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Random Sampling and Signal Bregman Reconstruction Based on Compressed Sensing

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Abstract

Compressed sensing (CS) sampling is a sampling method which is based on the signal sparse. Much information can be extracted from as little as possible of the data by applying CS, and this method is the idea of great theoretical and applied prospects. In the framework of compressed sensing theory, the sampling rate is no longer decided in the bandwidth of the signal, but it depends on the structure and content of the information in the signal. In this paper, the signal is the sparse in the Fourier transform and random sparse sampling is advanced by programing random observation matrix for peak random base. The signal is successfully restored by Bregman algorithm. The signal is described in the transform space, and a theoretical framework is established with new signal descriptions and processing. The case is maked to ensure that the information loss, signal is sampled at much lower than the Nyquist sampling theorem requiring rate, but also the signal is completely restored in high probability. The random sampling has following advantages：alias-free, sampling frequency need not obey the Nyquist limit, and there is higher frequency resolution. The random sampling can measure the signals which their frequencies component are close, and it can implement the higher frequencies measurement with lower sampling frequency.

Keywords: *compressed sensing, random sampling, nonuniformly sampling, sparse sampling, signal reconstruction*

1. Introduction

Compressive Sensing (Compressed Sensing, CS) literally looks as if the data compression means, and it is indeed for a completely different considerations. Classic data compression technology, whether it is audio compression (such as mp3), image compression (for example, jpeg), video compression (mpeg), or general coding compression (zip), is all from the characteristics of the data itself, to find and eliminate the implied redundancy in data, so as to achieve the purpose of compression. There are two characteristics in such compression : First, it occurs after the data has been collected completely; second, which itself requires a complex algorithm to be completed. In contrast, the decoding process is relatively simple in computation in general, for example, in the audio compression, the calculation amount of compression is much larger than the calculation amount of an mp3 file playback (ie decompress) .

This asymmetry of the compression and decompression is just the opposite with the people's needs. This asymmetry of the compression and decompression is just the opposite with the people's needs. In most cases, data acquisition and processing equipment often be low-cost, low power, low capacity portable computing devices, such as shoot camera or voice recorder, the remote control monitors. The process of dealing with (and decompression) information often be on large computers , it has more computing power, and there are often not portable and power requirements. In other words, low-cost energy-saving equipment are used to handle complex computing tasks, but a large and efficient equipment is used to process relatively simple computing tasks. This contradiction is even more acute in some cases, for example, work in the field or in the case of military operations, data acquisition devices are often exposed to the natural environment, energy supply may be lost at any time or there even be

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partial loss of performance, in which case, the traditional data acquisition - compression - transmission - decompression mode is basically ineffective.

Compressed sensing is to resolve such conflicts arising. After collecting the data anyway, you want to compress them out redundancy, and this compression process is relatively difficult, why do not we direct "acquisition" compressed data? How much lighter collection tasks, but the need is also eliminated for compression trouble. This is called "compressed sensing", that is, direct perception of compressed information.

Traditional Nyquist sampling theorem requires a sampling rate of not less than twice the highest frequency signal, with the development of signal processing technology and the surge in the amount of processed data. This sampling method has been far can not keep up the high-speed signal processing requirements. In 2006, Donoho put forward compressed sensing (Compressed sensing, CS) theory [1,2], if the signal has a sparse nature, it can take advantage of its sparse features, based on the points less than the number of signal sampling point, it can be approximated to restore the original signal[3]. This theory has greatly promoted the process of signal processing theory, and there are broad application prospects. Currently, compressive sensing theory have a very good application in image compression, converting analog information, bio-sensing, signal detection and classification, wireless sensor networks, data communications and geophysical data analysis and other fields[4]. If you want to capture a small part of the data and expect to decompres a lot of information from these few data , it is necessary to ensure that: First, the amount of collected data contains global information of the original signal, and the second, there is an algorithm which is able to restore the original information from the amount of data.

In this study, a compressed sensing random sampling and high resolution signal reconstruction is researched. The method utilizes non-uniform sampling from the sampling frequency limits, there are the advantages of high frequency resolution and anti-aliasing [5], at a low sampling frequency, the signal spectrum is obtained based on non-uniform sampling of the Fourier transform, and then split Bregman method is used to reconstruct signal and to reduce the noise spectrum of the non-uniform sampling[6,7].

