Enhancing stochastic optimization: investigating fixed points of chaotic maps for global optimization

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ABSTRACT

Chaotic maps, despite their deterministic nature, can introduce controlled randomness into optimization algorithms. This chaotic map behaviour helps overcome the lack of mathematical validation in traditional stochastic methods. The chaotic optimization algorithm (COA) uses chaotic maps that help it achieve faster convergence and escape local optima. The effective use of these maps to find the global optimum would be possible only with a complete understanding of them, especially their fixed points. In chaotic maps, fixed points repeat indefinitely, disrupting the map's characteristic unpredictability. While using chaotic maps for global optimization, it is crucial to avoid starting the search at fixed points and implement corrective measures if they arise in between the sequence. This paper outlines strategies for addressing fixed points and provides a numerical evaluation (using Newton's method) of the fixed points for 20 widely used chaotic maps. By appropriately handling fixed points, researchers and practitioners across diverse fields can avoid costly failures, improve accuracy, and enhance the reliability of their systems.

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1. INTRODUCTION

In the last two decades, tremendous usage of chaotic maps was showcased in literature. The dramatic growth in e-commerce, sensitive media transfers, and online banking industries paved the way for developing secure data transfer measures. It was found that chaotic hash functions help prevent data hacking by unauthorized parties [1], [2]. Chaotic maps play a crucial role in generating pseudo-random numbers for encryption algorithms. Pseudo-random number generators [3]-[6] find applications across various fields, including e-health applications like medical imaging systems. Chaotic maps also help in stochastic simulations, improved single/multi-objective evolutionary algorithms, lossy image compression, encryption of digital signatures, hashing seed vectors, one-time passwords (OTPs), and video-game animations. Chaotic maps have been used in data science to protect big data and solve feature selection problems by selecting the most relevant features and reducing redundant and irrelevant ones. Optimization of neural network parameters and training efficiency can be improved by applying chaos optimization algorithms to the recently more popular deep learning algorithms. Chaotic time series approaches are used to forecast energy and power [7]. Image segmentation based on computer vision is improved by using chaotic maps [8], [9]. Our work deals with pseudo-random number generation using chaotic maps, especially for applications in global optimization.

Generally, optimization algorithms can be characterized into deterministic [10] and stochastic [11], [12] algorithms. Stochastic optimization algorithms involve heuristic and evolutionary methods based on the random number generation of feasible points. While effective in finding global solutions, these algorithms often need more mathematical validation due to their reliance on random number generation. However, this drawback can be addressed by replacing traditional random number generators with chaotic maps. Chaotic maps, characterized by deterministic yet unpredictable behaviour, can introduce controlled randomness (regularity, pseudorandomness) into the optimization process, offering a path towards mathematical validation. Chaotic numbers generated by these chaotic maps have properties like pseudo-randomness, sensitivity, regularity and ergodicity that strengthen the chaos optimization algorithm (COA). The ergodicity of the chaotic numbers helps the COAs to explore the entire search space over time. The sensitive nature of chaotic numbers is due to their sensitivity to the initial conditions that contribute to their unpredictability. Chaotic maps, used to generate chaotic numbers, often occur in studying discrete dynamical systems [13] and also help in giving mathematical strength to global optimization algorithms. In evolutionary algorithms, chaotic maps can be incorporated by replacing random numbers with chaotic numbers. For example, Alatas [14] applied a chaotic search to enhance the artificial bee colony (ABC) algorithm. Rani *et al.* [15] introduced a chaotic golden section algorithm, a chaotic pattern search algorithm is introduced in [16], and [17] introduces a PSO algorithm with chaos. An ant colony algorithm is combined with chaotic sequences in [18] to improve it's efficiency. Chaotic maps have been integrated into SMA [19] for faster convergence and higher accuracy. Inspired by chaos theory, chaos game optimization (CGO) [20] is developed, which is a novel method for tackling optimization problems. [21] investigates the use of modified chaotic adaptive invasive weed optimization to design high-directivity planar antenna arrays, finding a critical "chaos factor" for balancing convergence and exploration in the optimization process [22] reveals that chaos-based control of exploration/exploitation rates in optimization algorithms (like GWO, ALO, and MFO) outperform systematic control for machine learning feature selection, leading to improved performance.

The following challenges were identified in the literature regarding the use of chaotic maps:
−   Different chaotic maps exhibit varying properties, and selecting the most suitable one for a specific problem can be crucial for optimal performance.
−   The effectiveness of chaotic maps often depends on carefully adjusting their internal parameters to achieve the desired balance between exploration and exploitation.
−   A deeper theoretical understanding of chaotic maps concerning their lyapunov exponent (LE) and their fixed points (both of which effects the chaotic behaviour of the map) is missing in the literature, which may lead to a failure of the chaotic algorithm developed. This is the major drawback that is addressed in this paper.

A chaotic map's performance can be analyzed using its LE, probability distribution, histogram, and fixed point [16], [23]. A chaotic map may or may not have a fixed point. If a chaotic map has fixed points, it may affect the performance of the algorithm in which the map is used, mainly when it is used for solving optimization problems. All the chaotic global optimization algorithms start with a random number as an initial point. If this random number is a fixed point of the incorporated chaotic map, then the chaotic map cannot exhibit chaos, and hence, the algorithm fails to get the global optimum. Hence, care must be taken during the experiment to see that the initial point is not a fixed point.

