TELKOMNIKA Indonesian Journal of Electrical Engineering
Vol. 12, No. 10, October 2014, pp. 7059 ~ 7069
DOI: 10.11591/telkomnika.v12i8.5901

Necessary Conditions of the Wave Packet Frames with Several Generators

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Abstract

The main goal of this paper is to consider the necessary conditions of wave packet systems to be frames in higher dimensions. The necessary conditions of wave packet frames in higher dimensions with several generators are established, which include the corresponding results of wavelet analysis and Gabor theory as the special cases. The existing results are generalized to the case of several generators and general lattices.

Keywords: *wavelet, the wave packet frame, necessary condition*

1. Introduction

Frames were first introduced by Duffin and Schaeffer (1952) in the context of nonharmonic Fourier series. Outside of signal processing, frames did not seem to generate much interest until the ground breaking work of Daubechies *et al*. (1986). Since then, the theory of frames began to be more widely studied. Traditionally, frames have been used in signal processing, image processing, data compression, and sampling theory. Recently, frames are also used to mitigate the effect of losses in packet-based communication systems and hence to improve the robustness of data transmission [Casazza and Kovaevi, 2003; Goyal *et al*., 2001], and to design high-rate constellation with full diversity in multiple-antenna code design [Hassibi *et al*., 2001]. We refer to the monograph of Daubechies (1992) or the research-tutorial [Christensen, 2002] for basic properties of frames.

An important example about frame is wavelet frame, which is obtained by translating and dilating a finite family of functions. Wavelets were introduced relatively recently, in the beginning of the 1980. They attracted considerable interest from the mathematical community and from members of many diverse disciplines in which wavelets had promising applications. Daubechies *et al*. (1986) combined the theory of the continuous wavelet transform with the theory of frames to define wavelet frames for $L^2(R^n)$. In 1990, Daubechies (1990) obtained the first result on the necessary conditions for affine frames, and then in 1993, Chui and Shi (1993) obtained an improved result. After about ten years, Casazza and Christensen (2001) established a stronger condition which also works for wavelet frame. Recently, Shi and his co-authors [Shi and Chen, 2005; Shi and Shi, 2005] obtained the necessary conditions and sufficient conditions of wavelet frames.

Another most important concrete realization of frame is Gabor frame. Gabor systems (Weyl-Heisenberg systems) were first introduced by Gabor (1946). They are generated by modulations and translations of a finite family of functions. In 2007, Shi and Chen (2007) established some new necessary conditions for Gabor frames. These conditions are also sufficient for tight frames. In paper [Li and Wu, 2001], Li and Wu presented two new sufficient conditions for Gabor frame via Fourier transform. The conditions they proposed were stated in terms of the Fourier transforms of the Gabor system's generating functions, and the conditions were better than that of Daubechies. Furthermore, in paper [Li *et al*., 2001], Li *et al*. established a necessary condition and two sufficient conditions ensuring that the shift-invariant system is a frame for $L^2(R^n)$. As some applications, the results are used to obtain some known conclusions about wavelet frames and Gabor frames.

Received March 2, 2014; Revised June 24, 2014; Accepted July 14, 2014

In paper [Cordoba and Fefferman, 1978], authors introduced wave packet systems by applying certain collections of dilations, modulations and translations to the Gaussian function in the study of some classes of singular integral operators. In paper [Labate *et al.*, 2004], authors adopted the same expression to describe any collections of functions which are obtained by applying the same operations to a finite family of functions. In fact, Gabor systems, wavelet systems and the Fourier transform of wavelet systems are special cases of wave packet systems. Wave packet systems have recently been successfully applied to some problems in harmonic analysis and operator theory [Lacey and Thiele, 1997; Lacey and Thiele, 1999].

The main goal of this paper is to consider the necessary conditions of multiwave packet frames in higher dimensions. We establish some necessary conditions conditions for the wave packet frames of the different operator order in $L^2(R^n)$ with matrix dilations of the form $(Df)(x)=\sqrt{q}\,f(Ax)$, where A is an arbitrary expanding $n\times n$ matrix with integer coefficients and $q=|detA|$, which is a generalization of classical wavelet frame and Gabor frame. Of course, our way combines with some techniques in wavelet analysis and time-frequency analysis. In particular, we use some thoughts of Chui and Shi (1993) in classifying the necessary condition for the Gabor frame. Also, we discuss necessary conditions for other wave packet frames with the different operator order. Also, we fuse some ways in wavelet analysis and Gabor theory and we mainly borrow some thoughts in classifying the sufficient conditions of the wavelet frame in papers [Shi and Chen, 2007; Li and Wu, 2001; Li *et al.*, 2001; Shi and Shi, 2005].