2. Random Sampling and Analysis

2.1. Uniform Sampling Drawbacks

Uniformly sampled time function is a linear function of the standard; it is equally spaced sampling time. The sample signal is defined as $x(t)$, the sampling interval is Δt, the sampling function of time is $t_n = n\Delta t$, and the sampling frequency is $f_s = \frac{1}{\Delta t}$ and it is to satisfy the sampling theorem, it is greater double. Than the highest frequency of signal. For the sampled signal discrete finite length, ie $x[n] = x([1:N]\Delta t)$, N is the number of sampling points, the sampling duration $T = N\Delta t$.

Fourier transform is used to analyze the sampled signal $x(t) = \sin(2\pi ft), (f = 185Hz, N = 256, f_s = 256Hz)$. The analysis result of signal Spectrum shows Figure 1.

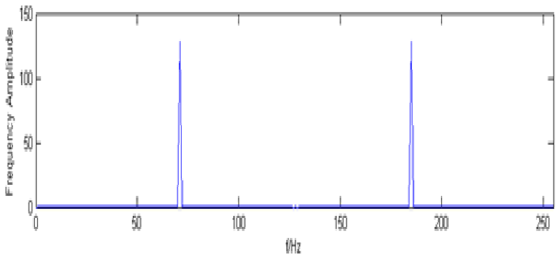

Figure 1. Signal Spectrum Analysis of Uniformly Sampling (f_s=256Hz)

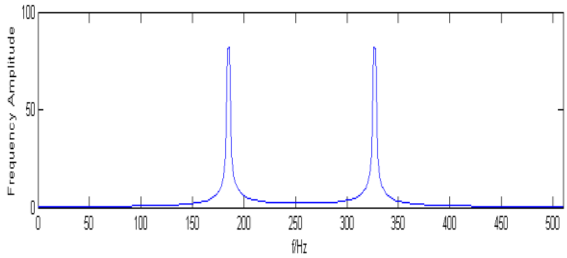

Figure 2. Signal Spectrum Analysis of Uniformly Sampling(f_s=512Hz)

As can be seen from Figure 1, since the sampling frequency is less than twice the frequency value of the sampling signal, there has been 71Hz frequency aliasing signal. Since the spectrum of the real signal are equal with the spectrum of aliasing signals, the true signal can be not distinguished. Note also that there is 1Hz frequency resolution in this case, signal frequency is an integer multiple of the frequency resolution, it is possible to accurately measure the frequency value.

Of the cases, the other parameters are constant; the sampling frequency is changed for to meet the limit of the sampling theorem. Analysis results of signal spectral are shown in Figure 2.

Figure 2 shows, there is no aliasing signal in the (0, $f_s/2$) band, but due to a change of the sampling frequency, the frequency resolution becomes 2Hz, the true frequency of signal is 185 Hz, it is not an integer multiple of the frequency resolution, and thus it leads to a spectrum leakage and barrier phenomenon, so that the measured frequency is $f = 188Hz$, it is deviated from the correct value.

As can be seen from the above analysis, uniform sampling is subject to the limitations of sampling frequency; there is aliasing frequency; frequency resolution is not high; there spectrum leakage and fence phenomena and other issues.

2.2. Random Sampling and its Fourier Transform

Random sampling is sometimes known as non-uniform sampling, as opposed to evenly sampling, it os a sampling mode. The sampling interval of random sampling is random, the time interval is generally set at an unequal interval, and number of samples is not a linear function with the sampling time. Random sampling is not limited by the sampling theorem, it increases the detection range of the frequency, and it can be achieved in the short data length and a low sampling frequency to detect the frequency of the higher order, which can meet fast the requirements of real-time specific occasions. Most importantly, due to random sampling, non-uniform sampling is used to eliminate aliasing problems which are caused by uniform sampling; there are also the advantages of high frequency resolution, and it can reduce the spectrum leakage, eliminating the phenomenon of fences and other issues.

In the example above, the other parameters are constant, it is instead of random sampling, $t_n = rand(0,1)T = g(n), x(n) = x(t_n)$, Where rand (0,1) is random value between (0,1) , $n = 1,2,\cdots,N$, g(n) is a nonlinear function of n. Its Fourier transform is equation (1):

$$X(\omega) = \sum_{n=1}^{N} x(n)\exp(-j\omega t_n) \tag{1}$$

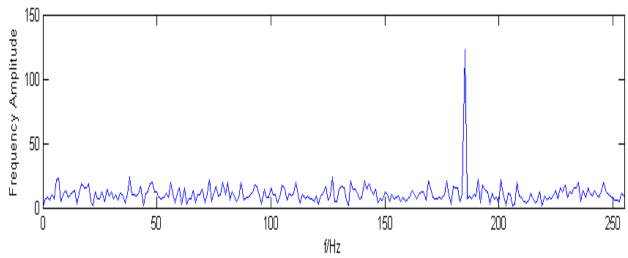

Figure. 3. Signal Spectrum Analysis of Random Sampling (Average f_s=256Hz)

