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A Complete Combinatorial Solution for a Coins Change Puzzle and Its Computer Implementation

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Abstract

In this paper, we study a combinatorial problem encountered in monetary systems. The problem concerned is to find an optimal solution $R(k, n)$ of a combinatorial problem for some positive integers k and n. To the authors' knowledge, there is no efficient solutions for this problem in the literatures so far. We first show how to find an efficient recursive construction algorithm based on the backtracking search strategy. Furthermore, we can give an explicit formula for finding the maximal elements of the solution. Our new techniques have improved the time complexities of the search algorithm dramatically.

Keywords: *Coins Change Puzzle, combinatorial solution, linear time, optimal algorithm*

1. Introduction

In this paper, we consider the following combinatorial problem encountered in monetary systems. Suppose $C(k)$ is a monetary system that divides the currency denomination into $k + 1$ decimal levels: $\{1, 2, 5\}; \{10, 20, 50\}; \cdots ; \{10^i, 2 \times 10^i, 5 \times 10^i\}; \cdots ; \{10^k\}$. For example, China's currency system (RMB) can be classified as $C(4)$.

Notation: $c(i, j), 0 \leq i \leq k, 0 \leq j \leq 2$ denote the levels of monetary values.

The monetary value of level i can be written as $c_i = (c(i, 0), c(i, 1), c(i, 2))^\top, 0 \leq i \leq k$. In particular, when $i = k$, $c_k = (10^k, 0, 0)^\top$.

For any integer $n \in I^+$ we can obviously express n by the above currency system as follows

$$n = \sum_{i=0}^{k} \sum_{j=0}^{2} a(i, j)c(i, j) \tag{1}$$

where $a(i, j) \in I^+, 0 \leq i \leq k, 0 \leq j \leq 2$.

Denote $a_i = (a(i, 0), a(i, 1), a(i, 2))^\top, g(a_i, c_i) = a_i^\top c_i, 0 \leq i \leq k$ and $a = (a_0, a_1, \cdots , a_k)^\top$. Then, the integer n can be expressed by

$$n = \sum_{i=0}^{k} a_i^\top c_i = \sum_{i=0}^{k} g(a_i, c_i) \triangleq f(k, a) \tag{2}$$

For a given $n \in I^+$, the above representation is obviously not unique in general. The different values of a satisfying (1) will give different representations of the positive integer n. Set $A(k, n) = \{a \mid f(k, a) = n\}$ constitutes all representations of a positive integer n in the given currency system. For example, when $k = 4, n = 3$ we have

$$A(4, 3) = \left\{ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

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Definition 1. Let a and b be two-dimensional arrays. $b \leq a$ if and only if $b(i,j) \leq a(i,j)$, $0 \leq i \leq k$, and $0 \leq j \leq 2$; $b < a$ if and only if both $b \leq a$ and $b \neq a$.

Definition 2. Let

$$s(k,a) = \{f(k,b) \mid f(k,a) = n, 0 < b \leq a\} \tag{3}$$

Set $s(k,a)$ is defined as an implication set of the positive integer n, which is the collection of all the money under the representation a. For example, when

$$a = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in A(4,3)$$

we have $s(4,a) = \{1,2,3\}$.

Definition 3. Set

$$R(k,n) = \bigcap_{a \in A(k,n)} s(k,a) \tag{4}$$

is defined to be an accurate implication set of the positive integer n in the given currency system[1].

For any $x \in R(k,n)$, regardless of the kind of par value of the currency that composes the positive integer n, it certainly contains x. For example, suppose the currency system is in RMB. A person has money \$5.27 ($n = 527$). If his money is composed of one \$5 piece ($c(2,2) = 500$), one 2 angle piece ($c(1,1) = 20$) , one 5 cent coin ($c(0,2) = 5$), and one 2 cent coin ($c(0,1) = 2$). In our definition, $k = 4$ and

$$a = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in A(4,527).$$

In this case, he cannot come up with \$0.17. That is, $17 \notin s(4,a)$. However, regardless of the kind of par value of the currency, he can certainly take out \$0.02 because without one 2 cent coin or two 1 cent coins he cannot scrape together \$5.27. In other words, $2 \in R(4,527)$. In addition to \$0.02, he can certainly take out \$5.00, \$0.2, \$0.07, \$5.2, \$0.27, and so on. These amounts of money, as they are called, are certainly taken out of the \$5.27.