2. Prelimilaries

In this section, some notations and some results which will be used later are introduced. Throughout this paper, the following notations will be used. R^n and Z^n denote the set of n-dimensional real numbers and the set of integers, respectively. $L^2(R^n)$ is the space of all square-integrable functions, and $<\cdot,\cdot>$ and $\|\cdot\|$ denote the inner product and norm in $L^2(R^n)$, respectively, and $l(Z^n)$ denotes the space of all square summable sequences.

For $x=(x_1,x_2,\cdots,x_n)\in R^n$, define:

$$|x|=\sqrt{x_1^2+x_2^2+\cdots+x_n^2}$$

We denote by T^n the n-dimensional torus. By $L^p(T^n)$ we denote the space of all Z^n-periodic functions f (*i.e.*, f is 1-periodic in each variable) such that $\int_{T^n}|f(x)|^p\,dx<+\infty$.

We use the Fourier transform in the form:

$$\hat{f}(\omega)=\int_{R^n} f(x)e^{-2\pi ix\cdot\omega}dx,$$

Where $\cdot$ denotes the standard inner product in R^n, and we often omit it when we can understand this from the background. Sometimes, $\hat{f}(\omega)$ is defined by $\mathrm{F}f$.

The expanding matrices mean that all eigenvalues have magnitude greater than 1. We denote the set of the expanding matrices as E_n. Let $GL_n(R)$ denote the set of all $n\times n$ non-singular (or invertible) matrices with real entries. For $A\in GL_n(R)$, we denote by A^* the transpose of A. It is obvious that $A^*\in E_n$. For $B\in GL_n(R)$ we denote by B^{-1} the invertible matrix of B. For the sake of simplicity, we denote $(A^*)^{-1}$ by $A^{\sharp}$.

Let us recall the definition of frame.

Definition 1. Let H be a separable Hilbert space. A sequence $\{f_i\}_{i\in N}$ of elements of H is a frame for H if there exist constants $0 < C \le D < \infty$ such that for all $f \in H$, we have:

$$C\|f\|^2 \le \sum_{i\in N}\|\langle f, f_i\rangle\|^2 \le D\|f\|^2. \tag{1}$$

The numbers C, D are called lower and upper frame bounds, respectively (the largest C and the smallest D for which (1) holds are the optimal frame bounds). Those sequences which satisfy only the upper inequality in (1) are called Bessel sequences. A frame is tight if $C = D$. If $C = D = 1$, it is called a Parseval frame.

Let T_f denote the synthesis operator of $f = \{f_i\}_{i\in N}$, i.e., $T_f(c) = \sum_i c_i f_i$ for each sequence of scalars $c = (c_i)_{i\in N}$. Then the frame operator $Sh = T_f T_f^*(h)$ associated with $\{f_i\}_{i\in N}$ is a bounded, invertible, and positive operator mapping of H on itself. This provides the reconstruction formula:

$$h = \sum_{i=1}^{\infty} <h, \tilde{f}_i> f_i = \sum_{i=1}^{\infty} <h, f_i> \tilde{f}_i, \forall\ h \in H. \tag{2}$$

Where $\tilde{f}_i = S^{-1} f_i$. The family $\{\tilde{f}_i\}_{i\in N}$ is also a frame for H, called the canonical dual frame of $\{f_i\}_{i\in N}$. If $\{g_i\}_{i\in N}$ is any sequence in H which satisfies:

$$h = \sum_{i=1}^{\infty} <h, g_i> f_i = \sum_{i=1}^{\infty} <h, f_i> g_i, \forall\ h \in H, \tag{3}$$

It is called an alternate dual frame of $\{f_i\}_{i\in N}$.