The contributions of our research are listed below:
−   How to select the internal parameters and provide the information on parameters of each map.
−   Listing the fixed points of 20 popular chaotic maps.
−   Suggesting ways to avoid the fixed points while taking the start point of the algorithm.

The paper is structured in the following manner. Section 2 provides a study of chaotic maps and fixed points. In the method section 3, fixed points are evaluated for 20 chaotic maps (numerically and graphically) that are popular for global optimization. Section 4 gives a brief discussion of the results, and finally, the paper is concluded in section 5.

2. PRELIMINARIES
2.1. Chaotic maps
One dimensional chaotic map is a simpler model of chaos. For the function f, general one-dimensional chaotic map [15], [16], [23]-[29] is given by $x_{n+1} = f(x_n)$. With the initial guess x_0, the sequence $x_0, x_1, x_2, ...$ is obtained. There are several popular chaotic maps, such as logistic map, sinusoidal map, sine map, neuron map, tent map, Chebyshev map, and circle map. The best value for the parameters in these maps can be evaluated by the understanding of the LE [16], [23] and scatter diagrams. LEs are vital tools for understanding the complex world of dynamical systems. They measure the average rate at which nearby points in a system's trajectory tend to stretch (diverge) or shrink (converge) over time. This sensitivity

to initial conditions is a key aspect in determining the system's predictability and chaotic nature. Positive LEs signify exponential divergence, leading to unpredictable and chaotic behaviour, while negative values indicate stable and predictable dynamics. The larger the value of LE, the larger the chaotic behaviour of the map, thus giving a good spread of the chaotic numbers over the region. To demonstrate the same, the example of logistic map, $x_{n+1} = \mu x_n(1 - x_n)$ is taken.

Figure 1 gives the LE and the scatter diagrams for various parameters for logistic map. Figure 1(a) has the parameter μ on X axis and the LE with that parameter on the Y axis. It can be seen from this figure that the LE is positive at many points when $\mu > 3.5$ and is the highest at $\mu=4$. Figure 1(b) is the scatter plot of 1,000 chaotic points generated for the logistic map: $x_{n+1} = 3.5x_n(1 - x_n)$, i.e., with parameter $\mu = 3.5$. It can be seen from Figure 1(a) that the LE for logistic map is negative at $\mu = 3.5$ and this is the reason why the distribution of points is not proper in Figure 1(b), thus explaining how parameters with negative LE can damage the chaos. Figures 1(c) and 1(d) shows the distribution of 1,000 chaotic points generated by the logistic map with $\mu = 3.8$ and $\mu = 4$, respectively. It can be seen from the LE graph, Figure 1(a)) that at both $\mu = 3.8$ and $\mu = 4$, the LE value is positive, but the distribution of the chaotic points is better in Figure 1(d) than in Figure 1(c). Hence, better is the chaos with a larger μ.
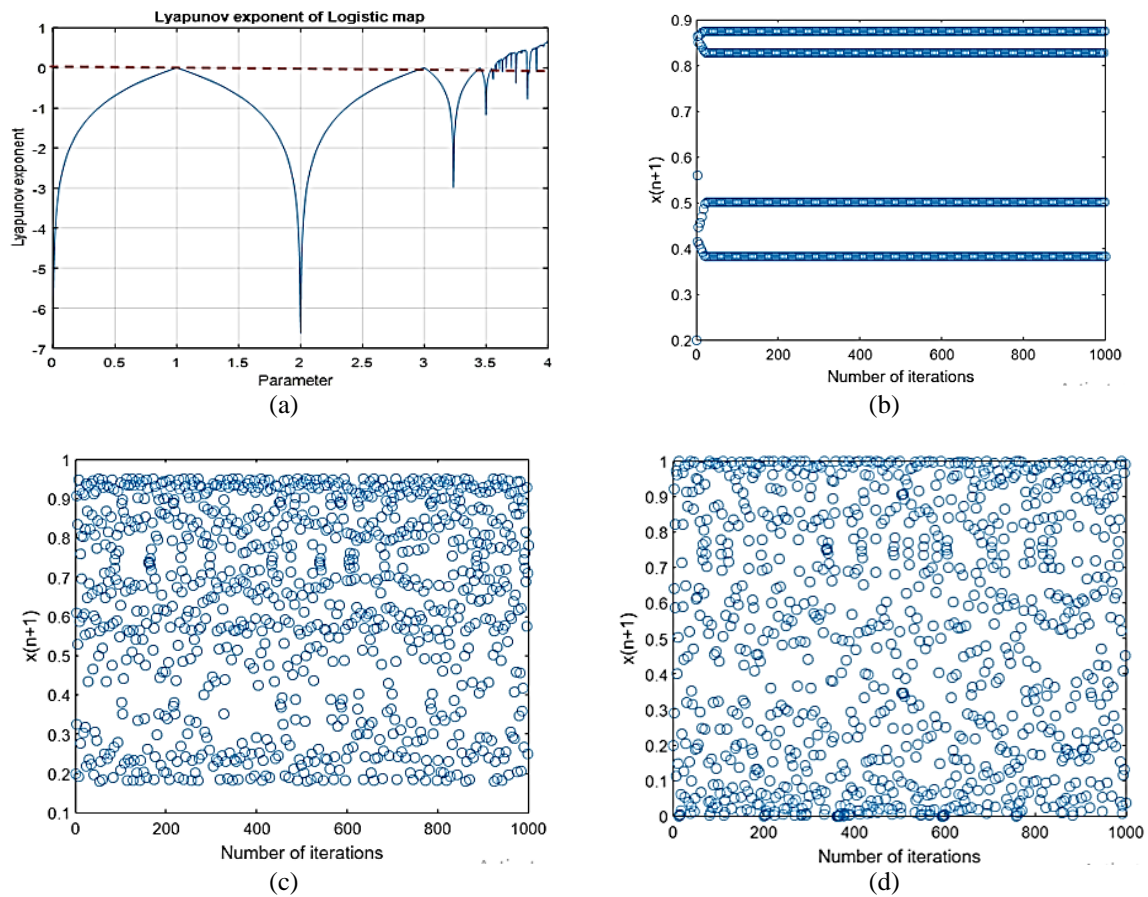