The main problem concerned in this paper is for the given positive integers k and n, how to find the corresponding accurate implication set $R(k,n)$ efficiently. To the authors' knowledge, there is no solutions for the problem in the literatures so far. A preliminary conference version of this paper was presented at Advances in Information Technology and Education Communications in Computer and Information Science[2]. In this paper the correctness and complexities are proved rigorously, but not just stated in intuitively. More experiment details are described in this version of the paper.

2. Backtracking Algorithm

2.1. A Simple Backtracking Algorithm

According to Definition 3, the accurate implication set of the given positive integers k and n in the currency system $C(k)$ can be formulated as (4). In the algorithm description, we use operations + and - for a set U and a positive integer v defined as follows

$$U + v = \{x + v \mid x \in U\},\ U - v = \{x - v \mid x \in U \text{ and } x \geq v\}$$

Based on this formula we can design a simple backtracking algorithm [**?**, 3, 4] to find $R(k,n)$ as follows. Initially, $R = \{1,2,\cdots,n\}$ and $S = \emptyset$. A recursive function call $Backtrack(n)$ will compute the set $R = R(k,n)$.

Algorithm 1 Backtrack(t)

if $t = 0$ **then**
  $R \leftarrow R \bigcap S$
  return R
else
  for all $c(i, j) \in C(k)$ such that $c(i, j) \leq t$ **do**
    $S \leftarrow S + c(i, j)$
    Backtrack($t - c(i, j)$)
    $S \leftarrow S - c(i, j)$
  end for
end if
return R

2.2. Backtrack Pruning

If par value 1, 2, and 5 are used to compose the money, then positive integer 10 can be one of the following 10 different representations.

Table 1. Representations of 10

$e_1 = (10, 0, 0)$	$e_2 = (8, 1, 0)$	$e_3 = (6, 2, 0)$	$e_4 = (4, 3, 0)$	$e_5 = (2, 4, 0)$
$e_6 = (0, 5, 0)$	$e_7 = (5, 0, 1)$	$e_8 = (3, 1, 1)$	$e_9 = (1, 2, 1)$	$e_{10} = (0, 0, 2)$

Let $E = \{e_i, i = 1, \cdots, 10\}$.

Lemma 1. For the positive integers $m = 10$, $m = 12$ and $m \geq 14$, if $m = g(a_0, c_0) = \sum_{j=0}^{2} a(0, j)c(0, j)$, then there must be an integer $d \in E$ such that $d \leq a_0$.

Lemma 2. For any $a \in A(k, n)$ we have,

(1) $\sigma(a, i) \in A(k, n), 0 \leq i \leq k$. (2) $s(k, \sigma(a, i)) \subseteq s(k, a), 0 \leq i \leq k$.

Theorem 3. Let $S_0 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 13\}$,

$B(k, n) = \{a \in A(k, n) \mid \sigma(a, i) = a, 0 \leq i \leq k\}$, $F(k, n) = \{a \in B(k, n) \mid a_i^\top c_0 \in S_0\}$.

Then, $R(k, n) = \bigcap_{a \in A(k,n)} s(k, a) = \bigcap_{a \in B(k,n)} s(k, a) \bigcap_{a \in F(k,n)} s(k, a)$.

By making use of the constraints of $F(k, n)$ in Theorem 3, we can add pruning condition in the backtracking algorithm to improve the searching speed as follows [5].

