

# Structure and Mechanical Analysis of Single Cantilever Piezoelectric Energy Harvester

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## Abstract

*Cantilever energy harvester has become the main structure in piezoelectric energy harvester. There are two different methods to build their model. One is lumped parameter model; the other is distributed parameter model. By building their govern equation and solving them, amplitude-frequency characteristics, power and natural frequency of the model are obtained. Comparison of model frequency and amplitude are made between two models. Problems, scope of application and correct method for energy harvester are also given which provides reliable theoretical reference and makes solid foundation for energy harvester design.*

**Keywords:** energy harvester, lumped mass, Euler Bernoulli beam, single cantilever

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## 1. Introduction

With the application of wireless sensor networks become more and more widespread, supply energy for these wireless sensors proves to be a significant issue. At the same time, owing to the application of MEMS technology, the power consumption of wireless sensors become more and more low. Moreover, for the ambient vibration lies everywhere, the vibration can supply energy for the low-power consumption wireless sensor via energy harvester which transforms vibration into electricity. It is a favorable approach to deal with the long time power supply for the wireless sensor [1]. Therefore, numerous scholars have researched energy harvesting technology. The piezoelectric energy harvester becomes focus for simple structure and higher energy conversion efficiency [2].

Cantilevers is the most simple, effective and generally used geometry. Study and analyze the structure and mechanical relationship is the basis of the research of energy harvester. Generally, there are two mechanical models for energy harvester. One is lumped parameter model (Spring vibrator model), the other is distributed parameter model (Euler Bernoulli model) [3].

## 2. Lumped Parameter Model for Single Cantilever Energy Harvester

Non coupled lumped parameter model is lumped parameter model. It is a convenient way to model. Acquired the parameter of the mechanical part of the harvester, the mechanical equilibrium equation and electrical balance equation can be build up by piezoelectric constitutive relation. And the transforming relationship is build up. The simplified model can interpret some feature of the energy harvester more accurately.

### 2.1. Model Structure

The sketch of the single cantilever beam energy harvester is shown in Figure 1. It is constituted of polarized piezoelectric plate along the depth of the layer and an elastic layer. The whole structure is clamped end of the beam and form into a cantilever structure. The energy harvester vibrates harmonically in the ambient vibration. The amplitude of it is A, and frequency

is  $\omega$ . And the cantilever vibration belongs to bending vibration mode. Moreover, a lumped mass is fixed on the free end of the beam that adjust the resonate frequency and increase the output power. The piezoelectric plate is covered with electrode on the upper and lower surface. The electrode is connected with load circuit and form a close circuit. The impedance of load circuit is represented with  $Z_L$ .

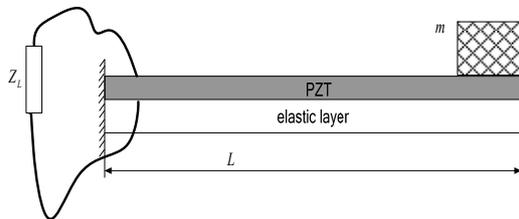


Figure 1. Model of Single Cantilever Energy Harvester with Lumped Mass on the End

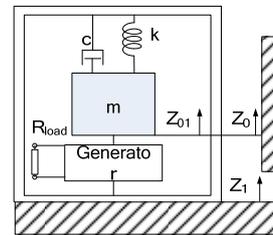


Figure 2. Equivalent System of Single Cantilever Energy Harvester

Single cantilever energy harvester can be simplified as single degree of freedom system if mass of the beam is regardless and lumped mass and beam vibrate in the vertical direction. Then, the energy harvester model can be replaced by the spring-mass system shown in Figure 2. The spring-mass system is more sensitive to ambient vibration and generate forced vibration.

In the model above mentioned, main mechanical components are inertia mass and support spring. Mass is connect to the base through the spring. The stiffness of the energy harvester can be expressed by the spring stiffness  $k$ . The system mass can be replaced by the lumped mass. With the act of vibrate acceleration, the harvester will vibrate. Displacement of the base is represented with  $z_1$ , relative displacement of the mass to the base is represented with  $z_0$ , and then mass displacement relative to the frame is represented with  $z_{01} = z_0 - z_1$ . (Only the relative displacement can produce deformation in spring.) Reserved energy is in form of elastic potential energy in the process of transform of system. The output and dissipate energy is reflect on the damping. Damping is represented with  $c$ .

## 2.2. Differential Equation of Spring Vibrator Model

If displacement of the measured base is  $z_1$  (velocity is  $\frac{dz_1}{dt}$ , acceleration is  $\frac{d^2z_1}{dt^2}$ ) that used as input,  $\frac{d^2z_{01}}{dt^2}$  of the mass can be used as output. So, differential equation of the mass can be written as follow [4].

$$\frac{d^2z_{01}}{dt^2} + 2\varepsilon \frac{dz_{01}}{dt} + \omega_n^2 z_{01} = -\frac{d^2z}{dt^2} \quad (1)$$

$\varepsilon = c/2m$  is damping coefficient,  $\omega_n = \sqrt{k/m}$  is natural frequency of the system.