Figure 1. Chaotic nature of logistic map: $x_{n+1} = \mu x_n(1 - x_n)$ with (a) lyapunov exponent graph and scatter diagrams with (b) $\mu = 3.5$, (c) $\mu = 3.8$, and (d) $\mu = 4$

2.2. Fixed points

The fixed point [24] x^* of a map $x_{n+1} = f(x_n)$ is a point at which $x^* = f(x^*)$. For example, for the chaotic map given by $x_{n+1} = 4x_n(1 - x_n)$, $x^* = 0.75$ is a fixed point. With, $x_0 = 0.75$, the sequence $x_1 = x_2 = x_3 = \cdots = 0.75$. Hence a chaotic sequence is not getting generated with $x_0 = 0.75$, and so a COA with this map will fail.

For the previous example, when $x_0 = 0.25$ is chosen, then also $x_1 = x_2 = x_3 = \cdots = 0.75$, that is the fixed point appears later in the sequence. This scenario also affects the chaos and leads to the failure of the COA. The appearance of the fixed point in the sequence of chaotic numbers at any time in the algorithm will

alter the chaotic behaviour of the algorithm, and it will fail to give the optimum solution. Thus, knowing fixed points and how to tackle them efficiently in the algorithm will help prevent such failures. Whenever a chaotic map is used, care must be taken to ensure that the initial point in the sequence is not a fixed point. In case, if a fixed point x^* appears in the chaotic sequence that is generated for global search for optimum in a COA, then it must be replaced by $x^* + \varepsilon$, where ε can be any small number (that can be selected based on the map).

Graphically, the point of intersection of the line $y = x$ and the curve $y = f(x)$ gives the fixed points. Numerically, the fixed points can be evaluated by finding the root of the equation, $x = f(x)$. The Newton's method has been applied [30] to find the root of $x = f(x)$ to obtain the fixed points. The following section evaluates fixed points of 20 popular chaotic maps.

A chaotic map may also have eventually fixed points x_e^*. These are points that will, in the long run, generate a fixed point in the sequence. For example, x = 0.25 is an eventually fixed point for the example taken above. Eventually fixed points can be analytically found by solving the equation $f(x) = x^*$, where x^* is the fixed point. Also, all solutions of $f(x) = x_e^*$ will be eventually fixed points.

3. METHOD
3.1. Parameter values of chaotic maps
Table 1 provides a list of 20 chaotic maps including their equations, best parameter values found through the study of lyapunov exponents (LE), and their applications in various fields such as physics, biology, and cryptography. This comprehensive overview aims to assist researchers in understanding and utilizing these maps effectively.

3.2. Logistic map
Invented by the biologist Robert May in 1976, the map stands analogous to Pierre Francois Verhulst's logistic equation, representing discrete-time demographic changes. The logistic map is commonly used in [24], [31]-[35], [36] cryptography, health care, and COAs. A logistic map is the most straightforward quadratic non-linear map.

$$x_{n+1} = \mu x_n (1 - x_n), x_n \epsilon (0,1) \text{ here, } \mu \in [0,4]$$

The optimal parameter value for the logistic map is μ =4, where the fixed points for $x_{n+1} = 4x_n(1 - x_n)$ are determined to be 0 and 0.75, as depicted in Figure 2. It has to be ensured that these two numbers do not appear in the sequence of chaotic numbers, that are generated during the search for a global minimum. A logistic map with μ=4 has eventually fixed points too, that means there are few points that will, in the long run, generate a fixed point in the sequence. Two eventually fixed points of logistic map are 1 and 0.25. These were obtained by solving, $4x(1 - x) = 0$ and $4x(1 - x) = 0.75$. By solving $4x(1 - x) = 1$, (1 which is an eventually fixed point), $x = 0.5$ is obtained, which is also an eventually fixed point of $x_{n+1} = 4x_n(1 - x_n)$. This manner by solving equations, $4x(1 - x) = x_e^*$, for all eventually fixed points, x_e^*, many other eventually fixed points for the map can be found. But as it is a tedious process to obtain all eventually fixed points, it is advised to address the fixed points whenever it appears in the chaotic sequence. This is explained in detail in the next section of this paper.
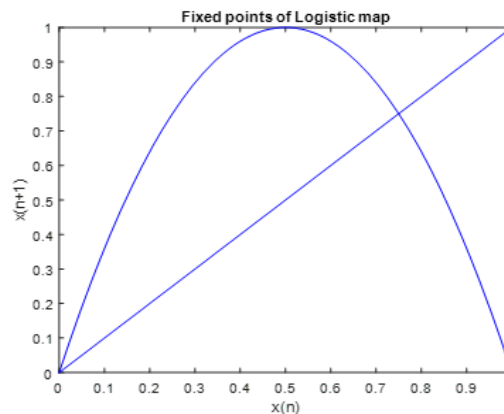

