TELKOMNIKA Indonesian Journal of Electrical Engineering
Vol. 12, No. 11, November 2014, pp. 7920 ~ 7926
DOI: 10.11591/telkomnika.v12i11.6488

A SIR Mathematical Model of Dengue Transmission and its Simulation

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Abstract

The Mathematical model that was developed is a SIR model human-mosquito-mosquito eggs, the rate of displacement of latent mosquitoes become infected mosquito was assumed constant and non-infected eggs were produced by infected mosquitoes and susceptible mosquitoes, while infected eggs were produced by infected mosquitoes. In addition, the temperature factor used in producing susceptible mosquitoes and infected mosquitoes from eggs. The analysis shows two equilibrium state, disease-free equilibrium and endemic equilibrium. The simulation was conducted to show dynamic population where $R_o<1$ and $R_o>1$. The result shows the disease-free equilibrium which is stable when $R_o<1$ and the endemic equilibrium which is stable when $R_o>1$. This also shows mosquito mortality rate towards the desease in population. If mosquito mortality rate is increased, the basic reproduction number is decreasing, so it can prevent spread in population.

Keywords: *mathematical models, basic reproductive number, disease-free quilibrium, endemic equilibrium, numerical simulations*

1. Introduction

Based on regular appearance of dengue, there are several solution come intertwined with the case. One of them is to provide mathematical model, or just model. This is not a brand new way to solve problem, in 1911 Ross developed a simple model to adept the spread of malaria which became famous Ross Model. Then in 1957 it was fortified by MacDonald which is known as Ross-MacDonald [5]. At the end of 20 century, the modeling is accepted globally, especially in the community health sector because its strategic and tactical features [3], as an example of a mathematical model that takes into account the temperature factor. Dengue spread model with the consideration of temperature was introduced by [4] and [1]. They develop formula for *SIR* human-mosquito- mosquito eggs.

In [4] assumed laten mosquitoes spreading rate cause the infected mosquitoes not constant and delayed (τ) as incubation period of mosquito. Furthermore, [4] assumed the non-infected eggs was reproduced by laten mosquitoes and infected mosquitoes, while infected egg was only reproduced by infected mosquitoes. Temperature factor introduced by [4] in susceptible mosquitoes and infected mosquitoes is not constant.

In [1] assumed laten mosquitoes spreading rate cause the infected mosquitoes constant and the non-infected eggs was reproduced by three compartment mosquitoes, while the infected eggs was produced by infected mosquitoes and latent mosquitoes. Meanwhile, [1] temperature was constant.

Based on those research, *SIR* model human-mosquito- mosquito eggs is developed more with the assumption transmission of laten mosquitoes spreading rate cause the infected mosquitoes is constant as in [1] and the non-infected eggs was reproduced by laten mosquitoes and infected mosquitoes, while infected egg was only reproduced by infected mosquitoes as in [4]. Furthermore, the temperature factor used in the production of susceptible mosquitoes and infected mosquitoes from eggs. Temperature was applied as in [4]. In this paper, equilibrium state and basic reproduction numbers were determined. Next, simuation was done to show dynamic population on model.

Received July 14, 2013; Revised September 17, 2014; Accepted October 4, 2014

2. Mathematical Model

In *SIR* model human-mosquito-eggs, human population is divided into susceptible humans (S_H), infected humans (I_H), dan recovered humans (R_H), with total human population $N_H = S_H + I_H + R_H$. Mosquito is divided into susceptible mosquitoes (S_M), latent mosquitoes (L_M), and infected mosquitoes (I_M), with total mosquito $N_M = S_M + L_M + I_M$. Egg population is divided into non-infected eggs (S_E) and infected eggs (I_E), with total egg population $N_E = S_E + I_E$. system differential equations of treatment model is:

$$
\begin{aligned}
\frac{dS_H}{dt} &= r_H N_H\left(1-\frac{N_H}{k_H}\right)-\left(ac_H\frac{I_M}{N_H}+\mu_H\right)S_H\\
\frac{dI_H}{dt} &= ac_H\frac{I_M}{N_H}S_H-(\alpha_H+\mu_H+\gamma_H)I_H\\
\frac{dR_H}{dt} &= \gamma_H I_H-\mu_H R_H\\
\frac{dS_M}{dt} &= PF(t)S_E-ac_M\frac{I_H}{N_H}S_M-\mu_M S_M\\
\frac{dL_M}{dt} &= ac_M\frac{I_H}{N_H}S_M-\gamma_M L_M-\mu_M L_M\\
\frac{dI_M}{dt} &= \gamma_M L_M-\mu_M I_M+PF(t)I_E\\
\frac{dS_E}{dt} &= (r_M S_M+(1-g)r_M I_M)\left(1-\frac{(S_E+I_E)}{k_E}\right)-\mu_E S_E-PF(t)S_E\\
\frac{dI_E}{dt} &= gr_M I_M\left(1-\frac{(S_E+I_E)}{k_E}\right)-\left(\mu_E+PF(t)\right)I_E
\end{aligned}
\qquad (1)
$$