In this paper, we will work with three families of unitary operators on $L^2(R^n)$. Let $A \in E_n$ and $B, C \in GL_n(R)$. The first one consists of the dilation operator $D_A : L^2(R^n) \to L^2(R^n)$ defined by $(D_A)(x) = q^{\frac{1}{2}} f(Ax)$ with $q = |detA|$. The second one consists of all translation operators

$$T_{Bk} : L^2(R^n) \to L^2(R^n), k \in Z^n,$$

Defined by $(T_{Bk} f)(x) = f(x - Bk)$. The third one consists of the modulation operator.

$$E_{Cm} : L^2(R^n) \to L^2(R^n), m \in Z^n,$$

Defined by $(E_{Cm} f)(x) = e^{2i\pi Cm\cdot x} f(x).$

Let $P \subset Z$ and $Q \subset R^n$. Let $S = P \times Q$. Then, we have $S \subset Z \times R^n$. Again, let:

$\{A_p : A_p \in P\} \subset E_n$ and $B \in GL_n(R)$.

For the functions $\psi^l \in L^2(R^n), l = 1, 2, \cdots, L$, we will consider the wave packet system Ψ defined by the following:

$$\Psi = \{D_{A_p} E_\nu T_{Bm} \psi^l(x)\}_{l=1,2,\cdots,L,\ m\in Z^n, (p,\nu)\in S} \tag{5}$$

Let $A_p = A^j (j \in Z), S = Z \times \{0\}$. Then, we obtain the wavelet systems. On the other side, we can get the Gabor systems when the set $\{A_p : A_p \in P\}$ only consists of the elementary matrix E. This simple observation already suggests that the wave packet systems provide greater flexibility than the wavelet systems or the Gabor systems.

By changing the order of the operators, we can also define the following one-to-one function systems from $S \times Z^n$ into $L^2(R^n)$:

$$\Psi^1 = \left\{\psi_{p,\nu,m}(x) \mid D_{A_p} T_{Bm} E_\nu \psi^l(x), m \in Z^n, (p,\nu) \in S\right\},$$
$$\Psi^2 = \left\{\psi_{p,\nu,m}(x) \mid E_\nu D_{A_p} T_{Bm} \psi^l(x), m \in Z^n, (p,\nu) \in S\right\},$$
$$\Psi^3 = \left\{\psi_{p,\nu,m}(x) \mid E_\nu T_{Bm} D_{A_p} \psi^l(x), m \in Z^n, (p,\nu) \in S\right\},$$
$$\Psi^4 = \left\{\psi_{p,\nu,m}(x) \mid T_{Bm} D_{A_p} E_\nu \psi^l(x), m \in Z^n, (p,\nu) \in S\right\},$$
$$\Psi^5 = \left\{\psi_{p,\nu,m}(x) \mid T_{Bm} E_\nu D_{A_p} \psi^l(x), m \in Z^n, (p,\nu) \in S\right\}. \tag{6}$$

Then, we will give the definitions of wave packet multiwavelet frame and the frame wave packet multiwavelet.

Definition 2. We say that the wave packet system defined Ψ by (5) is a wave packet multiwavelet frame if it is a frame for $L^2(R^n)$. Then, the functions $\psi = (\psi^1, \psi^2, \cdots, \psi^M)$ is called a frame wave packet multiwavelet.

For other wave packet systems $\Psi^i (1 \le i \le 5)$ defined by (6), we can define the corresponding wave packet frames and the frame wave packets like definition 2.2.

In order to prove the main results to be presented in next section, we need the following lemmas.

Lemma 2.1. Suppose that $\{f_k\}_{k=1}^{+\infty}$ is a family of elements in a Hilbert space H such that there exist constants $0 < C \le D < +\infty$ satisfying (2) for all f belonging to a dense subset D of H. Then, the same inequalities (2) are true for all $f \in H$; that is, $\{f_k\}_{k=1}^{+\infty}$ is a frame for H.

For proof of Lemma 2.1, people can refer to the book [Daubechies, 1990].

Therefore, we will consider the following set of functions:

$$D = \left\{\begin{array}{l} f \in L^2(R^n):\ \hat{f} \in L^\infty(R^n) \text{ and } \hat{f} \\ \text{has compact support in } R^n,\ \{0\} \end{array}\right\}.$$

The following result is well known, we can find it in [Daubechies, 1992].

Lemma 2.2 D is a dense subset of $L^2(R^n)$.