The ambient vibration can be regarded as synthesis of many vibrations of different frequencies. The energy harvester natural frequency takes the main frequency of the environment into account during design process. So, single frequency vibration can be referred as research emphasis. The ambient vibration can be written as follow.

$$z_1(t) = A \sin \alpha t \quad (2)$$

$A$  is amplitude of vibration,  $\omega$  is frequency. The acceleration of the vibration can be obtain by differentiate on the displacement function.

$$\frac{d^2 z_1}{dt^2} = -A \sin \omega t \quad (3)$$

Substitute (3) into (1), then:

$$\frac{d^2 z_{01}}{dt^2} + 2\varepsilon \frac{dz_{01}}{dt} + \omega_n^2 z_{01} = A \omega^2 \sin \omega t \quad (4)$$

The solution of differential Equation (4) includes two parts. The first part is the free vibration which doesn't take ambient vibration excitation into considering. i.e. the right side of the differential equation equals zero that make a homogeneous second-order differential equation. If the attenuation vibration is think about, general solution of the equation can be written as follow:

$$z = e^{-\alpha t} (C_1 \cos \alpha t + C_2 \sin \alpha t) + \frac{mA\omega^2}{\sqrt{(k - \omega^2 m)^2 + \omega^2 c^2}} \sin(\omega t - \phi) \quad (5)$$

$$\phi = \arctan \frac{\omega c}{(k - \omega^2 m)} \quad (6)$$

Equation explain that the lumped parameter system that make up of spring, mass and damping vibrate under forced vibration. The first item  $e^{-\alpha t} (C_1 \cos \alpha t + C_2 \sin \alpha t)$ , when  $t \rightarrow +\infty$ , it trends to zero and known as transient terms. It expresses that the amplitude of the vibration gradually attenuates. The second item is known as steady item. And it expresses that the amplitude and the cycle are invariant with the time.

Therefore, when the ambient is resonant excitation, the spring-mass vibration system is a stable periodic vibration. If  $c$  and  $(k - \omega^2 m)$  are too small, the vibration attenuates mildly. If

the ambient frequency is close to natural frequency  $f = \frac{1}{2\pi} \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}$ , amplitude may be tremendous large, this situation is called as resonance.

Thus, the relative vibration of spring-mass system is given as below.

$$z_{01} = A' \sin(\omega t - \phi) \quad (7)$$

Where,

$$A' = \frac{A(\omega / \omega_n)^2}{\sqrt{[1 - (\omega / \omega_n)^2]^2 + [2\zeta(\omega / \omega_n)]^2}} \quad (8)$$

$$\phi = -\arctan \frac{2\zeta(\omega / \omega_n)}{1 - (\omega / \omega_n)^2} \quad (9)$$

The displacement  $z_{01}(t)$  of the mass  $m$  relative to the frame is harmonic vibration obviously i.e.  $z_{01}(t) = A' \sin(\omega t - \phi)$ , but the phase angle is differ  $\phi$ .  $\zeta$  is damping ratio and

$\zeta = \frac{c}{2\sqrt{km}}$ .  $\omega_n$  is the natural frequency of the system and  $\omega_n = \sqrt{k/m}$ . The difference of amplitude

and phase is depend on  $\zeta$  and  $\frac{\omega}{\omega_n}$ .

Because of relative displacement of the inertia mass is  $z_{01}$ , according to Newton's second law, the deformation force that inertia system produced under the excitation can be written as below.

$$F = m\ddot{z}_{01} = -mA'\omega^2 \sin(\omega t - \phi) \quad (10)$$

### 2.3. Amplitude-frequency Characteristic of the Spring-vibrator Model

According to the result that mentioned previously, amplitude-frequency  $A_x(\omega)$  characteristic and phase-frequency characteristic  $\varphi(\omega)$  are shown as below.

$$A_x(\omega) = \frac{Z_{01}}{Z_1} = \frac{(\omega/\omega_n)^2}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + [2\zeta(\omega/\omega_n)]^2}} \quad (11)$$

$$\varphi(\omega) = -\arctan \frac{2\zeta(\omega/\omega_n)}{1 - (\omega/\omega_n)^2} \quad (12)$$

Amplitude-frequency curves and phase-frequency curves are shown in Figure 3 and Figure 4 according to the two equation that mention previously.

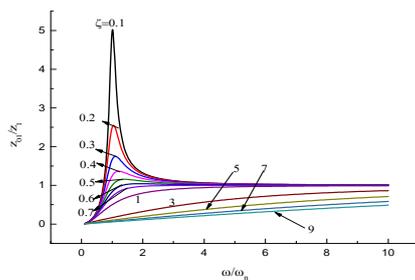


Figure 3. Amplitude under Different Damping Ratio

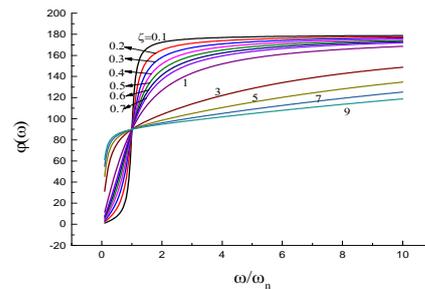


Figure 4. Phase Angle under Different Damping Ratio

As can be seen from Figure 3, the amplitude of the cantilever is larger 5 times than the ambient vibration when  $\zeta = 0.1$  and  $\omega = \omega_n$ . In other words, cantilever model is more sensitive to the ambient environment and amplify the amplitude. On the other hand, the amplitude is the biggest when vibration is resonance, and smaller damping can get bigger amplitude. Moreover, it can be concluded that damping is a critical parameter for cantilever energy harvester. It affects not only the amplitude of the energy harvester but also the phase that shown in Figure 4. Phase will affect both electric current and volt of the energy harvester circuit. Generally, the damping is between 0.01 and 0.05.