Algorithm 2 Backtrack(t)

if $t = 0$ **then**
  $R \leftarrow R \bigcap S$
  return R
else
  for all $c(i, j) \in C(k)$ **and** $c(i, j) \leq t$ **and** $a_i^\top c_0 \in S_0$ **do**
    $S \leftarrow S + c(i, j)$
    Backtrack($t - c(i, j)$)
    $S \leftarrow S - c(i, j)$
  end for
end if
return R

2.3. Recursive Constructing Algorithm

Definition 5.

$$\mathrm{div}(x,y)=\left\lfloor\frac{x}{y}\right\rfloor;\mathrm{mod}(x,y)=x-y\left\lfloor\frac{x}{y}\right\rfloor.$$

Lemma 4. Let

$$G_1(k,n)=\{a\in F(k,n)\mid a_0^T c_0=\mathrm{mod}(n,10)\}$$
$$G_2(k,n)=\{a\in F(k,n)\mid a_0^T c_0=10+\mathrm{mod}(n,10)\}$$

(1) If $\mathrm{mod}(n,10)\notin\{1,3\}$, then $F(k,n)=G_1(k,n)$.
(2) If $\mathrm{mod}(n,10)\in\{1,3\}$, then $F(k,n)=G_1(k,n)\bigcup G_2(k,n)$.

Theorem 5.

(1) If $\mathrm{mod}(n,10)\notin\{1,3\}$, then $R(k,n)=\bigcap_{a\in G1(k,n)}s(k,a)$.
(2) If $\mathrm{mod}(n,10)\in\{1,3\}$, then $R(k,n)=\left(\bigcap_{a\in G1(k,n)}s(k,a)\right)\bigcap\left(\bigcap_{a\in G2(k,n)}s(k,a)\right)$.

Proof. It can be readily proved by Theorem 3 and Lemma 4. ■

Lemma 6. Let

$$s_0(k,a)=\{f(k,b)\mid 0<b\le a, b_i=0, 0\le i\le k\},$$
$$s_1(k,a)=\{f(k,b)\mid 0<b\le a, b_0=0\},$$
$$s_2(k,a)=s(k,a)-s_0(k,a)-s_1(k,a).$$

Then
For any $a\in G_1(k,n)\bigcup G_2(k,n)$, we have $s(k,a)=s_0(k,a)\bigcup s_1(k,a)\bigcup s_2(k,a)$;
$\bigcap_{a\in G_1(k,n)}s(k,a)=\alpha_1\bigcup\beta_1\bigcup\gamma_1$; $\bigcap_{a\in G_2(k,n)}s(k,a)=\alpha_2\bigcup\beta_2\bigcup\gamma_2$.
where

$$\alpha_1=\bigcap_{a\in G_1(k,n)}s_0(k,a),\alpha_2=\bigcap_{a\in G_2(k,n)}s_0(k,a)$$

$$\beta_1=\bigcap_{a\in G_1(k,n)}s_1(k,a),\beta_2=\bigcap_{a\in G_2(k,n)}s_1(k,a)$$

$$\gamma_1=\bigcap_{a\in G_1(k,n)}s_2(k,n),\gamma_2=\bigcap_{a\in G_2(k,n)}s_2(k,n)$$

Definition 6. Let U and V be two sets of integer. The circle plus operation for sets U and V is defined as $U\oplus V=\{x+y\mid x\in U, y\in V\}$. The multiplication of a set U by an integer m is defined as $m\times U=\{mx\mid x\in U\}$.

