TELKOMNIKA Indonesian Journal of Electrical Engineering
Vol. 12, No. 9, September 2014, pp. 6999 ~ 7006
DOI: 10.11591/telkomnika.v12i9.4718

Global Stability of an SEIRS Model with Impulsive Vaccination and Saturating Incidence Rate

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Abstract

This paper introduces a impulsive vaccination SEIRS epidemic model with constant input, saturation incidence rate and exposed period, and the threshold value $\pm = 1$ *is obtained at which disease is eliminated. By using the impulsive differential equation theory, it is proved that when* $\pm < 1$*, the disease-free periodic solution is globally asymptotically stable, and that when* $\pm > 1$*,the system is uniformly persistent. On these grounds, it is shown that impulsive vaccination can bring obvious effects on the dynamics behaviors of the system.*

Keywords: *SEIRS model, impulsive vaccination, treatment rate, reproduction number, global stability*

1 Introduction

In recent 30 years, many scholars use the idea of Kermack and McKendrick's compartment modeling, building and analyzing a variety of epidemic models [1]. In order to control and prevent the infectious disease epidemics, people usually use vaccination and treatment as the preferred strategy. Immunization vaccine is a non-continuous process, so it can be treated as a transient behavior like impulse. To research the optimal strategy of the impulse for the inhibition of infectious diseases, undoubtedly, has a very important practical significance. At present, the existing main results on impulsive vaccination epidemic model are: B. Shulgin, L. Stone [2] discussed the local asymptotic stability of the disease-free periodic solution for the impulsive vaccination SIR model; Alberto d'Onfrio [3] studied the impulsive vaccination SIR model and prove the global stability of disease-free periodic solution; Jin zhen and Ma Zhien [4] considered the impulsive vaccination SIR model with standard incidence rate, given the algebraic criterion for the global stability of disease-free periodic solution; Pang Guoping [5] and Xu Kin [6] investigated the SIRS model and SIQRS model of impulsive vaccination, obtained the sufficient condition of the global stability of disease-free periodic solution. Recently, Liu Kaiyuan and Cheng Lansun [7] analyzed dynamic behavior of impulsive vaccination SIR model with vertical infection; Huang Canyun and An Xiaofeng [8] discussed the global attractivity and disease persistent of disease-free periodic solution for the impulsive vaccination SIR model with multiple time delay and nonlinear incidence rate; Zhu Lingfeng and Li Weide [9] considered the continuous and impulsive vaccination SIQVS model, gave the sufficient condition of the existence of disease-free periodic solution and globally asymptotically stable, discuss the impact of isolation rate, impulsive rate and impulsive vaccination rate for disease control and prevention. Moreover, Lu Xue Juan and Wang Weihua [10] discussed the double time delays SEIRS epidemic model of non-impulsive vaccination, analyze the globally asymptotically stability of the disease-free equilibrium and endemic diseases equilibrium, proved the persistence of disease. Based on the above work, we study impulsive vaccination SEIRS epidemic model with saturation incidence rate as following:

Received October 11, 2013; Revised June 18, 2014; Accepted July 8, 2014

$$\begin{cases} \left.\begin{array}{l} S'(t) = «A - \Psi(S)I(t) - «S(t) + ¢R(t) \\ E'(t) = \Psi(S)I(t) - q\Psi(S)I(t) - (« + ³)E(t) \\ I'(t) = q\Psi(S)I(t) - (« + d + ¦ + ¥)I(t) \\ R'(t) = ³E(t) + (¦ + ¥)I(t) - (« + ¢)R(t) \end{array}\right\} t \neq n², \\ \left.\begin{array}{l} S(t^+) = (1 - °)S(t) \\ E(t^+) = E(t) \\ I(t^+) = I(t) \\ R(t^+) = °S(t) + R(t) \end{array}\right\} t = n², n = 1, 2, \cdots. \end{cases} \tag{1}$$

Where, the total population $N(t) = S(t) + E(t) + I(t) + R(t)$. $S(t), E(t), I(t), R(t)$ represent the population of susceptible, exposed, infected, resistant respectively, $\Psi(S) = S(t)/(1 + \sim S(t))$ is the saturating infect rate from infected to susceptible, $«A$ is input rate, ° is vaccination rate, ² is vaccination cycle, « is natural mortality rate, ³ , ¥ are the natural recovery rate of exposed and infected respectively, d is mortality rate of illness, ¦ is cure rate, ¥ is natural recovery rate, q is the scale factor for the exposed transforming to infected. ¢ is immunodeficiency rate coefficient, and assuming the healer and the natural recover have temporary immunity.