The rate of infection from infected mosquitoes to susceptible human is $\beta_H = mac_H$, where m is total ratio of mosquito population compared to total human population, then β_H multiplied by the proportion of infected mosquitoes so that human transmission becomes $ac_H I_M/N_H$. The rate of infection from infected humans to susceptible mosquitoes is $\beta_M = ac_M$, where β_M is multiplied by total infected human population so the mosquito transmission is $ac_M I_H/N_H$. Temperature factor is introduced as,

$$F(t) = (c - d\,sin(2\pi ft + \varphi)) \qquad (2)$$

In order to avoid equation (2) not becoming negative, Heaviside function is applied, where Heaviside is valued whether 0 or 1.

$$\theta(c - d\,sin(2\pi ft + \varphi)) = \begin{cases} 1\,; & \text{jika } (c - d\,sin(2\pi ft + \varphi)) \geq 0 \\ 0\,; & \text{jika } (c - d\,sin(2\pi ft + \varphi)) < 0 \end{cases} \qquad (3)$$

Equation (2) is determined as,

$$F(t) = (c - d\,sin(2\pi ft + \varphi))\theta(c - d\,sin(2\pi ft + \varphi)) \qquad (4)$$

Several parameters used in modificated model are, fraction of infective bites from an infected human (c_H), fraction of infective bites from an infected mosquitoes (c_M), dengue induced mortality in humans (α_H), humans recovery rate(γ_H), latency rate in mosquitoes (γ_M), humans natural mortality rate (μ_H), natural mortality rate of eggs (μ_E), birth rate of humans (r_H), humans carrying capacity (k_H), infected eggs hatching rate (P), infected eggs hatching rate (g), oviposition rate (r_M), eggs carrying capacity (k_E), climatic factor modulating winters (c), climatic factor modulating winters (d), frequency of the seasonal cycles(f), [1] and [4]

To facilitate Equation (1) analysis, empirical equation was made to compare each subpopulation towards total population, ie:

$$S_h = \frac{S_H}{N_H};\, I_h = \frac{I_H}{N_H};\, R_h = \frac{R_H}{N_H};\; S_m = \frac{S_M}{N_M};\, L_m = \frac{L_M}{N_M};\; I_m = \frac{I_M}{N_M};\; S_e = \frac{S_E}{N_E};\; I_e = \frac{I_E}{N_E} \qquad (5)$$

in correlation with,

$$S_h + I_h + R_h = 1,\; S_m + L_m + I_m = 1, \text{and } S_e + I_e = 1$$

From Equation (5) we get,

$$\frac{dS_h}{dt} = \frac{1}{N_H}\left[\frac{dS_H}{dt} - \frac{dN_H}{dt}\right] \text{ and } \frac{dI_h}{dt} = \frac{1}{N_H}\left[\frac{dI_H}{dt} - \frac{dN_H}{dt}\right].$$

The same thing is occurred fro other variables so that a new differential equation in eight dimensional state is made with two dimensions for human population (S_h, I_h), two dimensions for mosquito population (L_m, I_m), one dimension for eggs population(I_e), and three dimensions for total population $\left(N_{h,}N_m, N_e\right)$.

$$\begin{aligned}
\frac{dS_h}{dt} &= r_H\left(1-\frac{N_h}{k_H}\right) - \left(ac_H\frac{N_m}{N_h}I_m + r_H\left(1-\frac{N_h}{k_H}\right) - \alpha_H I_h\right)S_h \\
\frac{dI_h}{dt} &= ac_H\frac{N_m}{N_h}I_m S_h - \left(\alpha_H + \gamma_H + r_H\left(1-\frac{N_h}{k_H}\right)\right)I_h + \alpha_H I_h^2 \\
\frac{dL_m}{dt} &= ac_M I_h(1-L_m-I_m) - \left(\gamma_M + PF(t)\frac{N_e}{N_m}\right)L_m \\
\frac{dI_m}{dt} &= \gamma_M L_m + PF(t)\frac{N_e}{N_m}(I_e - I_m) \\
\frac{dI_e}{dt} &= \left(\frac{N_m}{N_e}\right)(gr_M I_m)\left(1-\frac{N_e}{k_E}\right) - \left(\frac{N_m}{N_e}\right)(r_M S_m + r_M I_m)\left(1-\frac{N_e}{k_E}\right)I_e \\
\frac{dN_h}{dt} &= r_H N_h\left(1-\frac{N_h}{k_H}\right) - \alpha_H I_h N_h - \mu_h N_h \\
\frac{dN_m}{dt} &= PF(t)N_e - \mu_M N_m \\
\frac{dN_e}{dt} &= r_M\left(1-\frac{N_e}{k_E}\right)(1-L_m-I_m) - r_M\left(1-\frac{N_e}{k_E}\right)I_m - \left(\mu_E + PF(t)\right)N_e
\end{aligned} \quad (6)$$