Definition 7.

$$T_0=R(k,\mathrm{mod}(n,10));T_1=10\times R(k-1,\mathrm{div}(n,10));T_2=\bigcap_{a\in G_2(k,n)}s_0(k,a);$$
$$T_3=10\times R(k-1,\mathrm{div}(n,10)-1);T_4=R(k,10+\mathrm{mod}(n,10)).$$

Lemma 7.

$$\bigcap_{a\in G_1(k,n)}s_0(k,a)=T_0 \tag{5}$$

$$\bigcap_{a\in G_1(k,n)}s_1(k,a)=T_1 \tag{6}$$

$$\bigcap_{a\in G_1(k,n)}s_2(k,a)=T_0\oplus T_1 \tag{7}$$

$$\bigcap_{a\in G_2(k,n)}s_1(k,a)=T_3 \tag{8}$$

$$\bigcap_{a\in G_2(k,n)} s_2(k,a) = T_2 \oplus T_3 \quad (9)$$

Lemma 8. Let $k > 1$, $x \in R(k,n)$ and $\mathrm{mod}(x,10) > 0$, then $\mathrm{mod}(x,10) \in R(k, \mathrm{mod}(n,10))$.
Theorem 9.
(1) If $\mathrm{mod}(n,10) \notin \{1,3\}$, then $R(k,n) = T_0 \bigcup T_1 \bigcup (T_0 \oplus T_1)$
(2) If $\mathrm{mod}(n,10) \in \{1,3\}$, then $R(k,n) = (T_0 \bigcup T_1 \bigcup (T_0 \oplus T_1)) \bigcap (T_3 \bigcup T_4 \bigcup (T_3 \oplus T_4))$.
(3) $R(0,n) = \{1,2,\cdots,n\}$.
Proof.
(1) It follows from Theorem 5 and Lemma 7 that if $\mathrm{mod}(n,10) \notin \{1,3\}$, then

$$R(k,n) = \bigcap_{a\in G_1(k,n)} s(k,a) = T_0 \bigcup T_1 \bigcup (T_0 \oplus T_1).$$

(2) It follows from Theorem 5, Lemma 6 and Lemma 7 that if $\mathrm{mod}(n,10) \in \{1,3\}$, then

$$R(k,n) = (T_0 \bigcup T_1 \bigcup (T_0 \oplus T_1)) \bigcap (T_3 \bigcup T_2 \bigcup (T_3 \oplus T_2)).$$

If $k > 1$ and $\mathrm{mod}(n,10) = 1$, then

$$R(k,11) = \{11\} \subseteq \{2,4,5,6,7,9,11\} = \bigcap_{a\in G_2(k,n)} s_0(k,a) = T_2$$

If $k > 1$ and $\mathrm{mod}(n,10) = 3$, then

$$R(k,13) = \{2,11,13\} \subseteq \{2,4,5,6,7,8,9,10,11,13\} = \bigcap_{a\in G_2(k,n)} s_0(k,a) = T_2$$

Therefore, $T_4 = R(k, 10 + \mathrm{mod}(n,10)) \subseteq T_2$
and thus, $T_3 \bigcup T_4 \bigcup (T_3 \oplus T_4) \subseteq T_3 \bigcup T_2 \bigcup (T_3 \oplus T_2)$.
It follows that

$$(T_0 \bigcup T_1 \bigcup (T_0 \oplus T_1)) \bigcap (T_3 \bigcup T_4 \bigcup (T_3 \oplus T_4)) \subseteq (T_0 \bigcup T_1 \bigcup (T_0 \oplus T_1)) \bigcap (T_3 \bigcup T_2 \bigcup (T_3 \oplus T_2)) = R(k,n)$$

On the other hand, for any $x \in R(k,n)$, we have $x \in T_3 \bigcup T_2 \bigcup (T_3 \oplus T_2)$. If $\mathrm{mod}(x,10) = 0$, then $x \in T_3$. If $\mathrm{mod}(x,10) > 0$, then $x \in T_2 \bigcup (T_3 \oplus T_2)$. It follows from Lemma 8 that $\mathrm{mod}(x,10) \in R(k, \mathrm{mod}(n,10))$.

