^5(88x^{10} + 2310x^7 - 5040x^6 + 16170x^4 - 93555x^3 - 349272)}{5239080} - \frac{x^4(2x^3+21)}{126} + 1$$

Also, numerical results and its comparison with the exact solution are given in Table 1.

Table 1. Absolute error between the exact and approximate solutions of example (1)

x	$\|y_1(x) - y(x)\|$	$\|y_2(x) - y(x)\|$	$\|y_3(x) - y(x)\|$
0	0	0	0
0.1	1.60E-05	6.45E-07	0
0.2	2.47E-04	2.00E-05	0
0.3	1.21E-03	1.48E-04	0
0.4	3.68E-03	6.10E-04	0
0.5	8.71E-03	1.83E-03	0
0.6	0.018	4.51E-03	0.001
0.7	0.032	9.72E-03	0.002
0.8	0.053	0.019	0.006
0.9	0.082	0.035	0.011
1	0.123	0.06	0.022

Example (2): Consider the 2×2 Riccati matrix differential equation with:

$$A = \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix},\ B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix},\ C(t) = \begin{pmatrix} t^2+4t+1 & 2t^2+2t+2 \\ 2t^2 & t^3-3 \end{pmatrix}$$

Which has the exact solution:

$$X(t) = \begin{pmatrix} t & 1 \\ 0 & t \end{pmatrix}$$

Then using the variational iteration formula to find $X_1, X_2, ..., X_5$ with initial solution $X_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. The results of the absolute error for each components with the exact solution had given in Table 2-5.

Table 2. Absolute error of the first components solutions $X_1, X_2, ..., X_5$

t	X_1	X_2	X_3	X_4	X_5
0	0	0	0	0	0
0.1	0.02	3.59E-03	3.15E-04	1.09E-05	2.89E-07
0.2	0.083	0.031	5.84E-03	5.08E-04	2.66E-06
0.3	0.189	0.108	0.033	5.24E-03	2.26E-04
0.4	0.341	0.266	0.116	0.028	3.15E-03
0.5	0.542	0.532	0.299	0.103	0.02
0.6	0.792	0.925	0.631	0.289	0.087
0.7	1.094	1.453	1.135	0.651	0.274
0.8	1.451	2.1	1.783	1.221	0.666
0.9	1.863	2.821	2.483	1.947	1.299
1	2.333	3.527	3.12	2.697	2.083

Table 3. Absolute error of the second components solutions $X_1, X_2, ..., X_5$

t	X_1	X_2	X_3	X_4	X_5
0	0	0	0	0	0
0.1	0.011	8.96E-04	3.81E-05	9.53E-06	5.44E-07
0.2	0.045	8.91E-03	9.82E-05	2.47E-04	3.54E-05
0.3	0.108	0.035	2.26E-03	1.26E-03	3.66E-04
0.4	0.203	0.095	0.016	2.04E-03	1.55E-03
0.5	0.333	0.203	0.058	5.10E-03	2.67E-03
0.6	0.504	0.373	0.152	0.041	4.80E-03
0.7	0.719	0.608	0.321	0.14	0.047
0.8	0.981	0.901	0.572	0.337	0.171
0.9	1.296	1.222	0.896	0.643	0.421
1	1.667	1.511	1.282	1.026	0.795

Table 4. Absolute error of the third components solutions $X_1, X_2, ..., X_5$

t	X_1	X_2	X_3	X_4	X_5
0	0	0	0	0	0
0.1	6.67E-04	1.23E-03	2.04E-04	1.25E-05	1.59E-07
0.2	5.33E-03	8.93E-03	3.24E-03	4.51E-04	2.13E-05
0.3	0.018	0.027	0.016	3.71E-03	3.72E-04
0.4	0.043	0.056	0.048	0.016	2.77E-03
0.5	0.083	0.096	0.106	0.049	0.013
0.6	0.144	0.144	0.193	0.113	0.04
0.7	0.229	0.198	0.297	0.21	0.098
0.8	0.341	0.261	0.391	0.318	0.185
0.9	0.486	0.343	0.434	0.395	0.271
1	0.667	0.467	0.369	0.395	0.304

Table 5. Absolute error of the fourth components solutions $X_1, X_2, ..., X_5$

t	X_1	X_2	X_3	X_4	X_5
0	0	0	0	0	0
0.1	3.33E-04	6.60E-04	1.55E-05	4.92E-06	4.80E-07
0.2	2.67E-03	5.14E-03	5.54E-04	9.98E-05	2.55E-05
0.3	9.00E-03	0.017	3.09E-03	3.29E-04	2.14E-04
0.4	0.021	0.037	0.011	2.70E-04	7.06E-04
0.5	0.042	0.068	0.03	4.97E-03	7.34E-04
0.6	0.072	0.108	0.06	0.019	3.06E-03
0.7	0.114	0.16	0.101	0.048	0.017
0.8	0.171	0.224	0.145	0.089	0.047
0.9	0.243	0.309	0.177	0.13	0.086
1	0.333	0.429	0.182	0.146	0.113

6. Conclusion

In the current study, we have applied He's VIM to solve Riccati matrix differential equation, which is a powerful tool that has the ability to solve different types of linear and nonlinear operator equations. The general form of the approximate solution has been derived depending on the general criteria of the VIM for solving n-th order ODE's. Also, the convergence of the obtained approximate solution have also been proved by proving the convergence of the error term between the approximate and exact solutions to be tend to zero as the sequence of iterations increased. The obtained results of the illustrative examples seem to be reliable in which the absolute error between the approximate solution and the exact solution has been used also to check the accuracy of the obtained results.

In addition, different approach based on the theory of matrix algebra may be used to derive the approximate solution however with different rate of convergence to the exact solution and with more restrictions on the matrices A, B and C of the Riccati matrix differential equation, where the approximate solution in this case has the form as follows:

$$X_{n+1}(t) = X_n(t) - e^{-(A+A^T)t}\int_{t_0}^{t} e^{(A+A^T)s}[\dot{X}_n(s) + X(s)A + A^T X(s) - X(s)BX(s) + C]ds$$

n=0,1.

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