Figure 2. Fixed points of the logistic map with μ=4

Table 1. List of chaotic maps along with their best parameter value

Chaotic map	Governing equation	Parameter value
Logistic	$x_{n+1} = \mu x_n (1 - x_n), x_n \epsilon (0,1)$	$\mu = 4$
Sinusoidal	$x_{n+1} = \alpha x_n^2 \sin(\pi x_n), x_n \epsilon (0,1)$	$\alpha = 2.3$
Sine	$x_{n+1} = \alpha \sin(\pi x_n), x_n \epsilon (0,1)$	$\alpha = 0.95$
Neuron	$x_{n+1} = \alpha - 2 \tanh\tanh(\beta) e^{-3x_n^2}, \quad x_n \epsilon (-1.5, 0.5)$	$\alpha = 0.5$
Tent	$x_{n+1} = \begin{cases} \mu x_n & 0 \le x_n < 0.5 \\ \mu(1 - x_n) & 0.5 \le x_n \le 1 \end{cases}$	$\mu = 2$
Chebyshev	$x_{n+1} = \cos(\alpha \cos^{-1}(x_n)), x_n \epsilon (-1,1)$	$\alpha = 8$
Circle	$x_{n+1} = x_n + a - \frac{b}{2\pi}\big(\sin(2\pi x_n)\big) \bmod 1, \quad x_n \epsilon (0,1)$	$a \epsilon (0,1), b \epsilon (0,4\pi)$
Cubic	$x_{n+1} = \beta x_n (1 - x_n^2), x_n \epsilon (0,1)$	$\beta = 2.59$
ICMIC	$x_{n+1} = \sin\left(\frac{a}{x_n}\right), a \in (0, \infty), x_n \in (-1,1)$	$a = 3$
Bernoulli shift	$x_{n+1} = \begin{cases} bx_n - a & x_n \ge 0 \\ bx_n + a & x_n < 0 \end{cases} \quad x_n \in [-1,1]$	$a = 1, b = 1.4$
Liebovitch	$x_{n+1} = \begin{cases} \alpha x_n & 0 < x_n \le d_1 \\ \frac{d_2 - x_n}{d_2 - d_1} & d_1 < x_n \le d_2 \quad x_n \in (0,1) \\ 1 - \beta(1 - x_n) & d_2 < x_n < 1 \end{cases}$ $\&\ d_1, d_2 \in (0,1)$	$d_1 = 0.5, d_2 = 0.6,$ $\alpha = 1.08, \beta = 1.125$
Intermittency	$x_{n+1} = \begin{cases} \epsilon + x_n + c\, x_n^m & 0 < x_n \le d \\ \frac{x_n - d}{1 - d} & d < x_n < 1 \end{cases} \quad x_n, d \in (0,1)$	$m = 2, d = 0.5,$ $\epsilon = 0.49$
Piecewise	$= \begin{cases} \frac{x_n}{d}, & 0 \le x_n < d \\ \frac{x_n - d}{0.5 - d}, & d \le x_n < \frac{1}{2} \\ \frac{1 - d - x_n}{0.5 - d}, & \frac{1}{2} \le x_n < 1 - d \\ \frac{1 - x_n}{d}, & 1 - d \le x_n < 1 \end{cases} \quad x_n, d \in (0,1)$	$d = 0.3$
Singer	$x_{n+1} = \mu(7.86x_n - 23.31x_n^2 + 28.75\, x_n^3 - 13.3\, x_n^4) \quad , x_n \epsilon (0,1)$	$\mu = 1.073$
Kent	$x_{n+1} = \begin{cases} \frac{x_n}{m} & 0 < x_n \le m \\ \frac{(1 - x_n)}{1 - m} & m < x_n \le 1 \end{cases} \quad x_n \in (0,1)$	$m = 0.8$
Iterative	$x_{n+1} = \sin\left(\frac{a\pi}{x_n}\right), x_n \in (0,1)$	$a = 0.7$
1-D Sine powered chaotic	$x_{n+1} = (x_n(\alpha + 1))^{\sin(\beta\pi + x_n)}$	$\alpha = 4.4926, \beta = 0.3306$
Sine-circle	$x_{n+1} = x_n + \alpha - \frac{\mu}{2\pi} \sin(2\pi x_n) \bmod 1$	$\alpha = 0.606661,$ $\mu = 2.1$
Sinus	$x_{n+1} = 2.3(x_n)^{2\sin(\pi x_n)}\ x_n \in (0,1)$	
Dyadic or saw tooth	$x_{n+1} = 2x_n \bmod 1$ Or $x_{n+1} = \begin{cases} 2x_n & 0 \le x_n < \frac{1}{2} \\ 2x_n - 1 & \frac{1}{2} \le x_n \le 1 \end{cases} \quad x_n \in (0,1)$	

3.3. Sinusoidal map

The sinusoidal map [18], [29] that generates the chaotic series in the interval $x_n \epsilon (0,1)$ is given by:

$$x_{n+1} = \alpha x_n^2 \sin(\pi x_n)$$

The sinusoidal map's optimal parameter, determined by LE, is $\alpha = 2.3$. At this value, the map's fixed points are found to be 0, 0.4421, and 0.8228. Figure 3 illustrates these fixed points.