(2.1) If $\mathrm{mod}(n,10) = 1$, then $R(k, \mathrm{mod}(n,10)) = \{1\}$. If $x \in T_2$, then $x \in \{11\} = R(k,11) = T_4$; If $x \in T_3 \oplus T_2$, then $x = 11 + y$,$y \in T_3$ and thus, $x \in T_3 \oplus T_4$. Therefore, $x \in T_4 \bigcup (T_3 \oplus T_4)$.

(2.2) If $\mathrm{mod}(n,10) = 3$, then $R(k, \mathrm{mod}(n,10)) = \{1,2,3\}$. If $x \in T_2$, then $x \in \{2,11\} = R(k,13) = T_4$; If $x \in T_3 \oplus T_2$, then $x = y + z$, $y \in T_4$, $z \in T_3$ and thus, $x \in T_3 \oplus T_4$. Therefore, $x \in T_4 \bigcup (T_3 \oplus T_4)$.

It follows from the arbitrariness of x that

$$R(k,n) \subseteq (T_0 \bigcup T_1 \bigcup (T_0 \oplus T_1)) \bigcap (T_3 \bigcup T_4 \bigcup (T_3 \oplus T_4)).$$

In summary, we have

$$R(k,n) = (T_0 \bigcup T_1 \bigcup (T_0 \oplus T_1)) \bigcap (T_3 \bigcup T_4 \bigcup (T_3 \oplus T_4)).$$

(3) $R(0,n) = \{1,2,\cdots,n\}$ is obvious. ■

According to Theorem 9, we can design a recursive constructing algorithm $RecurConst(k,n)$ for computing $R(k,n)$ as follows [4].

In the algorithm description above, the sub-algorithm $Direct(k,n)$ compute the set $R(k,n)$ directly by a pre-computed solution table.

Algorithm 3 RecurConst(k, n)

if $k = 0$ **or** $n < 14$ **then**
  return $Direct(k, n)$
end if
$T_0 \leftarrow Direct(k, \mathrm{mod}(n, 10))$
$T_1 \leftarrow RecurConst(k-1, \mathrm{div}(n, 10))$
$R \leftarrow T_0 \bigcup T_1 \bigcup (T_0 \oplus T_1)$
if $k = 0$ **or** $n < 14$ **then**
  $T_3 \leftarrow Direct(k, 10 + \mathrm{mod}(n, 10))$
  $T_4 \leftarrow RecurConst(k-1, \mathrm{div}(n, 10) - 1)$
  $R \leftarrow R \bigcap (T_3 \bigcup T_4 \bigcup (T_3 \oplus T_4))$
end if
return R

3. Finding the Maximal Elements

Definition 8.
$g(k,n) = \max\limits_{1 \le i \le n} \{|R(k,i)|\}$; $h(k,n)$ satisfying $g(k,n) = |R(k, h(k,n))|$[4].

Lemma 10. $g(0,n) = n$; $h(0,n) = n$.

Proof. It follows from $R(0,n) = \{1, 2, \cdots, n\}$. ■

Lemma 11. If $\mathrm{div}(n, 10^k) \le 1$ and $m \ge k$, then $R(k,n) = R(m,n)$.

Proof. It follows from $\mathrm{div}(n, 10^k) \le 1$ that for any $a \in A(k,n)$, we have $n = f(k,a) = \sum_{i=0}^{k} a_i^\top c_i$; $\mathrm{div}(n, 10^k) = \mathrm{div}(a_k^\top c_k, 10^k) = a_k^\top c_0 \le 1$, and thus, $A(k,1) = A(k,2) = 0$.

If $m \ge k$, then for any $a \in A(m,n)$, we have $n = f(m,a) = \sum_{i=0}^{m} a_i^\top c_i$; $\mathrm{div}(n, 10^k) = \mathrm{div}(a_k^\top c_k, 10^k) \le 1$, and thus, $a_i = 0$ for all $i > k$; $A(k,1) = A(k,2) = 0$.