### 2.4. Power of Spring-vibrator Model

As shown in Equation (7), the amplitude of the vibrator is  $Z_{01} = A'$ .

$$A' = \frac{A \cdot \lambda^2}{\sqrt{(1 - \lambda^2)^2 + (2\zeta\lambda)^2}} \quad (13)$$

Where  $\lambda = \frac{\omega}{\omega_n}$ , the power of it is shown as below.

$$P = \frac{\zeta m_e \omega^3 A^2 \lambda^3}{(1 - \lambda^2)^2 + (2\zeta\lambda)^2} \quad (14)$$

The output power will be max when the frequency of excitation equal to the resonance frequency. i.e.  $\lambda = 1$

$$P = \frac{m_e \omega^3 A^2}{4\zeta} \quad (15)$$

The affect factors can be concluded that as below [1].

(1) Bigger mass of energy harvester can harvest more power. So it is important to increase the mass of the energy harvester as possible. And add mass on the end of the beam can not only decrease the natural frequency but also increase output power.

(2) Higher frequency of the energy harvester can harvest higher power. However, higher frequency makes smaller amplitude. So it is important to consider the amplitude as well as the frequency is concerned.

(3) Smaller damping makes bigger power. But it is not pretty as small as possible. The output of energy harvester is depending upon the power that damping consumption.

(4) Increase of damping ratio is help to frequency sensitivity of energy harvester that near the resonance frequency.

## 2.5. Natural Frequency Solving of Spring-vibrator

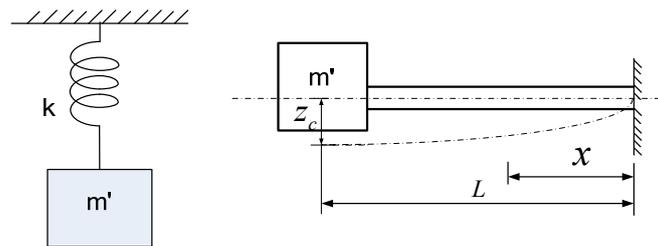


Figure 5. Deflection Curve of Cantilever

### 2.5.1. Lumped Mass is Taken into Account

When single cantilever beam simplified as spring-vibrator model, elastic element is cantilever structure. The mass of cantilever have significant proportion of the system. So it cannot be ignored. Otherwise, the calculated frequency will be obviously high. Generally, Rayleigh method is used to calculate the natural frequency of single cantilever piezoelectric energy harvester [5].

$$\omega_n = \sqrt{\frac{k_c}{(33/140)m + m'}} = \sqrt{\frac{420YI}{(33m + 140m')L^3}} \quad (16)$$

$m'$  is lumped mass on the free end,  $m$  is mass of the cantilever. Because of the assumed bending curve is different from the real vibration cure, the calculated natural frequency is slightly higher than the accurate.

### 2.5.2. Non Lumped Mass Condition

According to the derivation as above, if the mass on the end doesn't take into account, the natural frequency is as Equation (16). The equivalent mass is  $\frac{33}{140}m$ . Compare to the condition of lumped mass cantilever, natural frequency of the cantilever is obviously lower.

$$\omega_n = \sqrt{\frac{k_c}{(33/140)m}} = \sqrt{\frac{140YI}{11mL^3}} \quad (17)$$

### 3. Euler Bernoulli Model of Single Cantilever

Although lumped parameter model has given preliminary solution of the problem by simplification. But it is confined in single freedom vibration. It lacks of detailed deformation and vibration of the cantilever, such as vibration mode, accurate strain distribution and electric affect etc. Facts shown that for the transverse vibration cantilever, harmonic excitation of lumped parameter model may lead great error. The error depends on specific ratio of end mass and the mass of cantilever. As shown in Figure 6, for transverse vibration of slender beam, suppose principal axis of inertia of every cross sections are in the same plane XOZ. External load is also in the plane. The beam will vibrate in the plane. Then main deformation of the beam is bending. If length is great 5 times than the height of beams, shear deformations and the cross section spinning around their principal axis of inertia can be ignored. In this case, the beam is equivalent to Euler Bernoulli beam [6].

#### 3.1. Laminate Structure and Partial Equation of Single Cantilever

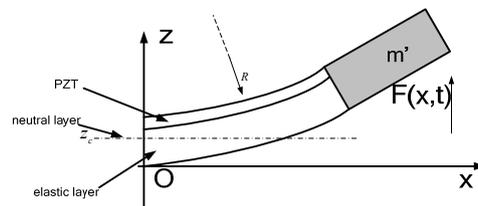


Figure 6. Euler Bernoulli Cantilever Model

The end fixed cantilever is made up two layers. The length of it is  $L$ , width is  $b$ , thickness are  $t_p$  (PZT layer) and  $t_m$  (elastic layer), the lumped mass is  $m'$  ( $x=L$ ), the end is fixed ( $x=0$ ). Top layer is piezoelectric layer and the bottom layer is elastic layer. Two layers are smooth continuous and have no relative sliding. It is supposed that the layers are uniform. Elastic model of PZT layer is  $Y_p$ , bending moment of inertia is  $I_p$ , thickness is  $t_p$ , and cross-section area is  $A_p (=bt_p)$ . Elastic model of elastic layer is  $Y_m$ , bending moment of inertia is  $I_m$ , thickness is  $t_m$ , and cross-section area is  $A_m (=bt_m)$ . Subscript  $p$  signifies PZT layer,  $m$  signifies elastic layer. Curvature is  $C=1/R$ , and the dimensionless couple effect of piezoelectric effect is  $k=(d_{31}^2 Y_p / \epsilon_0 \epsilon_p)^{1/2}$  that suppose to be less than  $1/L$ .  $d_{31}$  is polarization coupling coefficient in  $z$  direction when subjected to stress/strain in  $x$  direction.  $\epsilon_0$  is vacuum permittivity,  $\epsilon_p$  is relative permittivity of piezoelectric material. Bending stiffness of the beam about the neutral axis  $z_c$  is shown as below in Figure 7. ( $D_p$  is bending stiffness of unit width. [7])

$$YI = D_p \cdot b = \sum_{i=1}^N \left\{ A_i Y_i \left[ (z_i - z_N)^2 + \frac{t_i^2}{12} \right] \right\} = \sum_{i=1}^N \left\{ b t_i Y_i \left[ (z_i - z_N)^2 + \frac{t_i^2}{12} \right] \right\} \quad (18)$$