3. Result and Analysis

3.1. Equilibrium State

Equation (6) is taken to determine equilibrium state. Equilibrium state is based on [2]. There are two equilibrium state, disease-free equilibrium and endemic equilibrium,

3.1.1. Disease-Free Equilibrium

$$\boldsymbol{T_{dfe}\left(S_h, I_h, L_m, I_m, I_e, N_{H,}N_M, N_E\right)}$$

$$S_h = 1;\ I_h = L_m = I_m = I_e = 0;\ N_h = k_H - \frac{k_H\mu_H}{r_H};\ N_m = F(t)\,k_E P\left(\frac{1}{\mu_M} - \frac{1}{r_M}\right) - \frac{(k_E - \mu_E)}{r_M};\ N_e = \left(\frac{\mu_M}{F(t)r_M P}\right)\left(\frac{F(t)\,k_E P r_M}{\mu_M} - F(t)\,k_E P - k_E\,\mu_E\right)$$

3.1.2. Endemic Equilibrium

$$\boldsymbol{T_{ee}(S_h^*, I_h^*, L_m^*, I_m^*, I_e^*, N_H^*, N_M^*, N_E^*)}$$

$$\begin{aligned}
S_h^* &= \frac{(k_H - N_h)N_h r_H}{a\,c_H I_m k_H N_m - N_h(N_h r_H - k_H r_H + k_H I_h\,\alpha_H)} \\
I_h^* &= -\frac{N_h^2 r_H - k_H N_h(r_H+\alpha_H+\gamma_H) + \sqrt{N_h\left(-4a\,c_H I_m k_H^2 N_m \alpha_H S_h + N_h\left(N_h r_H - k_H(r_H+\alpha_H+\gamma_H)\right)^2\right)}}{2k_H N_h \alpha_H} \\
L_m^* &= -\frac{ac_M I_h(I_m - 1)N_m}{ac_M I_h N_m + F(t)N_e P + N_m\gamma_M} \\
I_m^* &= I_e + \frac{L_m N_m \gamma_M}{F(t)N_e P} \\
I_e^* &= \frac{gI_m}{1-L_m} \\
N_h^* &= \frac{k_H(r_H - I_h\alpha_H - \mu_H)}{r_H} \\
N_m^* &= \frac{F(t)N_e P}{\mu_M} \\
N_e^* &= \frac{k_E(1-L_m)N_m r_M}{F(t)k_E P + N_m(r_M - L_m r_M) + k_E\mu_E}
\end{aligned}$$

With,

$$F(t) = (c - d\,sin(2\pi ft + \varphi))\theta(c - d\,sin(2\pi ft + \varphi))$$

3.2. Basic Reproduction Numbers

The basic reproduction number, denoted $\mathcal{R}_0$. According to [7] if $\mathcal{R}_0 < 1$, then on average an infected individual produces less than one new infected individual over the course of its infectious period, and the infection cannot grow. Conversely, if $\mathcal{R}_0 > 1$, then each infected individual produces, on average, more than one new infection, and the disease can invade the population. To Determination of the basic reproduction number is used*the next generation matriks G. The next generation matrix G* is defined [7]:

$$G = FV^{-1} \quad (7)$$

With F as matrix coefficient of the rate infection, while V is matrix coefficient of desease transmission, either for mortality or recovery.

Diferential equations were used to determine basic reproduction number, ie (2), (3), (4), dan (5) which included in Equation (6). Equation (2), (3), (4), dan (5) in Equation (6) is constructed into matrix which evaluated in disease-free equilibrium, in order to obtain the matrix $G = FV^{-1}$. Furthermore determined eigenvalues matrix G, ie λ_i, $i = 1,2,\ldots n$.. $\mathcal{R}_0 = Max\{\lambda_i\}$, $i = 1,2,\ldots n$ is called the basic reproduction number.