3.4. Sine map

A sine map is a chaotic map [22], [26], [34], and [37] with a sequence generated between 0 and 1 is given by:

$$x_{n+1} = \alpha \sin(\pi x_n), x_n \epsilon (0,1)$$

Here $\alpha \in [0, 1]$ is the control parameter, but it was found that the sine map exhibits chaotic conduct when $\epsilon[0.87,1]$. This map is used mainly in image encryption and optimization. With calculations based on LE, $\alpha = 0.95$ can be selected as the parameter in this for which the fixed points of the sine map are 0 and 0.7241 as can be seen in Figure 4.
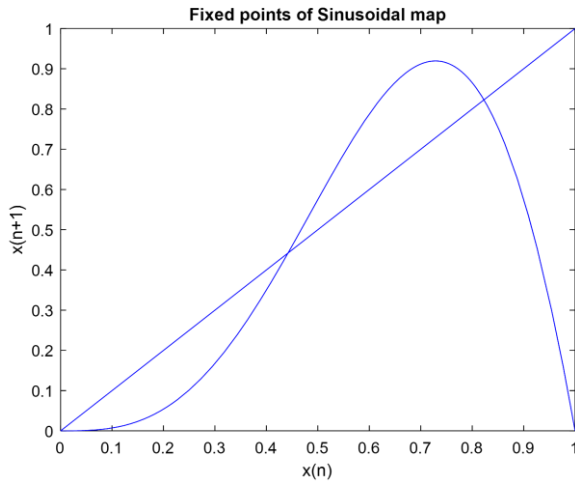
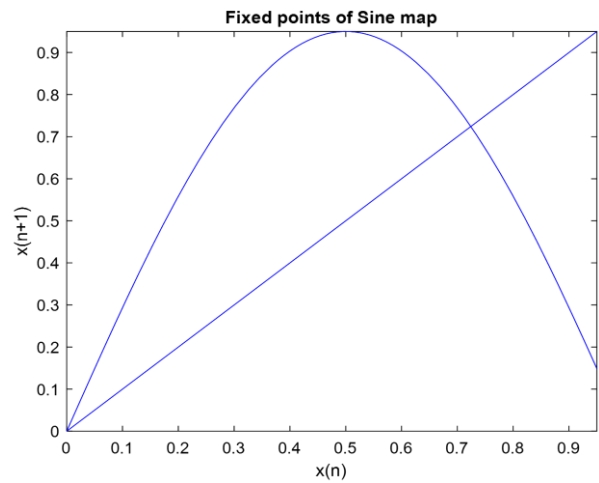Figure 3. Fixed points of the Sinusoidal map with α =2.3

Figure 4. Fixed points of the Sine map with α =0.95

3.5. Neuron map

A neuron map [23], [25] and [33] is a two-parameter map defined as:

$$x_{n+1} = \alpha - 2\tanh(\beta)\, e^{-3x_n^2}$$

This map is becoming popular lately due to its success in identifying the global optimum [16], [23], and [25]. Compared with many other chaotic maps, Neuron map has only one fixed point, which is one of the reasons for this success.

It was found that the Neuron map shows chaotic behaviour when $\beta = 5$ and $\alpha = 0.5, 0.8,$ and 0.9. For $\alpha = 0.5$ and $\beta = 5$ the sequence generated for neuron map is in the range $(-1.5, 0.5)$ [23] and the fixed point is -0.4856 as can be seen in Figure 5. For α=0.8 and β=5 neuron map generates a sequence in the range (-1.2,0.8) [23] and the fixed point in this case is -0.4094 as can be seen in Figure 6. For α=0.9 and β=5 the sequence generated for the Neuron map is in the range (-1.1,0.9) [18] For these parameters, the fixed point is found to be -0.3842 as can be seen in Figure 7.
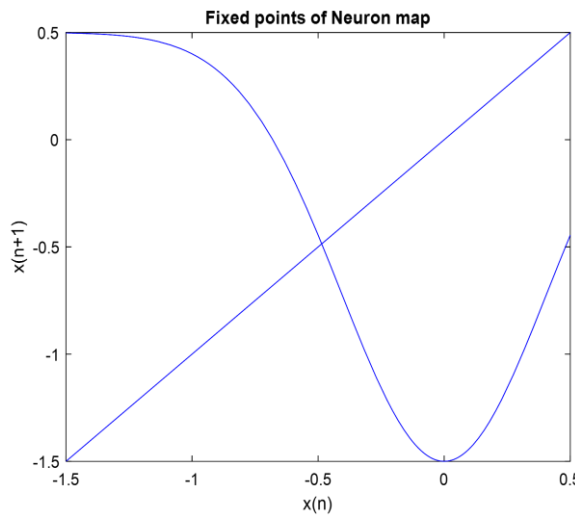
Figure 5. Fixed point of the neuron map with $\alpha = 0.5$ and $\beta = 5$
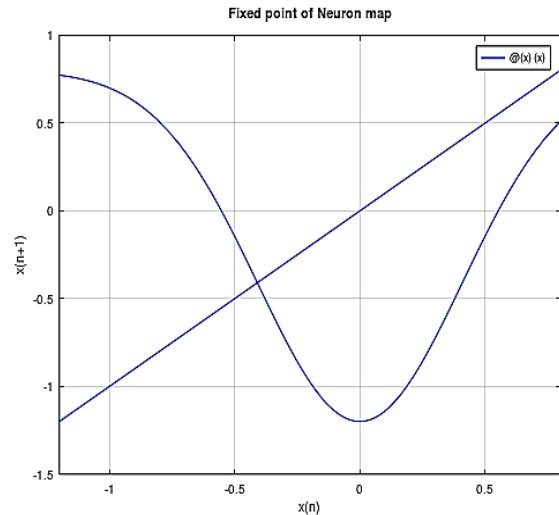
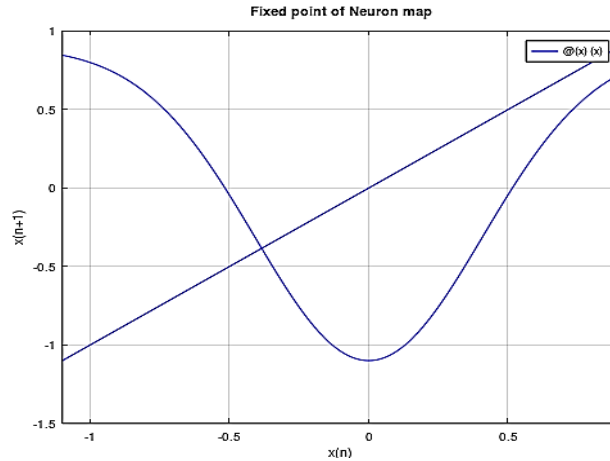Figure 6. Fixed point of the neuron map with $\alpha = 0.8 \; and \; \beta = 5$