If the beam is made up of two layers, the equation can be simplified as below. The upper is piezoelectric material and the lower is elastic material.

$$D_p = \frac{Y_p^2 t_p^4 + Y_m^2 t_m^4 + 2Y_p Y_m t_p t_m (2t_p^2 + 2t_m^2 + 3t_p t_m)}{12(Y_p t_p + Y_m t_m)} \quad (19)$$

For laminated layer,  $z_c = \frac{\sum_{i=1}^N Y_i t_i z_i}{\sum_{i=1}^N Y_i t_i}$ ,  $z_i$  is center axis of the coordinate of the  $i$  th layer.  $t_i$  is the thickness of  $i$  th layer that is shown in Figure 7.

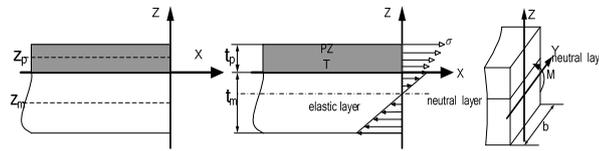


Figure 7. Model of Euler Bernoulli

$$z_c = \frac{\sum_{i=1}^N Y_i t_i z_i}{\sum_{i=1}^N Y_i t_i} = \frac{Y_p t_p^2 - Y_m t_m^2}{2(Y_m t_m + Y_p t_p)} \tag{20}$$

$$m = \sum_{i=1}^N \rho_i b t_i = \rho_p b t_p + \rho_m b t_m \tag{21}$$

$m$  is the mass of unit length, for the convenience of derivation, it can be concluded as below by principle of virtual work [5]:

$$YI \frac{\partial^4 w(x,t)}{\partial x^4} + m \frac{\partial^2 w(x,t)}{\partial t^2} + F(x,t) = 0 \quad (0 < x < L) \tag{22}$$

$$F(x,t) = -b_m \frac{\partial z(x,t)}{\partial x} - F(t) \tag{23}$$

The first item in equation (23) is damping force, the second item  $F(t)$  is produced force that caused by vibration along  $z$ .  $w(x,t)$  is the deflection when time is  $t$ .

**3.2. Free Vibration Solution that without Lumped Mass on the Beam End**

For the energy harvester is cantilever structure with single-end fixed, the vibration of it is forced vibration under base excitation. In this case, the forced base of single-end cannot be regarded as fixed. Base excitation (small deflection conditions) is taken base excitation that proposed by Erturk and Inman [3]. If the cross section of cantilever is uniform and have no lumped mass on the end, the translational motion is  $g(t)$ , tiny rotation of the root is  $h(t)$  which is shown in Figure 8.

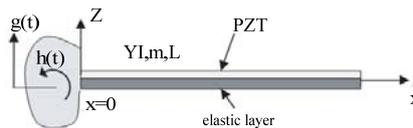


Figure 8. Free Vibration of Single Cantilever Consider the Rotation of Root

As description of Timoshenko [8], absolute displacement  $w(x,t)$  is sum of base displacement  $w_b(x,t)$  and transverse displacement  $w_{rel}(x,t)$  [9].

$$w(x,t) = w_b(x,t) + w_{rel}(x,t) \quad (24)$$

$w_b(x,t)$  represents the displacement of the fixed end,  $w_{rel}(x,t)$  is the transverse displacement to the fixed end.

$$w_b(x,t) = \delta_1(x)g(t) + \delta_2(x)h(t) \quad (25)$$

$\delta_1(x)$  and  $\delta_2(x)$  are translate affect function and rotate affect function. For cantilever which has no lumped mass on the end, the partial equation of it can be shown as below.

$$YI \frac{\partial^4 w_{rel}(x,t)}{\partial x^4} + m \frac{\partial^2 w_{rel}(x,t)}{\partial t^2} + b_m \frac{\partial}{\partial x} [w_b(x,t)] = F(t) \quad (26)$$

If  $F(t)=0$  in equation, the equation turn into corresponding homogeneous equation and can be solved by variables separation. General solution of deflection is superposition of every principal vibration mode and can be written as below.

$$w_{rel}(x,t) = \sum_{r=1}^{\infty} \phi_r(x) \eta_r(t) \quad (27)$$

After derivation,

$$\sum_{r=1}^{\infty} \{ \phi_r(x) [ \frac{d^2 \eta_r(t)}{dt^2} + 2\zeta_r \omega_r \frac{d \eta_r(t)}{dt} + \omega_r^2 \eta_r(t) ] \} = f(t) \quad (28)$$

Where,  $2\zeta_r \omega_r = \frac{b_m}{m}$ ,  $\omega_r^2 = \frac{YIk_r^4}{m}$ ,  $f(t) = \frac{F(t)}{m}$