Because $N(t) = S(t) + E(t) + I(t) + R(t)$ $N'(t) = «A - «N(t) - dI(t)$, so model (1) could be written as:

$$\begin{cases} \left.\begin{array}{l} S'(t) = «A - \Psi(S)I(t) - «S(t) + ¢(N(t) - S(t) - E(t) - I(t)) \\ E'(t) = \Psi(S)I(t) - q\Psi(S)I(t) - (« + ³)E(t) \\ I'(t) = q\Psi(S)I(t) - \mu I(t) \\ N'(t) = A - «N(t) - dI(t) \end{array}\right\} t \neq n², \\ \left.\begin{array}{l} S(t^+) = (1 - °)S(t) \\ E(t^+) = E(t) \\ I(t^+) = I(t) \\ N(t^+) = N(t) \end{array}\right\} t = n², n = 1, 2, \cdots. \end{cases} \tag{2}$$

Where $\mu = « + d + ¦ + ¥$. Based on $N'(t) \leq «A - «N(t)$ and comparison theorem, exist $t_0 > 0$, when $t \geq t_0$, all the solution of system (2) $(S(t), E(t), I(t), N(t))^T$ enter and remain in the area of

$$\Omega = \{(S(t), E(t), I(t), N(t))^T \in R^4 \mid S(t) \geq 0, E(t) \geq 0, I(t) \geq 0, 0 \leq N(t) \leq A\}$$

So, Ω is the positive maximum variant set of system (2).

Model (1) have universality, When $N(t) = 1, E(t) = 0, q = 1, ¦ = 0$, model (1.1) became the models of paper [5] considered. So, the result of this paper is the promotion and improvement of paper [5].

2. Lemma

To prove the following, the relevant conclusions in the theory of impulsive differential equations are introduced as the lemma of this paper.

Lemma 2.1 [1].Assuming constant $a > 0, b > 0, 0 < ° < 1,$ impulsive differential system.

$$\begin{cases} x'(t) = a - bx(t), t \neq n², \\ x(t^+) = (1- °)x(t), t = n², n = 1, 2, \cdots. \end{cases}$$

Exist a unique and globally asymptotically stable positive periodic solution.

$$x(t) = \frac{a}{b}\frac{1-(1- °)e^{-b²} - ° e^{-b(t-n²)}}{1-(1- °)e^{-b²}}, n² < t \leq (n+1)².$$

Lemma 2.2. Assuming funcation $m(t) \in PC^1(R_+, R)$ satisfied impulsive differential inequality.

$$\begin{cases} m'(t) \leq p(t)m(t) + q(t), t \neq t_k, \\ m(t_k^+) \leq d_k m(t_k) + b_k, 0 \leq t_0 < t_1 < \cdots, \end{cases}$$

Where $p(t), q(t) \in PC(R_+, R), d_k \geq 0, b_k (k = 1, 2, \cdots)$ is constant, then when $t \geq t_0$:

$$m(t) \leq m(t_0)\left(\prod_{t_0<t_k<t} d_k\right)\exp\left(\int_{t_0}^t p(s)ds\right) + \sum_{t_0<t_k<t}\left(\prod_{t_k<t_j<t} d_k \exp\left(\int_{t_k}^t p(s)ds\right)\right)b_k$$
$$+\int_{t_0}^t \prod_{s<t_k<t} d_k \exp\left(\int_s^t p(v)dv\right)q(s)ds.$$

3. Persistence and Local Stability

Theorem 3.1. System (2) exist disease-free periodic solution $(\tilde{S}(t), 0, 0, A)$, where:

$$\tilde{S}(t) = \frac{A[1-(1- °)e^{-(« + ¢)²} - ° e^{-(« + ¢)(t-n²)}]}{1-(1- °)e^{-(« + ¢)²}}, n² < t \leq (n+1)². \tag{3}$$

Proof When $E(t) = 0, I(t) = 0$, Because $N'(t) = « A - « N(t), \lim_{t\to+\infty} N(t) = A$, so the limiting system of system (2) is:

$$\begin{cases} S'(t) = (« + ¢)A - (« + ¢)S(t), t \neq n², \\ S(t^+) = (1- °)S(t), t = n², n = 1, 2, \cdots. \end{cases} \tag{4}$$

According to Lemma 2.1, system (4) exist a unique and globally asymptotically stable positive periodic solution, as formula (3). Completes the proof.