Figure 7. Fixed point of the neuron map with $\alpha = 0.9 \ and \ \beta = 5$

3.6. Tent map

The tent map [23], [34] that produces chaotic sequences in (0,1) is defined as follows:

$$x_{n+1} = \begin{cases} \mu x_n & 0 \leq x_n < 0.5 \\ \mu(1 - x_n) & 0.5 \leq x_n \leq 1 \end{cases}$$

where, $\mu = 2$ is found to be the apt value for the parameter, with which the fixed points of the tent map are 0 and 0.6667 as can be seen in Figure 8. This map has four eventually fixed points that are 0.25, 0.5, 0.7, and 0.75. Hence, care must be taken to avoid these points as initial points and to replace a fixed points $x^* = 0$ or 0.6667 if it appears in the sequence by $x^* + \varepsilon$, where $0 < \varepsilon \leq 0.09$.
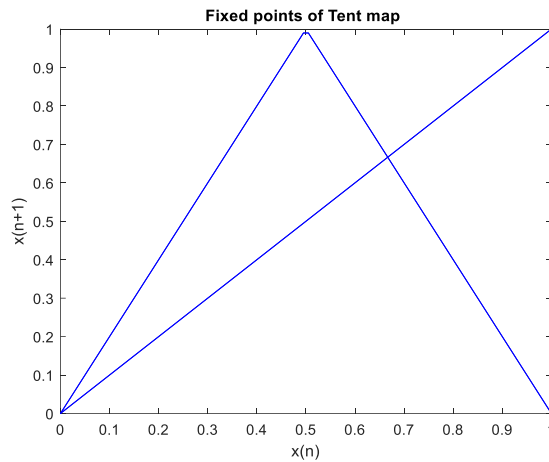
Figure 8. Fixed points of the tent map $\mu = 2$

3.7. Chebyshev map

Generally, Chebyshev chaotic map is used to deal with security issues, digital communication, and neural networks [16], [23], [34], and [38]. The definition of this map is as follows:

$$x_{n+1} = cos\big(\alpha \ cos^{-1}(x_n)\big), x_n \epsilon(-1,1)$$

on the basis of the LE, the apt value for α is 8. There are eight fixed points for Chebyshev map with $\alpha = 8$, which are -0.9397, -0.9010, -0.5, -0.2225, 0.1736, 0.6235, 0.7660 and 1 as can be seen in Figure 9. Notably, 0.5 emerges as an eventually fixed point of the Chebyshev map.
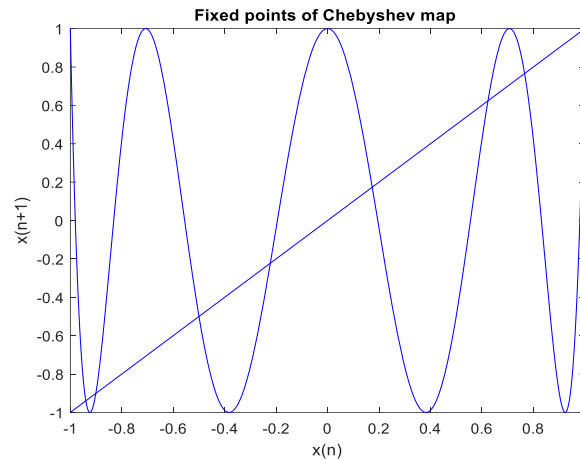
Figure 9. Fixed points of the chebyshev map with $\alpha = 8$