If  $\phi_r(x)$  is regular mode, the vibration mode satisfies:

$$\frac{d^4 \phi_r(x)}{dx^4} - k_r^4 \phi_r(x) = 0 \quad (29)$$

From  $\sum_{n=1}^{\infty} \{ \phi_r(x) [ \frac{d^2 \eta_r(t)}{dt^2} + 2\zeta_r \omega_r \frac{d \eta_r(t)}{dt} + \omega_r^2 \eta_r(t) ] \} = f(t)$

According to formula  $\sum_{n=1}^{\infty} \{ \phi_r(x) [ \frac{d^2 \eta_r(t)}{dt^2} + 2\zeta_r \omega_r \frac{d \eta_r(t)}{dt} + \omega_r^2 \eta_r(t) ] \} = f(t)$ , it can be obtained as:

$$\frac{d^2 \eta_r(t)}{dt^2} + 2\zeta_r \omega_r \frac{d \eta_r(t)}{dt} + \omega_r^2 \eta_r(t) = N_r(t) \quad (30)$$

Modal excitation function of  $r$  th is:

$$N_r(t) = -m[\gamma_r^w a_w(t) + \gamma_r^\theta a_\theta(t)] \quad (31)$$

$\gamma_r^w = \int_{x=0}^L \phi_r(x) dx$  and  $\gamma_r^\theta = \int_{x=0}^L \phi_r(x) x dx$  are  $r$  th integration constant.  $a_w(t) = \frac{d^2 g(t)}{dt^2}$   $a_\theta(t) = \frac{d^2 h(t)}{dt^2}$

are translate and rotate velocity respectively. If base translate and rotate are arbitrary function, it can be obtained by Duhamel integration.

$$\eta_r(t) = \frac{1}{\omega_{rd}} \int_{\tau=0}^t N_r(\tau) e^{-\zeta_r \omega_r (t-\tau)} \sin \omega_{rd} (t-\tau) d\tau \quad (32)$$

$\omega_{rd}$  is damping frequency of  $r$  th mode.  $\omega_{rd} = \omega_r \sqrt{1 - \zeta_r^2}$ , and  $\zeta_r$  is damping ratio of  $r$  th mode.

$$w_{rel}(x, t) = \sum_{r=1}^{\infty} \frac{\phi_r(x)}{\omega_{rd}} \int_{\tau=0}^t N_r(t) e^{-\zeta_r \omega_r (t-\tau)} \sin \omega_{rd} (t-\tau) d\tau \quad (33)$$

### 3.3. Forced Vibration Solution that without Lumped Mass on the Beam End

Back to non-homogeneous equation, focus on the lowest order mode ( $r = 1$ ) that closely relates to energy harvester [10].

$$f(t) = \frac{F(t)}{m} = \frac{d^2(Z_0 e^{i\omega t})}{dt^2} = Z_0 \omega_1^2 e^{i\omega t} \quad (34)$$

According to Equation (28), the following equation can be got.

$$\phi_1(x)[\ddot{\eta}_1(t) + 2\beta_1 \omega_1 \dot{\eta}_1(t) + \omega_1^2 \eta_1(t)] = Z_0 \omega_1^2 e^{i\omega t} \quad (35)$$

That is,

$$\phi_1(L) = \sqrt{\frac{1}{mL}} \frac{2[\cos(k_1 L) \sinh(k_1 L) - \cosh(k_1 L) \sin(k_1 L)]}{\sin(k_1 L) + \sinh(k_1 L)} \quad (36)$$

According to Equation (30), the response of vibration mode is changed as below.

$$\frac{d^2 \eta_r(t)}{dt^2} + 2\zeta_r \omega_r \frac{d \eta_r(t)}{dt} + \omega_r^2 \eta_r(t) = \gamma_r^w m \omega^2 Z_0 e^{i\omega t} \quad (37)$$

By way of Duhamel integration,

$$\eta_r(t) = \frac{\gamma_r^w m \omega^2}{\omega_r^2 - \omega^2 + 2i\zeta_r \omega \omega_r} Z_0 e^{i\omega t} \quad (38)$$

Substitute the integration constant of  $r$  th mode,

$$\gamma_r^w = \int_{x=0}^L \phi_r(x) dx = \frac{2r_r}{k_r L} \sqrt{\frac{L}{m}} \quad (39)$$

$$\eta_r(t) = \frac{2r_r \omega^2 \sqrt{mL}}{k_r L (\omega_r^2 - \omega^2 + 2i\zeta_r \omega \omega_r)} Z_0 e^{i\omega t} \quad (40)$$

Substitute the above into  $w_{rel}(x, t) = \sum_{n=1}^{\infty} \phi_r(x) \eta_r(t)$ , the response of cantilever under forced vibration can be obtained.

$$w_{rel}(L, t) = 2\omega^2 Z_0 e^{i\omega t} \sum_{r=1}^{\infty} \frac{r_r \{[\cos(k_r x) + \cosh(k_r x)] - r_r \cdot [\sin(k_r x) - \sinh(k_r x)]\}}{k_r L (\omega_r^2 - \omega^2 + 2i\zeta_r \omega \omega_r)} \quad (41)$$

### 3.4. Model Take the Lumped Mass into Account

The cantilever that previously calculated is uniform cross section and has no lumped mass. Nevertheless, for decrease natural frequency and diminish dimension of energy harvester, a lumped mass will be attached on the end of beam. The vibration mode and eigenvalue are not applicable for this model [9]. If the lumped mass is  $m'$  as shown in Figure 9.