Define a basic reproduction number:

$$± = \frac{²\sim[q \quad A - µ(1+\sim A)](« + ¢)}{q}\left[\ln\frac{K + °}{K + °(1+\sim A - \sim Ae^{(« + ¢)²})}\right]^{-1}. \tag{5}$$

where $µ = « + d + ¦ + ¥, K = (1+\sim A)(e^{(« + ¢)²} - 1).$

Theorem 3.2. When $± < 1$, the disease-free periodic solution $(\tilde{S}(t), 0, 0, A)$ of system (1) is locally asymptotically stable When $± > 1$, the disease-free periodic solution $(\tilde{S}(t), 0, 0, A)$ is locally unstable.

Proof Set $(S(t), E(t), I(t), N(t))^T$ is the any solution of system (2), make transformation.

$$x(t) = S(t) - \tilde{S}(t), y(t) = E(t), z(t) = I(t), w(t) = N(t) - A,$$

So, the approximate linear system of system (2) is:

$$\begin{pmatrix} x'(t) \\ y'(t) \\ z'(t) \\ w'(t) \end{pmatrix} = \begin{pmatrix} -(« + ¢) & -¢ & \tilde{S}(t)/(1+\sim\tilde{S}(t)) - ¢ & ¢ \\ 0 & -(« + ³) & (1-q)\ \tilde{S}(t)/(1+\sim\tilde{S}(t)) & 0 \\ 0 & 0 & q\ \tilde{S}(t)/(1+\sim\tilde{S}(t)) - µ & 0 \\ 0 & 0 & -d & -« \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \\ z(t) \\ z(t) \end{pmatrix}, 0 < t \le ². \quad (6)$$

It is not difficult to obtain the fundamental solution matrix which satisfied condition $\Phi(0) = E$ (E is unit matrix) is:

$$\Phi(t) = \begin{pmatrix} e^{-(«+¢)t} & C_1 & C_2 & C_3 \\ 0 & e^{-(«+³)t} & C_4 & 0 \\ 0 & 0 & \exp\{\int_0^t [q\ \ \tilde{S}(s)/(1+\sim\tilde{S}(s)) - µ]ds\} & 0 \\ 0 & 0 & C_5 & e^{-«t} \end{pmatrix}, 0 < t \le ².$$

Because $C_i (i = 1, 2, 3, 4, 5)$ is not used in the following calculation, we omit the exact expression**.** Accordingly, the pulse conditioning of system (2) is:

$$\begin{pmatrix} x(n²^+) \\ y(n²^+) \\ z(n²^+) \\ w(n²^+) \end{pmatrix} = \begin{pmatrix} 1-° & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x(n²) \\ y(n²) \\ z(n²) \\ z(n²) \end{pmatrix}.$$

Apply Floquet theorem get:

$$M = \begin{pmatrix} 1-° & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Phi(²)$$

$$= \begin{pmatrix} (1-°)e^{-(«+¢)²} & (1-°)C_1 & (1-°)C_2 & (1-°)C_3 \\ 0 & e^{-(«+³)²} & C_4 & 0 \\ 0 & 0 & \exp\{\int_0^² [q\ \ \tilde{S}(t)/(1+\sim\tilde{S}(t)) - µ]dt\} & 0 \\ 0 & 0 & C_5 & e^{-«²} \end{pmatrix}.$$

Because the necessary and sufficient condition of disease-free periodic solution $(\tilde{S}(t), 0, 0, A)$ which is locally asymptotically stable is the mode of matrix M 's eigenvalues.

$$ª_1 = (1-°)e^{-(«+¢)²}, ª_2 = e^{-(«+³)²}, ª_3 = \exp\{\int_0^² [q\ \ \tilde{S}(t)/(1+\sim\tilde{S}(t)) - µ]dt\}, ª_4 = e^{-«²}$$

Less than 1, so need $\int_0^{²}[q\ \ \tilde{S}(t)/(1+\sim\tilde{S}(t))-\mu]dt<0,$ this is equivalent to $± <1.$ Therefore, when $± <1$, the disease-free periodic solution of system (1.2) $(\tilde{S}(t),0,0,A)$ is locally asymptotically stable When $± >1$, the mode of matrix M 's eigenvalues $^{a}{}_{3}$ great than 1, disease-free periodic solution $(\tilde{S}(t),0,0,A)$ is unstable**.** Completes the proof.