The following useful facts can be found in paper [Christensen and Rahimi, 2008, Lemma 2.2].

Lemma 2.3 Let $A \in GL_n(R)$, $y, z \in R^n$ and $f \in L^2(R^n)$. Then the following holds:

(1) $(T_y f)^\wedge = E - y\hat{f}, (E_z f)^\wedge = T_z \hat{f}$,

$$(D_A f)^{\wedge} = D_{A^{\#}}\hat{f};$$

(2) $T_y E_z f = e^{-2\pi i z\cdot y} E_z T_y f$,

$$D_A E_y f = E_{A^* y} D_A f, D_A T_y f = T_{A^{-1}y} D_A f;$$

(3) $(T_y E_z f)^{\wedge} = e^{-2\pi i z\cdot y} T_z E_{-y}\hat{f};$

(4) $(D_A T_y f)^{\wedge}(\xi) = E_{-A^{\#}y} D_{A^{\#}}\hat{f}(\xi)$

$$=|detA|^{-\frac{1}{2}}\hat{f}(A^{\#}\xi)e^{-2\pi i A^{-1}y\cdot\xi}.$$

3. Necessary Conditions of Wave Packet Frames

Motivating by the fundament works [Christensen, 2002; Chui and Shi, 1993; Daubechies, 1990], we will give a necessary condition of wave packet frame Ψ defined by (5) for higher dimension with an arbitrary expansive matrix dilation in the following.

Theorem 3.1. Suppose that wave packet system:

$$\{D_{A_p} E_\nu T_{Bm}\psi^l(x)\}_{l=1,2,\cdots,L,\ m\in Z^n,(p,\nu)\in S} \quad (7)$$

Defined by (5) is a frame with frame bounds A_1 and A_2, then we have:

$$bA_1 \le \sum_{l=1}^{L}\sum_{(p,\nu)\in S}|\hat{\psi}^l(A_p^{\#}\omega-\nu)|^2 \le bA_2,\ a.e.\omega, \quad (8)$$

Where $b = |\det B|$.

Proof. Because wave packet system:

$$\{D_{A_p} E_\nu T_{Bm}\psi^l(x)\}_{l=1,2,\cdots,L,\ m\in Z^n,(p,\nu)\in S}$$

Is a frame with frame bounds A_1 and A_2, for all $f \in L^2(R^n)$, we have:

$$A_1\|f\|^2 \le \sum_{l=1}^{L}\sum_{(p,\nu)\in S}\sum_{m\in Z^n}|<f, D_{A_p}E_\nu T_{Bm}\psi^l>|^2 \le A_2\|f\|^2. \quad (9)$$

Let $\hat{f} \in C_c(R)$ and $\hat{f}$ have compact support.

Let $q_p = |detA_p|$. According to Lemma 2.3 and Plancheral theorem, we have:

$$\sum_{(p,\nu)\in S}\sum_{m\in Z^n}|<f, D_{A_p}E_\nu T_{Bm}\psi^l>|^2$$

$$= \sum_{(p,\nu)\in S}\sum_{m\in Z^n}|<\mathrm{F}f, \mathrm{F}D_{A_p}E_\nu T_{Bm}\psi^l>|^2$$

$$= \sum_{(p,\nu)\in S}\sum_{m\in Z^n}|<\hat{f}, D_{A_p^{\#}}T_\nu E_{-Bm}\hat{\psi}^l>|^2$$

$$= \sum_{p\in P} q_p^{-1} \sum_{\nu\in Q} \sum_{m\in Z^n} | \int_{R^n} \hat{f}(\omega) \overline{\hat{\psi}^l (A_p^{\#}\omega - \nu)}$$

$$e^{2\pi i Bm(A_p^{\#}\omega-\nu)} d\omega |^2$$

$$= \sum_{p\in P} q_p \sum_{\nu\in Q} \sum_{m\in Z^n} | \int_{R^n} \hat{f}(A_p^*(\omega+\nu))$$

$$\overline{\hat{\psi}^l(\omega)} e^{2\pi i Bm\omega} d\omega |^2 \quad (10)$$

Where we change variables by $\omega' = A_p^{\#}\omega - \nu$ in the last equality.