The Fourier transform spectral analysis results of random samples (N = 256) are shown in Figure 3. Random sampling time is used to increase the sampling interval, frequency resolution is improved, the phenomenon of the fence is eliminated. Due to random sampling, aliasing will no longer concentrate on a few specific points and sampling frequency-dependent, butit is evenly distributed to all the signal frequencies. In addition, the spectrum leakage can also cause noise spectrum. The spectral noise may is reduced with the increase number of sampling points.

3. Compressed Sensing Principle
3.1. Compressed Sensing Statements

The main idea of compressive Sensing (Compressive Sensing CS) theory is that: Suppose the coefficients of a signal x with length N are sparse (that is, only a small number of non-zero coefficient) on orthogonal basis or on a tight frame Ψ, if the coefficients are projected to another observation group $\Phi : M \times N$, $M << N$, which is not related to the transform group Ψ, the observation set $y : M \times 1$ is obtained. Then the signal x can rely on these observations to solve an optimization problem and accurate recovery. CS theory is the theoretical framework with a new sampling while achieving the compression purpose, its compressive sampling procedure is shown in Figure 4.

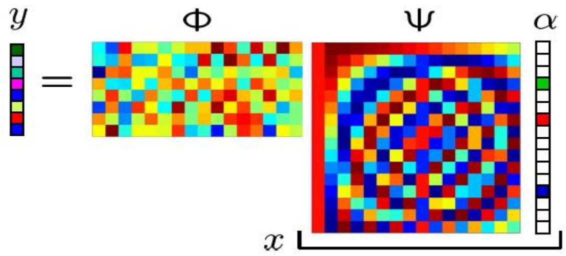

Figure 4. Compression Sampling Process

First, if the signal $x \in R^N$ is compressible on an orthogonal base or tight frame Ψ, the transform coefficients $\alpha = \Psi^T x$ are obtained, α is equivalent x or approximation sparse representation; The second step, a smooth measurement matrix Φ is designed, which is irrelevant to the transform base Ψ with $M \times N$ dimension, the observation x is projected to M-dimensional space to give the set of observations $y = \Phi x$, the sampling process is compressed, i.e. drawing sample [8]. Finally, the optimization problem solving $\hat{x}$ approach x exactly or approximately.

When the observations with noise z,

$$y = \Phi x + z \tag{2}$$

It can be converted for the sake:

$$\min_x \| \Psi^T x \|_1 \qquad s.t. \| y - \Phi x \|_2 < \varepsilon \tag{3}$$

or

$$\hat{x} = \underset{x}{\operatorname{argmin}} \frac{1}{2} \| y - \Phi x \|_2 + \lambda \| \Psi^T x \|_1 \tag{4}$$

3.2. Recovery signals separable Bregman iterative algorithm

Equestion (4) solving is first converted to sparse vector (5) solving, $A = \Phi \Psi$, then

$$\hat{\alpha} = \underset{\alpha}{\operatorname{argmin}} \frac{\lambda}{2} \| y - A\alpha \|_2^2 + \| \alpha \|_1 \tag{5}$$

Specific steps of Bregman algorithm are as follows [9, 10]:

Step 1: to calculate $B=(\lambda A^T A+I_N)^{-1}$, I_N is the N-order unit matrix, $F=\lambda A^T y$; b_0, d_0 are N-dimensional zero vector;

Step 2: $\lambda(=10)$ is given, the iteration terminated conditions $\delta(=0.001)$, the number of iterations $n=1$;

Step 3: to calculate $\alpha_n = B(F+d_{n-1}-b_{n-1})$,

$$d_n = sign(\alpha_n + b_{n-1})\max(|\alpha_n + b_{n-1}|-1,0) ,$$

$$b_n = b_{n-1} + \alpha_n - d_n ;$$

Step 4: if $\|\alpha_n - \alpha_{n-1}\| \geq \delta$, $n=n+1$, go to step 3; otherwise, the iteration stops, $\hat{\alpha}=\alpha_n$;

Step 5: $\hat{x}=\Psi\hat{\alpha}$.