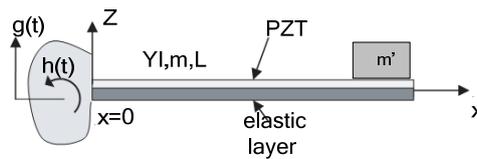


Figure 9. Single Cantilever with Lumped Mass

According to Equation (26) and (28), for single cantilever with lumped mass, the partial equation can be written as below.

$$YI \frac{\partial^4 w_{rel}(x,t)}{\partial x^4} + m \frac{\partial^2 w_{rel}(x,t)}{\partial t^2} = -[m + m' \delta(x-L)] \frac{\partial^2 w_b(x,t)}{\partial t^2} \quad (42)$$

The corresponding Eigenvalues will change with respect to the change of orthogonal condition. The vibration mode function will change as below.

$$\phi_r(x) = C_r [\cosh(k_r x) - \cos(k_r x) + s_r (\sin(k_r x) - \sinh(k_r x))] \quad (43)$$

$$s_r = \frac{mL [\sin(k_r L) - \sinh(k_r L)] + k_r L m' [\cosh(k_r L) - \cos(k_r L)]}{mL [\cosh(k_r L) + \cos(k_r L)] - k_r L m' [\sin(k_r L) - \sinh(k_r L)]} \quad (44)$$

If  $k_r L = \lambda$ , eigenvalue of the vibration mode can satisfy the equation as below. It is a transcendental equation and can be solved only by numerical methods.

$$1 + \cosh \lambda \cdot \cos \lambda + \frac{\lambda m'}{mL} (\cos \lambda \sinh \lambda - \sin \lambda \cosh \lambda) - \frac{\lambda^3 m' I_t}{mL^3} (\cosh \lambda \cdot \sin \lambda + \sinh \lambda \cos \lambda) + \frac{\lambda^4 m' I_t}{m^2 L^4} (1 - \cos \lambda \cosh \lambda) = 0 \quad (45)$$

$I_t$  is Moment of inertia to the centroid, all frequencies of every mode can be get by solves the above equation.

Similarly, according to equation,

$$\frac{d^2 \eta_r(t)}{dt^2} + 2\zeta_r \omega_r \frac{d \eta_r(t)}{dt} + \omega_r^2 \eta_r(t) = N_r(t) \quad (46)$$

$$N_r(t) = - \int_0^L \phi_r(x) [m + m' \delta(x-L)] \frac{\partial^2 w_b(x,t)}{\partial t^2} dx \quad (47)$$

The response of the lumped mass on the end of the cantilever is shown as below.

$$w_{rel}(x,t) = \sum_{r=1}^{\infty} \frac{\phi_r(x)}{\omega_{rd}} \int_{\tau=0}^t N_r(\tau) e^{-\zeta_r \omega_r (t-\tau)} \sin \omega_{rd} (t-\tau) d\tau \quad (48)$$

#### 4. Difference of Two Models

Spring-vibrator model is supposed to be single degree freedom system; actually, it is acontinuous syste. If the single degree freedom model is used, the error is inevitable. Thus, compare them will correct the system theoretically.

## 4.1. Frequency Difference

### 4.1.1. Single Frequency and Multi Frequency

Spring-vibrator model is supposed to be single degree freedom system; therefore one frequency can be calculated. But for Euler Bernoulli model, it is an infinite degree freedom system and has infinite frequency base point and multi-order mode [11].

### 4.1.2. Difference Ignored End Lumped Mass

If effect of end lumped is ignored for spring-vibrator model, natural frequency of is shown as below.

$$\omega_r = \sqrt{\frac{140 YI}{11 mL^3}} = \sqrt{\frac{3 YI}{(33/140) mL^3}} \quad (49)$$

For Euler Bernoulli model, multi-order mode frequencies are available. And the first order frequency is shown as below.

$$\omega_1 = 3.516 \sqrt{\frac{YI}{\rho AL^4}} \quad (50)$$

$$r = \frac{\omega_r}{\omega_1} = \frac{\sqrt{\frac{140 YI}{11 mL^3}}}{3.516 \sqrt{\frac{YI}{\rho AL^4}}} = \frac{\sqrt{\frac{140 YI}{11 mL^3}}}{3.516 \sqrt{\frac{YI}{mL^3}}} = \frac{\sqrt{\frac{140}{11}}}{3.516} = 1.015 \quad (51)$$

It is easily to find out that it is reasonable accurate to calculate the natural frequency of cantilever using spring-vibrator mode. The frequency is greater 1.5% than Euler Bernoulli model. The reason is that max potential energy curve under static load is different from the real curve [12].

## 4.2. Differences of Amplitude

### 4.2.1. Amplitude of Spring-vibrator Model

According to Equation (47), for the model of non lumped mass on the end of cantilever, equivalent mass is  $m_{eq} = \frac{33}{140} m$ , equivalent damping is  $c_{eq} = 2\zeta \sqrt{m_{eq} k_{eq}}$ . If the base excitation  $z(t) = Z_0 e^{i\omega t}$  is substituted into Equation (6), the amplitude-frequency can be got and identical to Equation (10).

$$|H_{SOD}(\omega, \zeta)| = \frac{(\omega / \omega_r)^2}{\sqrt{[1 - (\omega / \omega_r)^2]^2 + [2\zeta(\omega / \omega_r)]^2}} \quad (52)$$

### 4.2.2. Amplitude of Euler Bernoulli Model

According to Equation (41), when the forced vibration excitation on the base is  $z(t) = Z_0 e^{i\omega t}$ , the max amplitude is on the end of the cantilever  $x = L$ .