Based on the analysis basic reproduction number is gained,

$$\mathcal{R}_0 = \sqrt{\frac{a^2 c_H c_M k_E r_H \gamma_M (\mu_E \mu_M + F(t)P(\mu_M - r_M))}{(g-1)k_H r_M (r_H - \mu_H)(\alpha_H + \mu_H + \gamma_H)\mu_M^2(\gamma_M + \mu_M)}} \quad (8)$$

3.3. Population Dynamics Simulation of Dengue Transmission

Simulation was done when $\mathcal{R}_0 < 1$ dan $\mathcal{R}_0 > 1$, whre $\mathcal{R}_0$ is defined from equation (8). This simulation was to show that system will be stabilized to disease-free equilibrium when $\mathcal{R}_0 < 1$ and stabilized to endemic equilibrium when $\mathcal{R}_0 > 1$. Besides, this simulation was needed to know mortality rate in mosquito population (μ_M) to transmission in population.

Parameter values were used in simulation are on the below table with initial value $S_H = 10$, $I_H = 15$, $R_H = 75$, $S_M = 50$, $L_M = 30$ $I_M = 20$, $S_E = 50$, $I_e = 50$, and each total population $N_H = N_M = N_E = 100$.

Table 1. Parameters Value

Parameter	Value	Parameter	Value
c_H	1	k_H	2×10^5
c_M	1	P	0.15
α_H	0.001	g	0.1
γ_H	0.143	r_M	50
γ_M	0.143	k_E	10^6
μ_H	4×10^{-5}	f	2.8×10^{-3}
μ_E	0.1	c	0.07
r_H	2.5	d	0.06
φ	π/2		

3.3.1. Population Dinamics0020sxqfor $\mathcal{R}_0 < 1$

System (6) has a single equilibrium for $\mathcal{R}_0 < 1$ which can be shown by *Computation Program.* In this simulation, the parameter of average daily biting rate (a) is 1.2, natural mortality rate of mosquito (μ_M) is 1, and another parameter can be shown on Table 4. The Initial number for the simulation is $S_H = 10$, $I_H = 15$, $R_H = 75$, $S_M = 50$, $L_M = 30$, $I_M = 20$,

$S_E = 50$, $I_e = 50$, and each total population $N_H = N_M = N_E = 100$. Based on the simulation, the result is given as shown in the charts below.

Basic reproduction number is denoted as $(\mathcal{R}_0)$ and is used to show the population of mosquito, human and eggs is stabilized at the equilibrium state when $\mathcal{R}_0 < 1$. Besides, according to [7] stated that when $\mathcal{R}_0 < 1$ means there is no endemic, that means infected population in the system will be perished. To show this case, the simulation was done in order to show population changes in mosquito, human and eggs.

The susceptible mosquito population (S_M) oscillated to a periodic value. The latent Mosquito Population (L_M) was decreasing and stabilized at $L_m = 0$, then infected mosquito population (I_M) was decreased from initial value and stabilized at $I_m = 0$. The value added in this parameter is $a = 1.2$, $\mu_M = 1$, and other parameter made the value on Tabel 4 cause $\mathcal{R}_0 < 1$, so that latent mosquito population and infected mosquito are decreased and vanished from system. Furthermore, the increasing or decreasing value of each mosquito population correlate to the human population as well as eggs population.

Susceptible human population (S_H) was increased from initial value and stabilized at nearly $S_h = 1$ or 199997 people. Infected human population (I_H) was increased from initial stage and stabilized at $I_h = 0$, while recovered human population (R_H) was increased from initial point then buffered at $R_h = 1 - S_h - I_h = 1 - 1 - 0 = 0$. The increase of susceptible human population was caused by transfer rate from infected mosquito population, so that, susceptible human population was increased, as the result, the transfer rate from susceptible human to infected human was decreased. The decrease in infected human population causes transfer rate from infected human to recovered human was decreased too.

Susceptible egg population (S_E) oscillated periodically, while infected egg population (I_E) increased from initial stage and then decreased until stabilized at $I_e = 0$. Non-infected egg population was produced by susceptible mosquito and Infected mosquito. In a relatively long period of time, non-infected egg population was dominantly reproduced by susceptible mosquitos than those from infected mosquito, this is due to the fact that infected mosquito vanished from the system relatively fast. In the mean time, non-infected egg population oscillated periodically corresponded to that on susceptible mosquito population. When the susceptible mosquito population increased, then non-infected egg population would increase and vise versa.

Total numbers of human population (N_H) was increased from initial point and stabilized at $N_H = 199997$, while numbers of mosquito population (N_M) dan and egg population (N_E) oscillated to a periodic value. Can be said that the equilibrium without desease occurs when $\mathcal{R}_0 < 1$.

