

A Compressed Sensing Signal Processing Research

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Abstract

In the classical Shannon/Nyquist sampling theorem, information is not lost in uniformly sampling a signal, signal must be sampled at least two times faster than its bandwidth. Because of the restriction of the Nyquist rate, it end up with too many samples in many applications, and it becomes a great challenge for further transmission and storage. In recent years, an emerging theory of signal acquirement is a ground-breaking idea compared with the conventional framework of Nyquist sampling theorem, it is called as compressed sensing (CS). Compressed sensing is considered in the sampling as an novel way, and a brand new field is opened up for signal sampling process. It also reveals a promising future of application. The background of compressed sensing development is reviewed in this study. The framework and the key technique of CS are introduced, and some naïve application is illustrated on image process.

Keywords: information sampling, compressed sensing, compressed measurement, optimization recovery

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1. Introduction

The rapid development of information technology makes it sharp increase in demand for information. Traditional signal acquisition techniques generally need to go through the A/D sampling and compression are two aspects which the Nyquist sampling theorem is the basis of the guidance on how the classical theory of sampling, the sampling rate must reach more than twice the signal bandwidth in order to accurately reconstruct the signal. However, with the increase of people to demand information, the information-carrying signal bandwidth grows ever wider, increasingly high sampling rate and processing speed as a signal processing framework based on the requirements, thus the broadband signal processing difficulties in intensifying. For example, high-resolution Geographic Resources Observation its massive amount of data transfer and storage is a tough job. The other hand, in practical application, in order to reduce the cost of the storage, processing and transmission, it is often used compression method represented by fewer number of bits of the signal, a large number of non-critical data is discarded. Process this high-speed sampling and then compressed sampling wasting a lot of resources, so naturally raises a question: the possibility of using other transform space to describe the signal, the establishment of a new theoretical framework of signal description and processing, so that in the case to ensure no loss of information, far lower than Chennai the Nyquist sampling theorem requirements rate sampling signal, and also can be fully restored signal? Whether that signal will be the sampled transformed into information sampling? If this problem is solved, you can greatly reduce the sampling frequency of the signal, and data storage and transmission cost significantly reduce the signal processing time and the cost of computing, and will lead the signal processing into a new era of revolution.

In recent years, there's a new theory compressed sensing provides a viable solution. With traditional Nyquist sampling theorem, it is a sampling and compression simultaneously technology. It pointed out that, as long as there is a sparse signal in a transform domain expressions, you can use a transformation group observation matrix transform domain signal projection to a low space, effectively reducing the large number of non-critical data. Reconstruct the original signal is through the establishment of a mathematical model for solving optimization problems, seek these converted signal, and reconstruct the original signal with high probability, the experiment proved the transformed sparse signal contains enough information about the

reconstructed signal. In this theory, the sampling rate is not determined by the bandwidth of the signal, and depends on the structure and content of the information in the signal.

Compressed sensing is a new information theory [1, 2]. The theory is that the signal can be much lower than the Nyquist criterion for data sampling, it is able to accurately recover the original signal. The theory is surprised and proposed, it is the high degree of concern in the field of information theory, such as image signal processing, pattern recognition, medical imaging, radar imaging, wireless communication. The CS theory still belong to the infancy, but it has shown strong vitality, and it has developed a distribution of CS theory (Baron, et al.) [3], 1-BIT CS theory (Baraniuk) [4, 5], Bayesian the CS theory [6, 7], infinite dimensional CS theory, deformation CS theory, and so on, it has become a research focus in the field of mathematics and engineering applications.

2. Compression Perception Theory Framework

2.1. Mathematical Model of Compressed Sensing

The discrete-time implementation of the one-dimensional signal x , it can be expressed as $N \times 1$ column vector. The real signal x in a set of orthogonal basis $\{\psi_i\}_{i=1}^N$ (ψ_i for the N -dimensional column vector) are expanded, $\Psi = [\psi_1 \psi_2 \dots \psi_N] \in \mathbb{R}^{N \times N}$ is the orthogonal basis matrix (meet $\Psi \Psi^H = I$), the expansion coefficients can be expressed as $S_i \leq x$, $\psi_i \geq \psi_i^H x$. It is written in matrix form can be obtained:

$$x = \Psi S \quad (1)$$

You can clearly see that x and S are equivalent representation of the same signal in the time domain and Ψ domain [7, 8].