Theorem 3.3. When $± >1$, system (1.2) is uniform persistence**.**

Proof. Use the method of paper[5,6], easy to get the uniform persistence of system (2), Be omitted**.**

4. The Globally Asymptotically Stability of Disease-free Periodic Solution

Theorem 4.1. When $± <1$, the disease-free periodic solution $(\tilde{S}(t),0,0,A)$ of system (1.2) is globally asymptotically stability**.**

Proof. When $± <1$, choise a sufficiently small $£ >0,$ make:

$$±_0=\int_0^{²}[\frac{q\ \ \tilde{S}(t)}{1+\sim\tilde{S}(t)}+q\ \ £-\mu]dt<0. \tag{7}$$

Use $0\le N\le A$, based on the first function of system (2):

$$S'(t)\le(« +¢)A-(« +¢)S(t).$$

Make impulsive comparison equation:

$$\begin{cases} u_1'(t)=(« +¢)A-(« +¢)u_1(t), t\neq n², \\ \left.\begin{array}{l} u_1(t^+)=(1-°)u_1(t), \\ u_1(0^+)=S(0^+), \end{array}\right\} t=n². \end{cases} \tag{8}$$

From the Lemma 2.1 and formula (3),

$$\tilde{u}_1(t)=\frac{A[1-(1-°)e^{-(«+¢)²}-°e^{-(«+¢)(t-n²)}]}{1-(1-°)e^{-(«+¢)²}}=\tilde{S}(t).$$

Based on comparison theorem of impulsive differential systems [1], for arbitrarily small $£ >0,$ exist a positive integer $T_1,$ when $t>T_1$, there is:

$$S(t)\le u_1(t)<\tilde{S}(t)+£. \tag{9}$$

Where,

$$\frac{q\ \ (\tilde{S}(t)+£)}{1+\sim(\tilde{S}(t)+£)}\le\frac{q\ \ \tilde{S}(t)}{1+\sim\tilde{S}(t)}+q\ \ £. \tag{10}$$

Use formula (9) and (10), because the third formula of system (2):

$$\begin{cases} I'(t)\le[\frac{q\ \ \tilde{S}(t)}{1+\sim\tilde{S}(t)}+q\ \ £-\mu]I(t), t\neq n², \\ I(t^+)=I(t), t=n². \end{cases} \tag{11}$$

According to Lemma 2.2, obtain:

$$I((n+1)^2) \le I(n^2)\exp\{\int_{n^2}^{(n+1)^2}[\frac{q \quad \tilde{S}(t)}{1+\sim\tilde{S}(t)}+q \quad £-\mu]dt\} = I(n^2)e^{\pm_0}, n^2 > T_1.$$

From those recursion formulas $I(n^2) \le I(0^+)e^{n\pm_0}, \lim_{n\to+\infty} I(n^2)=0.$ For any $n^2 \le t \le (n+1)^2$ have:

$$0 < I(t) \le I(n^{2+})\exp\{\int_{n^2}^{(n+1)^2}[\frac{q \quad (\tilde{S}(t)+£)}{1+\sim(\tilde{S}(t)+£)}-\mu]dt\} = I(n^2)\exp\{q \quad ^2/\sim\},$$

There upon $\lim_{t\to+\infty} I(t)=0.$ So, exist a positive integer $T_2 > T_1,$ when $t > T_2$, there is $I(t) < £.$ And according to the second fuction of system (2):

$$-(« + ^3)E(t) \le E'(t) \le \frac{(1-q)}{\sim}£ - (« + ^3)E(t). \tag{12}$$

Where £ is arbitrarily small , obtain $\lim_{t\to+\infty} E(t) = \lim_{t\to+\infty} E(n^2)\exp\{-(«+^3)(t-n^2)\}=0.$ Namely exist a positive integer $T_3 > T_2,$ when $t > T_3$, thereis $E(t) < £.$ Similarly, according to the fourth fuction of system (2).