3.3.2. Population Dinamics for $\mathcal{R}_0 > 1$

System (6) has one equilibrium state when $\mathcal{R}_0 > 1$ which can be shown by *computation program.* When conducting the simulation, the value added for all perimeter as follow, daily biting rate (a) is 1.2, mosquito natural mortality rate (μ_M) is 0.071, and another parameter can be seen on Tabel 4. Initial value in this simulation is the same as when $\mathcal{R}_0 < 1$. Based on the simulation, it can be seen the result as follow. Figure 7 shows that basic reproduction number when $a = 1.2$, $\mu_M = 0.071$, and other parameter can be seen on Table 4.

According to [7] when $\mathcal{R}_0 > 1$, endemic occurs. It means total infected population in the system increases. To show this happen, the simulation conducted to see each population changes for mosquito, human and egg population.

Susceptible mosquito population (S_M) oscillated periodically, latent mosquito population (L_M) increased from initial stage, if the simulation is conducted for a longer period of time, the latent mosquito population oscillates to a periodic state. Infected mosquito population (I_M)

increased at initial stage, if the simulation is conducted for a longer period, the infected mosquito population oscillate to a periodic state. The parameter set $a = 1.2$, $\mu_M = 0.071$, and other parameter can be seen on Table 4 causes $\mathcal{R}_0 > 1$, so that of latent mosquito population and infected one increases and oscillate periodically. Besides, each mosquito population gives impact to human and egg population.

Susceptible human population (S_H) was increased from initial stage and oscillated to a periodic state, Infected human population (I_H) increased from initial stage, if the simulation is conducted for a longer period of time, it will produce a periodic value, recovered human population (R_H) increased from beginning state and oscillated periodically. The downward of susceptible human population was due to the increase in transfer rate from infected mosquito population, so that susceptible human population is decreased, as the result, transfer rate from susceptible human population to infected human population is increased. The increase of infected human causes the rate of infected human becomes recovered human is increased too.

Non-infected egg population (S_E) oscillated periodically, while infected egg population (I_E) increased at initial stage, if the simulation is done for a longer period, so the infected egg population oscillates periodically. Non-infected egg was produced by susceptible mosquito and infected one. If susceptible mosquito and infected mosquito population increase so that non infected egg population is increase and vise versa. Throughout the time, non-infected egg population oscillates periodically as well as susceptible mosquito population and infected one.

Total Number of human population (N_H) increase from initial point and slowdown untl it oscillated periodically, while total number of mosquito population (N_M) and egg population (N_E) oscillated periodically. Can be seen that equilibrium occurs when $\mathcal{R}_0 > 1$. Based on the simulation on the model to show total number of human, mosquito, and egg population increase when natural mortality rate (μ_M) increased.

3.4. Mosquito Mortality Rate Simulation

This simulation is needed to see the impact on mortality rate in mosquito (μ_M) toward the endemic in population. Furthermore, the increase of parameter values will reduce basic reproduction rate ($\mathcal{R}_0$) derived from Equation (8). There are 4 value for μ_M to be examined, taken in the interval [1 , 2.5] by 0.5 step [6]. These parameter value are based on Table 2 with bite rate $a = 1.2$.

Increased of mosquito mortality rate causing the number of mosquito reduced, susceptible mosquitoes, latent mosquitoes and infected mosquito. The reduced of mosquito population number give effect to the human population and mosquito eggs.

The conditioning of this is the increase in susceptible humans, while infected humans and recovered humans are decreasing. This is due to the shrink of mosquito population that lower the population of infected mosquitoes. In order that the bite rate is slowing down. In other words, infected humans and recovered humans are decreased too.

Observed impact on the egg population is the decrease of non-infected eggs and infected eggs population. This is due to the decrease of latent mosquitoes and infected mosquitoes to lay egg. The decrease of infected eggs resulted in the decrease in infected mosquitoes from infected eggs.

4. Conclusion

Based on the discussion and result on modificated model it can be concluded, the equilibrium gained is disease-free equilibrium and endemic equilibrium. Disease-free equilibrium stable when $\mathcal{R}_0 < 1$, while endemic equilibrium stable when $\mathcal{R}_0 > 1$. Human population, mosquito, and infected egg drawn to zero when $\mathcal{R}_0 < 1$, while increases and oscillated periodically when $\mathcal{R}_0 > 1$. Meanwhile, with the mortality rate in mosquito, the basic reproduction rate is decreasing, so we can reduce the rate of infection and desease.

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