Assuming that the coefficient vector $S = [s_1, s_2, s_3 \dots s_N]^T$ is K -sparse, i.e., wherein the number of the non-zero coefficient $K \ll N$, then the signal x is compressible. Compressible signal refers using K a large coefficient good approximation signal by a certain magnitude, i.e. it Expand the coefficients in an orthogonal base exponential decay, with a very few large coefficient and many small coefficient this transform compression method known as transform coding.

Traditional transform coding first need to sample the N signal samples, followed by converting these signals to calculate, extraction where K is one of the most important signal encoding, the remaining $(N-K)$ a signal will be discarded. Obviously then $K \ll N$, the traditional transform coding algorithm in terms of storage and computing spending will be great. Skip intermediate perception compression step Get N signal, and direct access to a signal after compression. Consider a general linear observation process, we have adopted an orthogonal base Ψ Irrelevant observation matrix $M \times N$ ($M < N$) Φ (here Φ of each line can be viewed as a sensor, it is multiplied with the coefficient, to obtain a part of the information of the signal), a compression of the signal x perform observations:

$$y = \Phi x \quad (2)$$

Can obtain M linear observation $y \in \mathbb{R}^M$, the linear projection contains a small amount of the reconstructed signal enough information. Recover x from y is a solution of linear equations, but because $M < N$ that seems to be impossible, because a number of unknowns is greater than the number of the number of equations in the morbid equation, there are infinitely many solutions. However, the formula (1) is substituted into (2), can be obtained:

$$y = \Phi x = \Phi \Psi s = \Theta s \quad (3)$$

Where Θ operator called compressed sensing. Although recovered from y is also an ill-posed problem, but because of the coefficient s is K -sparse, and the $K < M$ observation matrix

to satisfy constraint equidistant nature (RIP), i.e., for any coefficient signals s and the constant $\varepsilon > 0$, and satisfies:

$$1 - \varepsilon \leq \frac{\|\Theta s\|_2^2}{\|s\|_2^2} \leq 1 + \varepsilon \quad (4)$$

Properties showed that: the RIP can keep the energy of the signal on the original time domain (1) observed wherein constant, holds K important component of the length, that is the observation process is stable (1) if it is sparse, Θ must be dense. Indeed, if we want to restore s accurately the signal from y , at least the required quantity $M \geq K \log_2 \left(\frac{N}{K} \right)$ and are linearly independent observational data. Construct these linearly independent data requirements and Φ is not similar observation matrix Ψ . Related studies have shown that, when the matrix Φ is a random matrix, both observation matrix may not similar, and it also has a universal, applicable to different observation matrix. The contribution of the Θ 's elements in the recovery s made are the same, in other words, each observables y_i are equally important or unimportant. Φ can meet the matrix of the Gaussian distribution of the white noise, or Bernoulli force distribution matrix ± 1 and so on [9-11].

When the compressed sensing information operator Θ to meet the above conditions, the reconstructed signal algorithm by solving the optimization problem to find a signal sparse coefficient vectors s . The optimization problem can be viewed as the p -norm problem solving, based on the 1 - norm optimization problem, namely:

$$\min \|s\|_1 \quad s.t. \quad \Theta s = y \quad (5)$$

Can accurately recover a K -sparse signal s , and as long as $M \geq K \log_2 \left(\frac{N}{K} \right)$ we can accurately reconstruct the original signal [12, 13].

2.2. Gradient-Based Recovery Algorithm

A quick, low spending and reconstruction algorithm to compress the core part of the perception theory. This paper describes the recovery of a gradient-based algorithm, and restore the image using this algorithm. Sparse image in the time domain, the process to address the 1 - norm constraint. For general images, it is seen as a constrained total variation (Total Variation) problem. The function $|S_{i,j}|$ is not differentiable at the origin, it should apply to the sub-gradient (subgradient) and smooth approximation method to solve [14, 15].