We assert:

$$\sum_{p\in P} q_p \sum_{\nu\in Q} \sum_{m\in Z^n} | \int_{R^n} \hat{f}(A^{*j}(\omega+\nu))$$

$$\overline{\hat{\psi}^l(\omega)} e^{2\pi i Bm\omega} d\omega |^2$$

$$= \sum_{(p,\nu)\in S} \frac{q_p}{b} \int_{B^{\#}([0,1]^n)} | \sum_{s\in Z^n} \hat{f}(A_p^*(\omega + B^{\#}s + \nu))$$

$$\overline{\hat{\psi}^l}(\omega + B^{\#}s) |^2 \, d\omega. \quad (11)$$

For fixed $(p,\nu)\in S$, we have:

$$\int_{B^{\#}([0,1]^n)} \sum_{s\in Z^n} | \hat{f}(A_p^*(\omega + B^{\#}s + \nu)) \overline{\hat{\psi}^l}(\omega +$$

$$B^{\#}s) | d\omega$$

$$= \sum_{s\in Z^n} \int_{B^{\#}([0,1]^n)} | \hat{f}(A_p^*(\omega + B^{\#}s + \nu)) \overline{\hat{\psi}^l}(\omega + B^{\#}s) | d\omega$$

$$= \sum_{s\in Z^n} \int_{B^{\#}s + B^{\#}([0,1]^n)} | \hat{f}(A_p^*(\omega+\nu)) \overline{\hat{\psi}^l}(\omega) | d\omega = \int_{R^n} | \hat{f}(A_p^*(\omega+\nu)) \overline{\hat{\psi}^l}(\omega) | d\omega$$

$$\le \left(\int_{R^n} | \hat{f}(A_p^*(\omega+\nu)) |^2 \, d\omega\right)^{\frac{1}{2}}$$

$$\left(\int_{R^n} |\overline{\hat{\psi}^l}(\omega)|^2 \, d\omega\right)^{\frac{1}{2}}. \quad (12)$$

Where the fourth inequality is obtained by using Cauchy-Schwarz's inequality.

Thus we can define a function $F_p : R \to C$ by:

$$F_p(\omega) = \sum_{s\in Z^n} \hat{f}(A_p^*(\omega + B^{\#}s + \nu))$$

$$\overline{\hat{\psi}^l}(\omega + B^{\#}s), a.e.\ \omega. \quad (13)$$

$F_p(\omega)$ is $B^{\#}T^n$-periodic, and the above argument gives that $F_p(\omega) \in L^1(B^{\#}[0,1]^n)$. In fact, we even have $F_p(\omega) \in L^2(B^{\#}[0,1]^n)$. To see this, we first see that:

$$|F_p(\omega)|^2 \le \sum_{s\in Z^n} |\hat{f}(A_p^*(\omega + B^{\#}s + \nu))|^2 \sum_{s\in Z^n} |\hat{\psi}^l(\omega + B^{\#}s)|^2 . \tag{14}$$

Since $\hat{f} \in C_c(R)$, the function:

$$\omega \to \sum_{s\in Z^n} |\hat{f}(A_p^*(\omega + B^{\#}s + \nu))|^2$$

Is bounded. According to above argument, we easily get $F_p(x) \in L^2(B^{\#}[0,1]^n)$.

Then, according to the definition of $F_p(\omega)$, we have:

$$\begin{aligned}
&\int_{R^n} \hat{f}(A_p^*(\omega+\nu))\overline{\hat{\psi}^l(\omega)}e^{2\pi iBm\omega}d\omega \\
&= \sum_{s\in Z^n} \int_{B^{\#}s+B^{\#}([0,1]^n)} \hat{f}(A_p^*(\omega+\nu))\overline{\hat{\psi}^l}(\omega) e^{2\pi iBm\omega}d\omega \\
&= \sum_{s\in Z^n} \int_{B^{\#}([0,1]^n)} \hat{f}(A_p^*(\omega+B^{\#}s+\nu))\overline{\hat{\psi}^l}(\omega + B^{\#}s)e^{2\pi iBm\omega}d\omega \\
&= \int_{B^{\#}([0,1]^n)} \left(\sum_{s\in Z^n} \hat{f}(A_p^*(\omega+B^{\#}s+\nu)) \overline{\hat{\psi}^l}(\omega+B^{\#}s) \right) e^{2\pi iBm\omega}d\omega \\
&= \int_{B^{\#}([0,1]^n)} F_p(\omega)e^{2\pi iBm\omega}d\omega.
\end{aligned} \tag{15}$$