3.8. Circle map

Andrey Kolmogorov proposed a circle map to simplify mechanical motor rotation models. In electronics, phase-locked loops can also be described with circle maps [23], [34], and [38]. The circle map is defined as:

$$x_{n+1} = x_n + a - \frac{b}{2\pi}\big(sin(2\pi x_n)\big)\, mod \tag{1}$$

The apt values of the parameters are found to be $a = 0.2$ and $b = 0.5$. The circle map creates sequences in the interval (0, 1) and has a single fixed point at 0.7217 as can be seen in Figure 10.
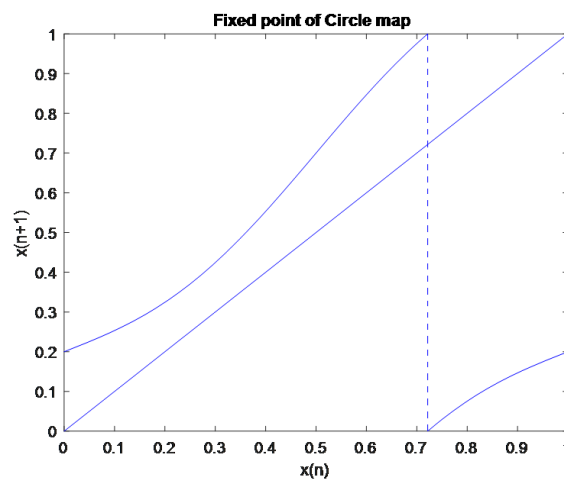
Figure 10. Fixed points of the circle map with $a = 0.2$ and $b = 0.5$

3.9. Cubic map

In several applications, like cryptography, cubic maps are commonly used to create chaotic sequences. The map is defined as:

$$x_{n+1} = \beta x_n(1 - x_n{}^2)$$

using a cubic map, chaotic sequences can be created in the interval (0, 1) with β=2.59 value [23], [39]. The fixed points of the cubic map are 0 and 0.7835 as can be seen in Figure 11.
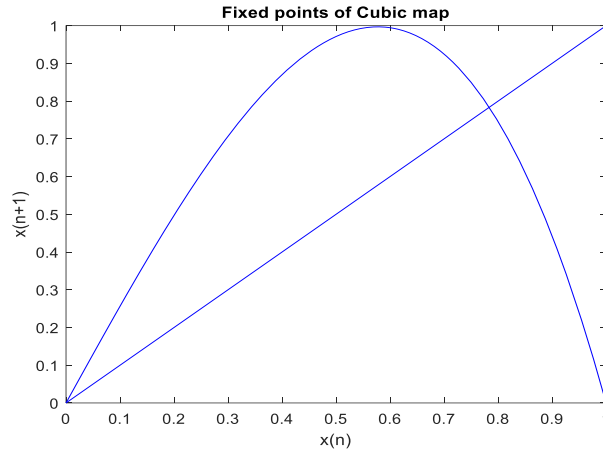
Figure 11. Fixed points of the cubic map with $a = 0.2$ and $b = 0.5$

3.10. ICMIC map

The chaotic map [23] with unlimited collapse named as iterative chaotic map with infinite collapses (ICMIC) is defined as:

$$x_{n+1} = sin\left(\frac{a}{x_n}\right), a \in (0, \infty)$$

and generates chaotic sequences of $(-1,1)$ with $a = 3$. The fixed points of this map are $\pm 0.0952, \pm 0.1065, \pm 0.1188, \pm 0.1373, \pm 0.1578, \pm 0.1934, \pm 0.2343, \pm 0.3301, \pm 0.4448$ and infinitely many fixed points in the interval $(-0.1, 0.1)$ as can be seen in Figure 12. Even with this many fixed points, the ICMIC map is very popular in optimization as well as image encryption applications. Figure 12 gives the ICMIC map along with the line $y = x$.
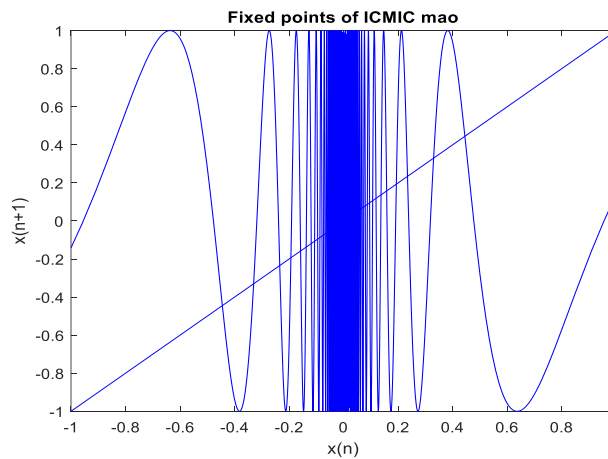

Figure 12. Fixed point of the ICMIC map with $a=3$

3.11. Bernoulli shift map

Bernoulli shift [23] map is a piece-wise linear map that is composed of piece-wise linear segments. In this map, two linear segments are shown in the following manner:

$$x_{n+1} = \begin{cases} bx_n - a & x_n \geq 0 \\ bx_n + a & x_n < 0 \end{cases}$$

For $a = 1$, $b = [1.4, 2)$, the sequence is generated in $x_n \in [-1, 1]$. The benefit of this map is that it does not have any fixed points in [-1, 1]. The linear segments along with the line $y = x$ are depicted in Figure 13.
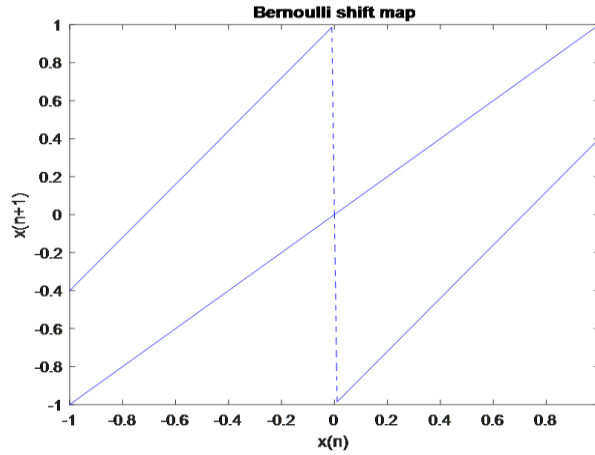
Figure 13. Bernoulli shift map (two parallel segments) and line $y = x$