2.2.1. Gradient Algorithm

Given a linear matrix equation:

$$\Theta s = y \quad (6)$$

s solution can be obtained by solving the following minimization problem:

$$\min_f L(f) = \frac{1}{2} \|\Theta s - y\|_2^2 \quad (7)$$

The gradient algorithm solving (7) can be expressed as:

$$s^{i+1} = s^i - \mu^i \nabla L(s^i) \quad (8)$$

These μ^i is iterative step, at the same time:

$$\nabla L(s^i) = \Theta^H (\Theta s^i - y) \quad (9)$$

$$\mu^i = \frac{\langle \nabla L(s^i), \nabla L(s^i) \rangle}{\langle \nabla L(s^i), \Theta^T \Theta \nabla L(s^i) \rangle + \varepsilon} \quad (10)$$

$$\mu^i = (\Theta \Theta^H + \varepsilon \mathbf{I})^{-1} \quad (11)$$

From Equation (10) add a small amount ε in order to prevent the situation of zero denominator. S of the formula (11) in order to avoid the result of the approximate singular matrix appears. Step of Newton's method on the convergence of the best extreme points, the steepest descent method began to converge speed quickly, but with the increase in the number of iterations, the convergence rate will slow down [16].

2.2.2. 1 - Norm Constraint

Assumed $s_{i,j}$ to represent the pixels of an $N \times N$ image on the i -th row and j -th column, the optimization problem of sparse image thus can be expressed as:

$$\min_s H(s) \equiv L(s) + \lambda \|s\|_1 \quad (12)$$

$\|s\|_1 = \sum_{i,j} |s_{i,j}|$, the above equation called the Lagrange multiplier. The first item is related on these undetermined matrix equation $\Theta s = y$, and the second item is 1-norm compensation term (l_1 -penalty term) ensure that the obtained sparse solution. Parameter λ is used as the relationship between the balance of the first and second weights.

$|s_{i,j}|$ is not differentiable at the origin, we can by defining a new sub-gradient to solve this problem:

$$\nabla_{i,j} H(s) = \begin{cases} \nabla_{i,j} L(s) + \lambda \operatorname{sgn}(s_{i,j}), & |s_{i,j}| \geq \varepsilon \\ \nabla_{i,j} L(s) + \lambda, & |s_{i,j}| < \varepsilon, \nabla_{i,j} L(s) < -\lambda \\ \nabla_{i,j} L(s) - \lambda, & |s_{i,j}| < \varepsilon, \nabla_{i,j} L(s) > \lambda \\ 0, & |s_{i,j}| < \varepsilon, |\nabla_{i,j} L(s)| \leq \lambda \end{cases} \quad (13)$$

$$s_{i,j}^{n+1} = s_{i,j}^n - \mu_j^n \nabla_{i,j} H(s^n) \quad (14)$$

Wherein $s_{i,j}$ is the pixels of an $N \times N$ image on the i -th row and j -th column, n is said the iteration n -th step, the μ_j^n values given in (10), (11). In the steepest descent method, λ is set to a small constant ($\lambda = 0.001 \sim 0.01$). In Newton's method, the parameter λ must be gradually decreases ($\lambda^{i+1} = (0.99 \sim 0.999) \times \lambda^i$) with increasing iteration steps.

2.2.3. Constrained Total Variation

For the general image is not sparse in the time domain, so that does not comply with the 1 - norm problem solving prerequisite. General image solving convex optimization problem, can be expressed as:

$$\min_s H(s) \equiv L(s) + \lambda TV(s) \quad (15)$$

$$D_{i,j}^v s = \begin{cases} s_{i,j} - s_{i+1,j} & 1 \leq i < N \\ 0 & i = N \end{cases} \quad (16)$$

$$D_{i,j}^h s = \begin{cases} s_{i,j} - s_{i,j+1} & 1 \leq j < N \\ 0 & j = N \end{cases} \quad (17)$$

$$TV(s) = \sum_{i,j} \sqrt{(D_{i,j}^v s)^2 + (D_{i,j}^h s)^2} = \sum_{i,j} |\nabla_{i,j} s| \quad (18)$$

$$\nabla_{i,j}(TV(s)) = \frac{D_{i,j}^v s}{|\nabla_{i,j} s|} + \frac{D_{i,j}^h s}{|\nabla_{i,j} s|} - \frac{D_{i-1,j}^v s}{|\nabla_{i-1,j} s|} - \frac{D_{i,j-1}^h s}{|\nabla_{i,j-1} s|} \quad (19)$$

$$|\nabla_{i,j} s| = \sqrt{(D_{i,j}^v s)^2 + (D_{i,j}^h s)^2} + \varepsilon \quad (20)$$