Therefore, $R(k,n) = R(m,n)$. ■

Theorem 12. If $\mathrm{div}(n, 10^k) \le 1$, then

(1) If $n \ge 40$, then

$$g(k,n) = 6g(k, \mathrm{div}(n+1, 10) - 1) + 5 \tag{10}$$

(2) If $n > 3$, then

$$g(k,n) \le \frac{3}{2} g(k, n-1) + \frac{1}{2} \tag{11}$$

Proof. The theorem will be proved by mathematical induction. Formula (2) can be verified directly if $3 < n \le 40$. Induction hypothesis: For all $40 \le m < n$, we have $g(k,m) = 6g(k, \mathrm{div}(m+1, 10) - 1) + 5$; For all $3 < m < n$, we have $g(k,m) \le \frac{3}{2} g(k, m-1) + \frac{1}{2}$.

(1) We first prove Formula (1) by induction.

(1.1) The case of $\mathrm{mod}(n, 10) = 9$. In this case, $\mathrm{div}(n+1, 10) - 1 = \mathrm{div}(n, 10)$. Let $g(k, \mathrm{div}(n, 10)) = |R(k,m)|$. Then, for any $40 \le i \le n$, we have $|R(k,i)| \le 6\,|R(k-1, \mathrm{div}(i, 10)|+5$. It follows from $\mathrm{div}(n, 10^k) \le 1$ and $i < n$ that $\mathrm{div}(\mathrm{div}(i, 10), 10^{k-1}) = \mathrm{div}(i, 10^k) \le \mathrm{div}(n, 10^k) \le 1$.

From Lemma 11, we know $R(k-1, \mathrm{div}(i, 10) = R(k, \mathrm{div}(i, 10)$. Therefore,

$$|R(k,i)| \le 6\,|R(k-1, \mathrm{div}(i, 10)| + 5 \le 6g(k, \mathrm{div}(n, 10)) + 5$$

On the other hand, from $m \le \mathrm{div}(n, 10)$, we know $10m + 9 \le n$. Thus,

$$|R(k, 10m+9)| = 6\,|R(k-1, m)| + 5 = 6g(k, \mathrm{div}(n, 10)) + 5$$

Therefore, $g(k,n) = 6g(k, \mathrm{div}(n, 10)) + 5 = 6g(k, \mathrm{div}(n+1, 10) - 1) + 5$.

(1.2) The case of $\mathrm{mod}(n,10) \neq 9$. In this case, $\mathrm{div}(n+1,10)-1 = \mathrm{div}(n,10)-1$. For any $10\mathrm{div}(n,10) \leq i \leq n$, we have $|R(k,i)| \leq 4\,|R(k-1,\mathrm{div}(i,10)| + 3 = 4\,|R(k,\mathrm{div}(i,10)| + 3 \leq 4g(k,\mathrm{div}(n,10)+3$.

It follows from $n \geq 40$ that $\mathrm{div}(n,10) \geq 4 > 3$. By induction hypothesis, $g(k,\mathrm{div}(n,10)) \leq \frac{3}{2}g(k,\mathrm{div}(n,10)-1)+\frac{1}{2}$. It follows that

$$|R(k,i)| \leq 4\left(\tfrac{3}{2}g(k,\mathrm{div}(n,10)-1)+\tfrac{1}{2}\right)+3 = 6g(k,div(n,10)-1)+5\ .$$

For any $1 \leq i < 10\mathrm{div}(n,10)$, let $g(k,\mathrm{div}(n,10)-1) = |R(k,m)|$, then $|R(k,i)| \leq 6\,|R(k,\mathrm{div}(i,10))|+5$. In this time, we have $\mathrm{div}(i,10) \leq \mathrm{div}(n,10)-1$. Thus, $|R(k,i)| \leq 6g(k,\mathrm{div}(n,10)-1)+5$. On the other hand, from $m \leq \mathrm{div}(n,10)-1$, we know $10m+9 \leq n$. Thus, $|R(k,10m+9)| = 6\,|R(k-1,m)|+5 = 6g(k,\mathrm{div}(n,10))+5$. $g(k,n) = 6g(k,\mathrm{div}(n,10)-1)+5 = 6g(k,\mathrm{div}(n+1,10)-1)+5$. Therefore, Formula (1) is held by induction.