Parseval's equality shows that:

$$\sum_{m\in Z^n} \left|\int_{B^{\#}([0,1]^n)} F_p(\omega)e^{2\pi iBm\omega}d\omega\right|^2 = \frac{1}{b}\int_{B^{\#}([0,1]^n)} |F_p(\omega)|^2\, d\omega; \tag{16}$$

Combining (15), (16) and the definition of $F_p(\omega)$, we obtain that:

$$\sum_{m\in Z^n} \left|\int_{R^n} \hat{f}(A_p^*(\omega+\nu))\overline{\hat{\psi}^l(\omega)}e^{2\pi iBm\omega}d\omega\right|^2 = \frac{1}{b}\int_{B^{\#}([0,1]^n)} \left|\sum_{s\in Z^n} \hat{f}(A_p^*(\omega+B^{\#}s+\nu)) \overline{\hat{\psi}^l}(\omega+B^{\#}s)\right|^2 d\omega. \tag{17}$$

So, we obtain (11). Thus, we complete the assertion.

Choose $\omega_0 \in R$ to be Lebesgue point of the function $\sum_{(p,\nu)\in S} |\hat{\psi}^l(A_p^{\#}\omega - \nu)|^2$.

Letting $B(\epsilon)$ denote the ball of radius $\epsilon > 0$ about the origin and ϵ be sufficiently small, define f_ϵ by:

$$\hat{f}_{\epsilon}(\omega)=\frac{1}{\sqrt{|B(\epsilon)|}}\chi_{B(\epsilon)}(\omega-\omega_0).$$

Therefore, we obtain:

$$\|f_{\epsilon}\|^2=\|\hat{f}_{\epsilon}\|^2=1.$$

Thus, we have:

$$\begin{aligned}&\sum_{(p,\nu)\in S}|\hat{\psi}^l(A_p^{\#}\omega_0-\nu)|^2\\&=\lim_{\epsilon\to 0}\int_{|\omega-\omega_0|<\epsilon}\frac{1}{|B(\epsilon)|}\sum_{(p,\nu)\in S}|\hat{\psi}^l(A_p^{\#}\omega-\nu)|^2\,d\omega.\end{aligned} \quad (18)$$

From the definition of f, (9), (10) and (11), we have:

$$\begin{aligned}&\int_{|\omega-\omega_0|<\epsilon}\frac{1}{|B(\epsilon)|}\sum_{l=1}^{L}\sum_{(p,\nu)\in S}|\hat{\psi}^l(A_p^{\#}\omega-\nu)|^2\,d\omega\\&=\sum_{l=1}^{L}\sum_{(p,\nu)\in S}\int_{B^{\#}([0,1]^n)}|\hat{f}_{\epsilon}(\omega)|^2|\hat{\psi}^l(A^{\#}\omega-\nu)|^2\,d\omega\\&=\sum_{l=1}^{L}\sum_{(p,\nu)\in S}q_p\int_{B^{\#}([0,1]^n)}|\sum_{s\in Z^n}\\&\hat{f}_{\epsilon}(A_p^{*}(\omega+B^{\#}s+\nu))\overline{\hat{\psi}^l}(\omega+B^{\#}s)|^2\,d\omega\\&=b\sum_{l=1}^{L}\sum_{(p,\nu)\in S}\sum_{m\in Z^n}|<f_{\epsilon},D_{A_p}E_{\nu}T_{Bm}\psi^l>|^2\\&\le bA_2,\end{aligned} \quad (19)$$

Where the third equality is obtained by changing variables $\omega'=A_p^{*}(\omega+\nu)$.

Let $\epsilon\to 0$, using the definition of Lebesgue point, we get:

$$\sum_{l=1}^{L}\sum_{(p,\nu)\in S}|\hat{\psi}^l(A_p^{\#}\omega_0-\nu)|^2\le bA_2. \quad (20)$$

According to the definition of Lebesgue point, by the similar technique of Chui and Shi [1993], we obtain:

$$\sum_{l=1}^{L}\sum_{(p,\nu)\in S}|\hat{\psi}^l(A_p^{\#}\omega_0-\nu)|^2\ge bA_1. \quad (21)$$

We leave the assertion to readers.
Comparing with (20) and (21), by changing variables by $\omega=\omega_0$, we have (9).