$$H_{euler}(\omega) = \frac{w_{rel}(L, t)}{w(t)} = \frac{w_{rel}(L, t)}{Z_0 e^{i\omega t}} \quad (53)$$

$$H_{euler}(\omega, \zeta) = 2\omega^2 \sum_{n=1}^{\infty} \frac{r_r \{[\cos(k_r L) + \cosh(k_r L)] - r_r \cdot [\sin(k_r L) - \sinh(k_r L)]\}}{k_r L (\omega_r^2 - \omega^2 + 2i\zeta_r \omega \omega_r)} \quad (54)$$

Perform modulo on the equation above,

$$|H_{euler}(\omega, \zeta)| = 2(\omega / \omega_n)^2 \sum_{n=1}^{\infty} \frac{r_r \{[\cos(k_r L) + \cosh(k_r L)] - r_r \cdot [\sin(k_r L) - \sinh(k_r L)]\}}{k_r L \sqrt{[1 - (\omega / \omega_r)^2]^2 + [2\zeta_r(\omega / \omega_r)]^2}} \quad (55)$$

According to dynamics, for continuous system, the effect of low order mode to amplitude is the largest. So the first term of the infinite series of above  $H_{euler}(\omega, \zeta)$  is approximate amplitude response. The two models can be compared when the damping is  $\zeta_1 = 0.01$ ,  $\zeta_1 = 0.02$  and  $\zeta_1 = 0.03$ .

**4.2.3. Comparisons and Results**

It can be seen from Figure 10 that Euler Bernoulli beam mode has just calculated the amplitude of the first-order modal and hasn't superposition other order modal. It's amplitude of the beam mode is higher than spring-vibrator. And the differences become greater with the frequency close to the natural frequency.

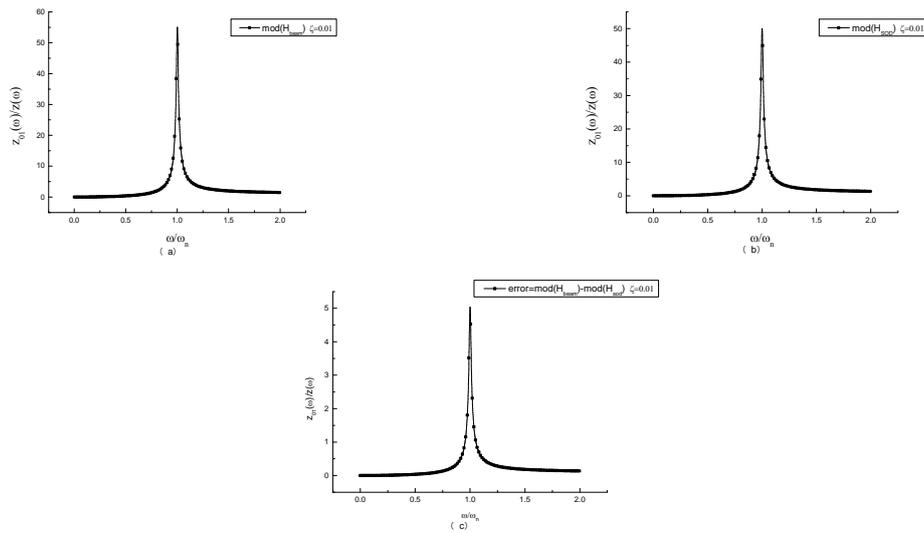


Figure 10. Amplitude and Error of Two Models ( $\zeta_1 = 0.01$ )  
 (a) is the Euler Bernoulli model, (b) is SOD model, (c) is the error between Euler Bernoulli and SOD model

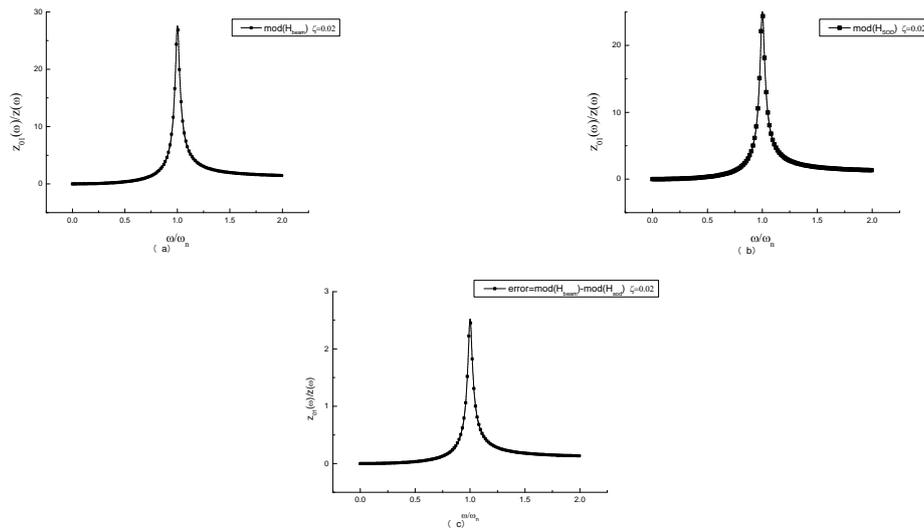


Figure 11. Amplitude and Error of Two Models ( $\zeta_1 = 0.02$ )  
 (a) is the Euler Bernoulli model, (b) is SOD model, (c) is the error between Euler Bernoulli and SOD model

It can be seen from Figure 11 that the amplitude has significantly decreased with the increase of damping. At the same time, amplitude of Euler Bernoulli beam model and spring-vibrator become closer. Furthermore, the difference become small with the frequency close to natural frequency.