(2) We now prove Formula (2) by induction.

From Formula (1), we know

$g(k,n) = 6g(k,\mathrm{div}(n+1,10)-1)+5$; $g(k,n-1) = 6g(k,\mathrm{div}(n,10)-1)+5$.

(2.1) The case of $\mathrm{mod}(n,10)=9$. In this case, $\mathrm{div}(n+1,10)-1 = \mathrm{div}(n,10)$. By induction hypothesis, $g(k,\mathrm{div}(n,10)) \leq \frac{3}{2}g(k,\mathrm{div}(n,10)-1)+\frac{1}{2}$. It follows that

$$\begin{aligned} g(k,n) &\leq 6\left(\tfrac{3}{2}g(k,\mathrm{div}(n,10)-1)+\tfrac{1}{2}\right)+5 = 9g(k,\mathrm{div}(n,10)-1)+8 \\ &= \tfrac{3}{2}\left(6g(k,\mathrm{div}(n,10)-1)+5\right)+\tfrac{1}{2} = \tfrac{3}{2}g(k,n-1)+\tfrac{1}{2} \end{aligned}$$

(2.2) The case of $\mathrm{mod}(n,10) \neq 9$. In this case, $\mathrm{div}(n+1,10)-1 = \mathrm{div}(n,10)-1$. From Formula (1), we know $g(k,n) = g(k,n-1) \leq \frac{3}{2}g(k,n-1)+\frac{1}{2}$. Therefore, Formula (2) is held by induction.■

Theorem 13. Suppose $m,n \in I^+,\ n=\sum_{i=0}^{m} a_i 10^i$, and

$$p = \begin{cases} n-1 & m=0 \\ \mathrm{div}(n+1,10^{m-1})-1 & 1 \leq m \leq k, a_k \leq 1 \\ 100 & k<m \quad \text{or} \quad k=m, a_k>1 \end{cases}$$. Then,

$$h(k,n) = \begin{cases} 1 & 0 \leq p \leq 1 \\ 3 & 2 \leq p \leq 7 \\ 9 & p=8 \\ 10^m-1 & 9 \leq p \leq 16 \\ 18\times 10^{m-1}-1 & 17 \leq p \leq 18 \\ 2\times 10^m-1 & 19 \leq p \leq 36 \\ 38\times 10^{m-1}-1 & 37 \leq p \leq 38 \\ 4\times 10^m-1 & 39 \leq p \leq 98 \\ n & p=99 \\ \mathrm{div}(n+1,10^k)\times 10^k-1 & p=100 \end{cases} \tag{12}$$

$$g(k,n) = \begin{cases} 1 & 0 \leq p \leq 1 \\ 3 & 2 \leq p \leq 7 \\ 5 & p=8 \\ 6^m-1 & 9 \leq p \leq 16 \\ 8\times 6^{m-1}-1 & 17 \leq p \leq 18 \\ 2\times 6^m-1 & 19 \leq p \leq 36 \\ 16\times 6^{m-1}-1 & 37 \leq p \leq 38 \\ 4\times 6^m-1 & 39 \leq p \leq 98 \\ 6^m-1 & p=99 \\ \mathrm{div}(n+1,10^k)\times 6^k-1 & p=100 \end{cases} \tag{13}$$

Proof.

If $m \leq 1$, we can compute the values of $h(k,n)$ and $g(k,n)$ directly as shown in Table 2.