4. Sparse Random Sampling Design

4.1. Signal Sparse Representation

Transform based of adaptive signal is ψ, the signal expression should be sparse at the base[11]. Transform coefficient vector of signal X is under the transform base ψ, if 0 <P <2 and R> 0, these coefficients satisfy: $\|\alpha\|_p = (\sum_i |\alpha_i|^p)^{\frac{1}{p}} \leq R$, the coefficient vector is sparse[12]. If the support domain $\{i:\alpha_i \neq 0\}$ of the transform coefficients $\alpha_i = <X,\psi_i>$ is equal or less K, namely there is K nonzero entries only in $\alpha \in R^N$. The number K of non-zero entries reflect the signal inherent freedom. Or sparseness is a measure of non-zero coefficients, and it constitutes a number of scales of the signal component. Typically, sparse representation of the signal can be measured by the 0-norm of the representing vector. A vector 0-norm is the number of non-zero elements in the vector. Fourier transform is our common one [13-14].

4.2. Irrelevant and Isometric Properties

The adaptive $M \times N$-dimensional measurement matrix Φ is designed which is not related to transformation base ψ. Observation matrix Φ design goal is to restore the original signal as little as possible from the observation values. In the specific design, it is need to consider the following two aspects:

1) The relationship between the observation matrix Φ and the base matrix ψ;

2) The relationship between the matrix $A=\Phi\psi$ and K- sparse signals α .

First, the observation matrix ϕ and the base matrix ψ have incoherence. Coherent between observation matrix ϕ and the base matrix ψ is defined as formula (6):

$$\mu(\Phi,\psi)=\sqrt{N}\max_{\substack{1\leq k\leq m\\1\leq j\leq N}}|<\phi_k,\psi_j>| \quad (6)$$

The degree of coherence μ gives the maximum coherence between any two vectors in Φ and ψ. When Φ and ψ contain coherent vectors, their coherence degree is coherence. The compressed sampling of the signal makes for each observation contains the different information of the original signal as much as possible, which requires orthogonal between the vectors of Φ and ψ, that degree of coherence μ is as small as possible, which is the reason of incoherence between the measurement matrix and base matrix [15]. Just to satisfy the following formula, signal can be reconstructed high probability [16].

$$M \geq C\mu^2(\Phi,\Psi)K\log N \quad (7)$$

Second, the relationship between the matrix $A = \Phi\psi$ and the K- sparse signals is about Restricted Isometry Property (RIP) [17], the matrix "equidistance constant": any K = 1, 2, ···, the matrix A isometric constant δ_K is defined to satisfy the minimum value of the following formula (8), which α is optionally K- sparse vector:

$$(1-\delta_K)\|\alpha\|_2^2 \le \|A\alpha\|_2^2 \le (1+\delta_K)\|\alpha\|_2^2 \quad (8)$$

If δ_K <1, Matrix A is called to satisfy the K-order RIP, matrix A is approximately to ensure in this time that the Euclidean distance of K- sparse signal α remains unchanged, which means that α is impossible in null space of A (otherwise there will be infinitely many solutions for α). In practice, random matrix is commonly used as observation; common are Gaussian measurement matrix, binary measurement matrix, Fourier observation matrix and irrelevant observation matrix. Random observations provide an effective way to achieve the compressed samples [18-19].

4.3. Low-rate Signal Sampling Design

In fact, the design of the observation partis is to design an efficient observation matrix, which is to design efficient observation (ie, sampling) agreement of useful information to capture a sparse signals, whereby the signal is compressed into a small number of sparse data. The agreement is non-adaptive, it requires only a small amount of link between the fixed waveform and the original signal, the fixed waveform is irrelevant to signal compact represented base. In addition, the observation process is independent of the signal itself. The reconstructed signal can be collected by optimization methods in a small number of observations.

Sampling interval is [0, T], M points were collected randomly in this interval, $t_i = rand(0,1)T, i = 1,2,\cdots,M$, $rand(0,1)$ is random point in interval (0,1); $x = [x(t_1), x(t_2), \cdots, x(t_M)]^T$, $y = x$, random measurement matrix Φ design is quite random spike base $\phi_k(t) = \delta(t-k)$, k is one of the M which are randomly selected in [1,2, ..., N] , ψ is Fourier base, $\psi_j(t_n) = N^{-1/2} e^{i2\pi j t_n}, j,n = 1,2,\cdots,N$, t_n belongs to domain collections of the reconstructed signal including a random collection. Such design of Φ and ψ meet incoherence in formula (6) and Restricted Isometry Property (RIP) in formula (8). A is composed of the M (M> K) row vectors which are randomly selected from the Fourier basis matrix ψ. A sampling method of the part Fourier transform is presented, the signal Fourier transform is first done, then the transform coefficients are randomly selected, the random selection makes each observation with random uncorrelated characteristics. A random related feature of observation Matrix is a sufficient condition for proper signal recovery, there is highly irrelevant between observation matrix and the signal; these can restore and ensure the effective signal.