$$A - « N(t) - d£ \le N'(t) \le A - « N(t). \tag{13}$$

Because £ arbitrarily small, $\lim_{t\to+\infty} N(t) = A.$ Simultaneously, exist positive integer when $T_4 > T_3, t > T_4,\ N(t) > A - £.$ And according to the first function of system (2).

$$S'(t) \ge (« + ¢)A - 3¢£ - (« + ¢ + \quad £)S(t). \tag{14}$$

Make pulse comparison equation:

$$\begin{cases} u_2'(t) = (« + ¢)A - 3¢£ - (« + ¢ + \quad £)u_2(t), t \ne n^2, \\ \left.\begin{array}{l} u_2(t^+) = (1-°)u_2(t), \\ u_2(0^+) = S(0^+), \end{array}\right\} t = n^2. \end{cases} \tag{15}$$

According to Lemma 2.1, system (15) exist unique globally asymptotically stable positive periodic solution:

$$\tilde{u}_2(t) = \frac{(«+¢)A-3¢£}{«+¢+\quad £}\frac{1-(1-°)e^{-(«+¢+\quad £)^2} - °e^{-(«+¢+\quad £)(t-n^2)}}{1-(1-°)e^{-(«+¢+\quad £)^2}}. \tag{16}$$

Based on comparison theorem of impulsive differential systems:

$$S(t) > u_2(t) > \tilde{u}_2(t) - £. \tag{17}$$

From fumula (7) to (17) obtain $\tilde{u}_2(t) - £ < S(t) < \tilde{S}(t) + £.$ Consider the arbitrariness of £ , have $\lim_{t\to+\infty} S(t) = \tilde{S}(t).$ Hereby, when $\pm < 1$, the disease-free periodic solution

$(\tilde{S}(t), 0, 0, A)$ of system (2) is globally attractive. And according to lemma 3.2, so $\lim_{t \to +\infty}(S(t), E(t), I(t), N(t)) = (\tilde{S}(t), 0, 0, A),$ Therefore the disease-free periodic solution $(\tilde{S}(t), 0, 0, A)$ of system (1.2) is globally asymptotically stable**.** Completes the proof.

5. Ecological Conclusion

Base on the disscusion above, The basic reproduction number $\pm = 1$ is the threshold of whether eliminate the disease or not**.** According to Theorem 3.3, when $\pm > 1$, system (2) is uniform persistence, disease will sustainable existence According to theorem 4.1, when $\pm < 1$, the disease-free periodic solution $(\tilde{S}(t), 0, 0, A)$ with a periodic ² of system (2) is globally asymptotically stable, at this point, the disease will gradually eliminate**.** Make:

$$\Lambda = \frac{²\sim[q \quad A - µ(1 + \sim A)](« + ¢)}{q}.$$

If the pulse vaccination rates satisfied:

$$° > \frac{(e^{\Lambda} - 1)K}{1 + e^{\Lambda}\sim A(e^{(«+¢)²} - 1) - e^{\Lambda}} := °_c$$

Then $\pm < 1$, disease will gradual extinction If $° < °_c$, then $\pm > 1$, disease will continue to develop and become endemic**.** This indicates whether the disease will form a epidemic endemic, pulse vaccination rates ° and vaccination period ² will significantly impact the dynamic behavior of the system. When consider system (2) only have pulse vaccination without treatment ($¦ = 0$), define the basic reproduction number:

$$\pm^{\Delta} = \frac{²\sim[q \quad A - µ^{\Delta}(1 + \sim A)](« + ¢)}{q}\left[\ln \frac{K + °}{K + °(1 + \sim A - \sim Ae^{(«+¢)²})}\right]^{-1}.$$

Where $µ^{\Delta} = « + d + ¥, K = (1 + \sim A)(e^{(«+¢)²} - 1).$ Similar conclusions can be obtained when $\pm^{\Delta} > 1$. Accordingly, the system is consistent with long-lasting when $\pm^{\Delta} < 1$, Accordingly, the disease-free periodic solution of system is globally asymptotically stable. And then, because $\pm < \pm^{\Delta}$, It is more effective by applying vaccination and treatment control strategies than a single application.

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