(1) The case of $m=0$ corresponds to $1 \leq n \leq 9$, $0 \leq p \leq 8$, and can thus be computed directly from Table 2. If $1 \leq m \leq k$ and $a_k \leq 1$, then from Theorem 12, we know that if $n \geq 40$,

Table 2. values of $h(k,n)$ and $g(k,n)$

n	1-2	3-8	9-16	17-18	19-36	37-38	39-98	99
$h(k,n)$	1	3	9	17	19	37	39	99
$g(k,n)$	1	3	5	7	11	15	23	35

then $h(k,n) = 10h(k, \mathrm{div}(n+1, 10) - 1) + 9; g(k,n) = 6g(k, \mathrm{div}(n+1, 10) - 1) + 5$. In this way, we can compute recursively that

$$h(k,n) = 10^{m-1}h(k, \mathrm{div}(n+1, 10^{m-1}) - 1) + 9\sum_{i=0}^{m-2} 10^i = 10^{m-1}h(k, \mathrm{div}(n+1, 10^{m-1}) - 1) + 10^{m-1} - 1$$
$$= 10^{m-1}\big(h(k, \mathrm{div}(n+1, 10^{m-1}) - 1) + 1\big) - 1;$$
$$g(k,n) = 6^{m-1}g(k, \mathrm{div}(n+1, 10^{m-1}) - 1) + 5\sum_{i=0}^{m-2} 6^i = 6^{m-1}g(k, \mathrm{div}(n+1, 10^{m-1}) - 1) + 6^{m-1} - 1$$
$$= 6^{m-1}\big(g(k, \mathrm{div}(n+1, 10^{m-1}) - 1) + 1\big) - 1.$$

Now, we have $9 \leq p = \mathrm{div}(n+1, 10^{m-1}) - 1 \leq 99$. The values of $h(k,n)$ and $g(k,n)$ can now be computed directly from Table 2. By substituting the values into the above formula, we get the results.

(2) If $m < k$ or $m = k$ and $a_k > 1$, then from the recursive formula of $h(k,n)$ and $g(k,n)$, we know $h(k,n) = 10^k h(0, \mathrm{div}(n+1, 10^k)-1) + 9\sum_{i=0}^{k-1} 10^i$, $h(0, \mathrm{div}(n+1, 10^k)-1) = \mathrm{div}(n+1, 10^k)-1$.

It follows from Lemma 10 that $h(0, \mathrm{div}(n+1, 10^k) - 1) = \mathrm{div}(n+1, 10^k) - 1$, $g(0, \mathrm{div}(n+1, 10^k) - 1) = \mathrm{div}(n+1, 10^k) - 1$.

By substituting them into the above formula we get

$$h(k,n) = 10^k(\mathrm{div}(n+1, 10^k) - 1) + 9\sum_{i=0}^{k-1} 10^i = 10^k(\mathrm{div}(n+1, 10^k) - 1) + 10^k - 1$$
$$= \mathrm{div}(n+1, 10^k) \times 10^k - 1;$$
$$g(k,n) = 6^k(\mathrm{div}(n+1, 10^k) - 1) + 5\sum_{i=0}^{k-1} 6^i = 6^k(\mathrm{div}(n+1, 10^k) - 1) + 6^k - 1 = \mathrm{div}(n+1, 10^k)6^k - 1$$

The proof is completed. ■

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References

[1] R. Bird. Pearls of Functional Algorithm Design. Cambridge University Press. 2010:258-274.
[2] Xiaodong Wang. Generation and Enumeration of Implication Sets. *Advances in Information Technology and Education Communications in Computer and Information Science*. 2011; 201: 87-92.
[3] TH Cormen, CE Leiserson, RL Rivest. Introduction to Algorithms. MIT Press, Cambridge, MA, 2001: 429-433.
[4] DL Kreher and D Stinson. Combinatorial Algorithms: Generation, Enumeration and Search, CRC Press, 1998: 125-133.
[5] D Zhu, X Wang. A Practical Algorithm and Data Structures for Range Selection Queries. *TELKOMNIKA Indonesian Journal of Electrical Engineering*. 2014; 12(3): 2406-2413.