5. Experimental Testing and Evaluation

Experimental select signal function is:

$$f(t) = \sin(40\pi t) + \sin(140\pi t) + \sin(300\pi t) \quad (9)$$

Sampling frequency $f_s = 256Hz$. The highest frequency of signal, $f_{max} = 150Hz, f_s = 256Hz < 2f_{max} = 300Hz$, The sampling frequency is less than twice the highest frequency of signal, it does not meet the Nyquist sampling theorem, it is referring to Figure 5, Fourier transform exist spectrum aliasing and disclosure; With the same sampling frequency, the random sampling of discrete Fourier transform (see Figure 5) is used to overcome both aliasing and disclosure; The random sampling and reconstruction of compressed sensing are proposed in this study, reconstruction is very consistentin either the frequency domain or the time domain. Figure 5 is compared to a signal compressed sensing sparse sampling and high resolution reconstruction results.

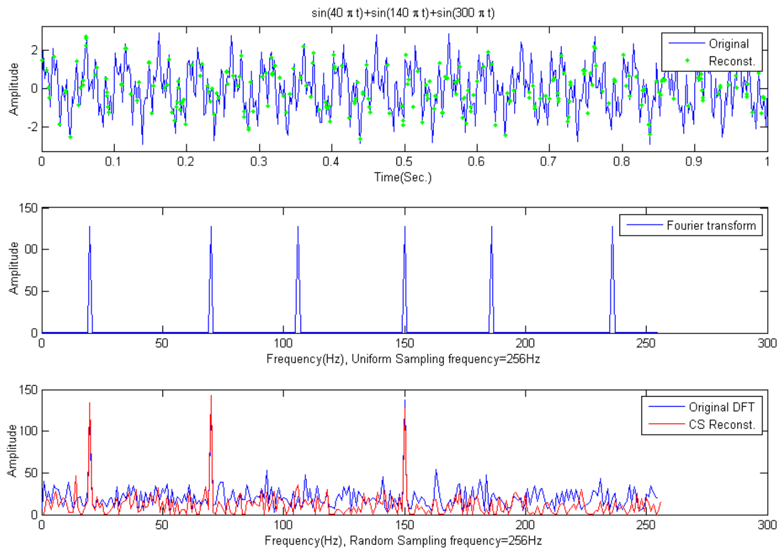

Figure 5. Compressed Sensing Sparse Sampling and Signal Reconstruction

The above picture of figure 5 shows the original signal and the random sampling reconstructed signal in time domain , the middle picture of figure 5 shows the Fourier transform of the uniform sampling, the below picture of figure 5 shows the original signal and its random sampling reconstructed signal recovery in the frequency-domain; the relative error of time domain reconstructed signal is Relative error = 0.1787.

6. Conclusions

Random sampling technique is as a non-uniform sampling method, it can effectively improve the sampling rate in sampling system [20-21]. In a random sampling, the non-uniform distribution characteristics of the sampling interval are not need to collect enough samples to signal reconstruction. In this study, the sparsity of signal is used in the Fourier transform, random observation matrix is designed as spikes random base, namely it is random sparse sampling; Bregman iterative algorithm is used successfully to restored signal. This method does not increase the costs on the basis of any hardware, and it is able to reconstruct the original signal by the limited random samples. Experimental results show that the frequency domain sparse signals are sampling far below the Nyquist frequency signal sampling rate, the original signal is accurately reconstructed by compressing the sensor signal reconstruction algorithms.

Compression sensing basic idea is to extract much information as little as possible from the data [22], there is no doubt that it has a great theory and promising idea. Compressed sensing is an extension of the traditional information theory, but it is beyond the traditional compression theory, it has become a new sub-branch. When the signal has a sparse feature, compressed sensing can accurately reconstruct the source signal by a small number of observations which is much smaller than the length of signal. Compressed sensing theory is that sampling and compressed signal are combined into a single step, the signal is encoded, it breaks the traditional Nyquist sampling theorem limit in a certain extent, the burden is reduced on the hardware processing.

In the framework of the compressed sensing theory, the sampling rate is no longer determined by the bandwidth of the signal, but depending on the structure and content of the information in the signal. It uses transform space to describe signal, a new theoretical framework is established for describing and signal processing, so that in the case to ensure that information is not lost, with far lower than signal sampling rate which the Nyquist sampling theorem requires, but also recovery signal is completed in high probability.

Acknowledgements

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