Therefore, we have completed the proof of Theorem 3.1.

Remark 3.1. In particular, let A the elementary matrix E in the Theorem 3.1, then, we obtain the necessary condition of the Gabor frames as the following, which is a generalization of the known result [Christensen, 2002] in higher dimensions.

Corollary 3.1 Let $B,C \in GL_n(R)$. Suppose that the Gabor system.

$$\{E_{Ck}T_{Bm}\psi^l(x)\}_{k,m\in Z^n}$$

Is a frame with frame bounds A_1 and A_2, then:

$$bA_1 \le \sum_{k\in Z^n} |\hat{\psi}^l(\omega - Ck)|^2 \le bA_2,\ a.e.\omega,$$

Where $b = |\det B|$.

On the other side, let :

$$P = \{A^j : j \in Z, A \in GL_n(R)\}$$

And $Q = \{0\}$ in the Theorem 3.1, then, we obtain the necessary condition of the wavelet frames as the following, which is a generalization of Chui and Shi [1993] in higher dimensions.

Corollary 3.2 Let $A \in E_n, B \in GL_n(R).$ Suppose that wavelet system.

$$\{D_A^j T_{Bm}\psi(x)\}_{j\in Z, m\in Z^n}$$

Is a frame with frame bounds \$A_1\$ and \$A_2\$, then:

$$bA_1 \le \sum_{j\in Z} |\hat{\psi}^l(A^{*j}\omega)|^2 \le bA_2,\ a.e.\omega,$$

Where $b = | detB |$.

Remark 3.2. In the following, we will discuss necessary conditions for other wave packet frames $\Psi^i\ (1 \le i \le 5)$ defined by (2.6) with the different operator order.

For wave packet systems Ψ^1, from Lemma 2.3, we have:

$$D_{A_p}T_{Bm}E_\nu\psi^l(x) = e^{-2\pi iBm\cdot\nu}D_{A_p}E_\nu T_{Bm}\psi^l(x).$$

If wave packet system.

$$\{D_{A_p}T_{Bm}E_\nu\psi^l(x)\}_{l=1,2,\cdots,L,\ m\in Z^n,(p,\nu)\in S}$$

Defined by (2.6) is a frame for with frame bounds A_1 and A_2, then, from Theorem 3.1 and (3.22), the inequality (3.4) holds.

For wave packet systems Ψ^2, from (2) of Lemma 2.3, we have:

$$E_\nu D_{A_p}T_{Bm}\psi^l(x) = D_{A_p}E_{A^{\#}_p\nu}T_{Bm}\psi^l(x).$$

If wave packet system

$$\{D_{A_p}T_{Bm}E_\nu\psi^l(x)\}_{l=1,2,\cdots,L,\ m\in Z^n,(p,\nu)\in S}$$

Defined by (2.6) is a frame for with frame bounds A_1 and A_2 , then, in the same way, the inequality (3.4) holds.

Then, from Theorem 3.1 and (3.23), we have:

Corollary 3.3 Suppose that wave packet system

$$\{E_\nu D_{A_p}T_{Bm}\psi^l(x)\}_{l=1,2,\cdots,L,\ m\in Z^n,(p,\nu)\in S}$$

Defined by (2.6) is a frame with frame bounds A_1 and A_2 , then we have:

$$bA_1 \le \sum_{l=1}^{L}\sum_{(p,\nu)\in S}|\hat{\psi}^l(A_p^{\#}(\omega-\nu))|^2 \le bA_2,\ a.e.\omega,\ \text{where } b=|\det B|.$$

4. Conclusion

Frames play an important role in signal processing, image processing, data compression, and sampling theory.

The main goal of this paper is to consider the necessary conditions of wave packet systems to be frames in higher dimensions. The necessary conditions for all kinds of wave packet frames of the different operator order in higher dimensions with arbitrary expanding matrix dilations are established, which include the corresponding results of wavelet analysis and Gabor theory as the special cases. Some techniques and ways in wavelet analysis and time-frequency analysis are combined.

Acknowledgment

The authors would like to express their gratitude to the referee for his (or her) valuable comments and suggestions that lead to a significant improvement of our manuscript.

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