It can be seen from Figure 12 that the amplitude has decreased further with the increase of damping. At the same time, amplitude of Euler Bernoulli beam model and spring-vibrator become closer. Furthermore, the differences become small with the frequency close to natural frequency.

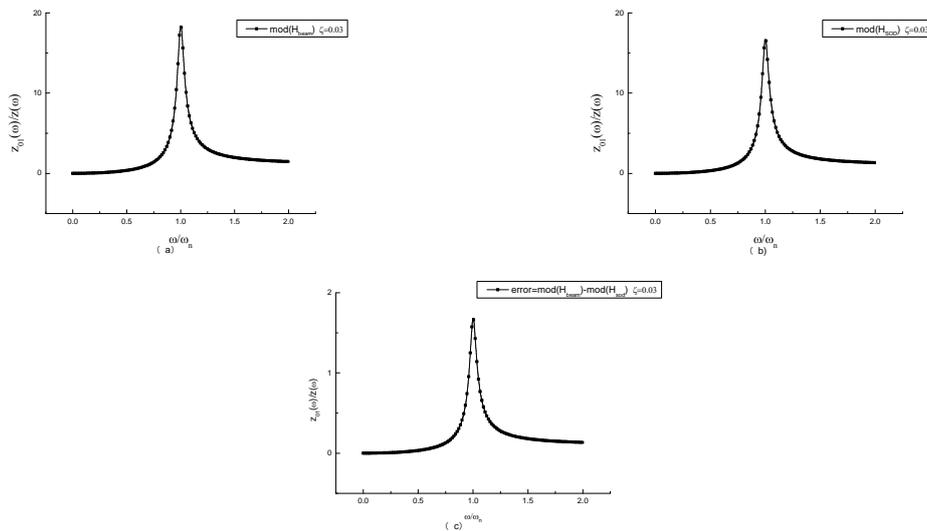


Figure 12. Amplitude and Error of Two Models ( $\zeta_1 = 0.03$ )

(a) is the Euler Bernoulli model, (b) is SOD model, (c) is the error between Euler Bernoulli and SOD model

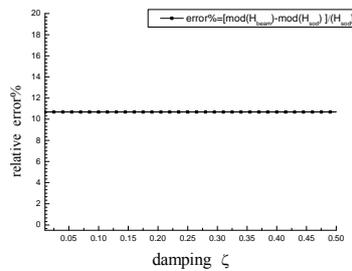


Figure 13. Relation of  $\zeta$  and Relative Errors

It can be seen from Figure 13 that the error of amplitude is a constant of 10.068% nearby the first-order frequency. The amplitude of first-order model of Euler Bernoulli beam model to spring-vibrator model is a constant value in which  $k_1L = 1.8751$ ,  $r_1 = 0.7341$  that can be got from previous narrative.

$$Ratio = \frac{|H_{euler}(\omega, \zeta_1)|}{|H_{SOD}(\omega, \zeta_1)|} = \frac{r_1 \{ [\cos(k_1L) + \cosh(k_1L)] - r_1 \cdot [\sin(k_1L) - \sinh(k_1L)] \}}{k_1L} \approx 1.566$$

It can be found that the amplitude of Euler Bernoulli beam model is greater 10% more than spring-vibrator model, and it is proportional. So the spring-vibrator model can be used to

simulate the energy harvester and some modification must be made. The presentation is shown as below.

$$H_{SOD}(\omega, \zeta) = \frac{z_{01}(t)}{z(t)} = \frac{Ratio \cdot (\omega / \omega_r)^2}{1 + 2i\zeta(\omega / \omega_r) - (\omega / \omega_r)^2} \quad (56)$$

#### 4.2.4. End Mass is Taken into Consideration

In most circumstance, for lower the first-order natural frequency, suit the environment vibration frequency and enlarge the deformation of the beam to generate more power. End mass is set on the end of the beam [13].

So the equation must take the end mass into the model equation. It makes the excitation function of the fixed end and eigenvalues of the eigenfunctions are changed. In this case, partial equation of the bending can be solved only by numerical method and voltage and power of the piezoelectric beam can be calculated. Some scholars use numerical method to correct spring-vibrator system and predict vibration of energy harvester which makes reasonable precision and errors less than  $4.5 \times 10^{-2}\%$  [14]. The correct method are shown as below.

$$z(t) = \frac{\mu_1 \cdot (\omega / \omega_r)^2}{1 + 2i\zeta(\omega / \omega_r) - (\omega / \omega_r)^2} Z_0 e^{j\omega t} \quad (57)$$

In which

$$\mu_1 = \frac{(m' / mL)^2 + 0.603(m' / mL) + 0.0895}{(m' / mL)^2 + 0.4637(m' / mL) + 0.05718} \quad (58)$$

## 5. Conclusion

In the research of energy harvester, the vibrators are always design as single cantilever beam. Single cantilever piezoelectric energy harvesters act as the research object. Lumped parameter model is use to build the vibration equation of spring-vibrator model. By solving the equation, amplitude-frequency characteristics, power and natural frequency of the model is obtained. Distributed parameter model of Euler-Bernoulli model also be introduced to study single cantilever energy harvester. Free vibration solution and forced vibration solution are derived in the circumstance of having end mass on the cantilever and having none end mass.

Comparison of model frequency and amplitude are made between two models. Problems, scope of application and correct method for energy harvester are also given which provides reliable theoretical reference and makes solid foundation for energy harvester design.

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