3. Image Restoration Based on Compressed Sensing Image Restoration

If the image is sparse in the time domain is not possible to convert the image to the Fourier domain or wavelet domain. Next, we recover the two 256×256 images, the observation matrix size is 100×256 . The first image will be converted to the Fourier domain (Figure 1), and generates a random observation matrix random measurement, the number of measurements is 30% of the number of pixels of the image. Then restore the image by calculating the constrained total variation (TV) method. Figure 2(c) total variation restore the image of the peak signal-to-noise ratio is 34.46dB.

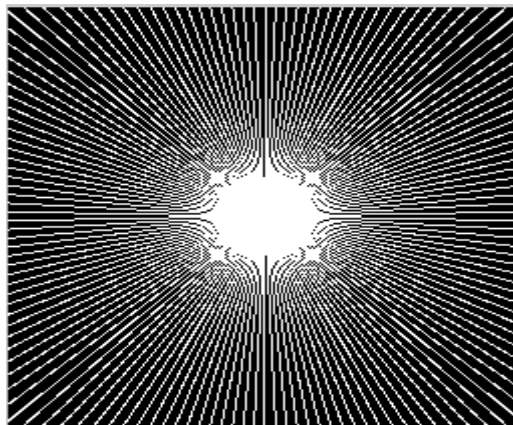


Figure 1. Fourier Domain Sampling Matrix

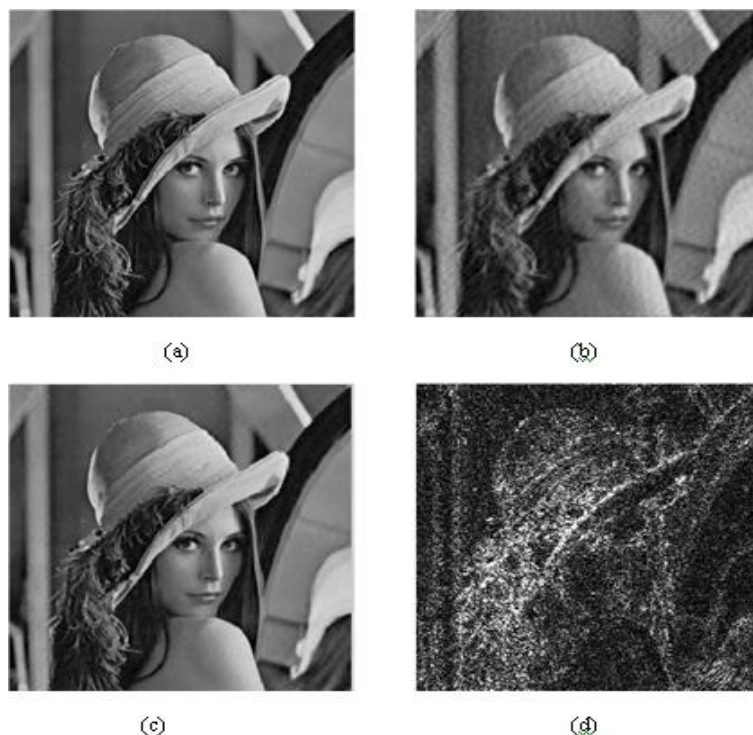


Figure 2. (a) The Original Image; (b) 2 - Norm Recovery Image; (c) Total Variation Restore the Image; (d) The Original Image and the Total Variation between the Images Of Error

4. Conclusion

Compression perception theory advantage of the sparse characteristics of the signal, based on the Nyquist sampling theorem original signal sampling process is transformed into the observation process optimization calculation based recovery signal. This paper reviews the compression perception theory development background, progress and results of research at home and abroad. Compression perception theory framework described compression perception theory of mathematical model, a detailed description of the key technology of compressed sensing recovery. And to restore the image, showing the specific application of compressed sensing technology in the field of image signal processing.

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