3.12. Liebovitch map

This is one more illustration of a piece-wise linear map, [23], [34]. Liebovitch and Toth proposed the Liebovitch map and it shows three non-overlapping piece-wise linear pieces between (0, 1) and is defined by:

$$x_{n+1} = \begin{cases} \alpha\, x_n & 0 < x_n \le d_1 \\ \frac{d_2 - x_n}{d_2 - d_1} & d_1 < x_n \le d_2 \\ 1 - \beta(1 - x_n) & d_2 < x_n < 1 \end{cases}$$

$x_n \in (0,1)$ and $d_1, d_2 \in$, d_1, d_2 are the parameters.

$$\alpha = \frac{d_2}{d_1}(1 - (d_2 - d_1))$$

$\beta = \frac{1}{d_2 - 1}((d_2 - 1) - d_1((d_2 - d_1))$ are the slopes of the linear maps and d_1, d_2 are the end points of the sub-intervals. As per LE, $\alpha = 1.08$ and $\beta = 1.125$ is obtained with $d_1 = 0.5$, $d_2 = 0.6$. The three fixed points of the Liebovitch map are 0, 0.5455, and 1 as can be seen in Figure 14.
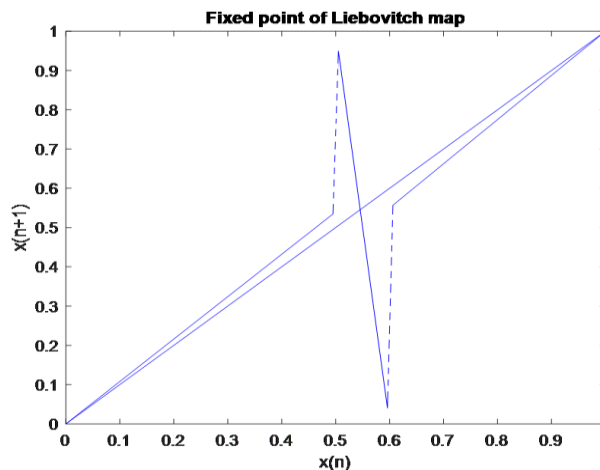

Figure 14. Fixed points of the liebovitch map with $d_1 = 0.5$, $d_2 = 0.6$

3.13. Intermittency map

The intermittency map [23], [34] extends Bernoulli Shift by replacing a piece-wise linear segment with a nonlinear segment as:

$$x_{n+1} = \begin{cases} \epsilon + x_n + c\,x_n{}^m & 0 < x_n \le d \\ \frac{x_n - d}{1-d} & d < x_n < 1 \end{cases} \quad , \; d \in (0,1)$$

Here ϵ, d, m are the parameters and $c = \frac{1-\epsilon-d}{1-d^m}$.

The suitable values of the parameters are found to be $m = 2, d = 0.5$ and $\epsilon = 0.49$. With these parametric values, it is found that the intermittency map has only one fixed point, which is 1 as depicted in Figure 15.
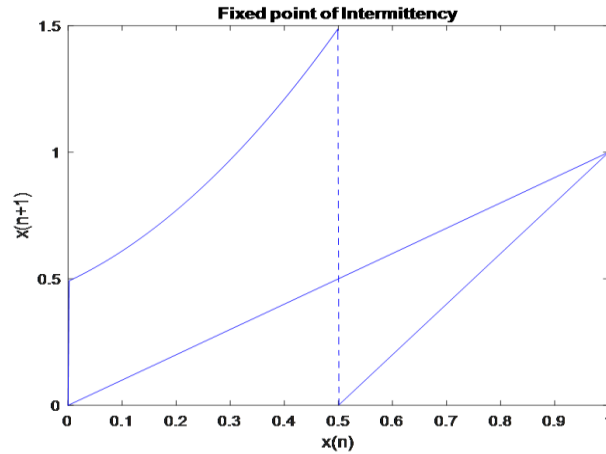

Figure 15. Fixed point of the intermittency map with $m = 2, d = 0.5$ and $\epsilon = 0.49$

3.14. Piecewise map

Following is the definition of the chaotic piece-wise map [23]:

$$x_{n+1} = \begin{cases} \frac{x_n}{d}, & 0 \le x_n < d \\ \frac{x_n - d}{0.5 - d}, & d \le x_n < \frac{1}{2} \\ \frac{1 - d - x_n}{0.5 - d}, & \frac{1}{2} \le x_n < 1 - d \\ \frac{1 - x_n}{d}, & 1 - d \le x_n < 1 \end{cases}$$

Using this map, chaotic sequences are obtained at $(0,1)$ where d represents the control parameter that lies in the range $(0, 0.5)$. For $d = 0.5$, the LE is high and for this d the fixed chaotic map are 0, 0.3750, 0.5833, and 0.7692. Figure 16 portrays these points.
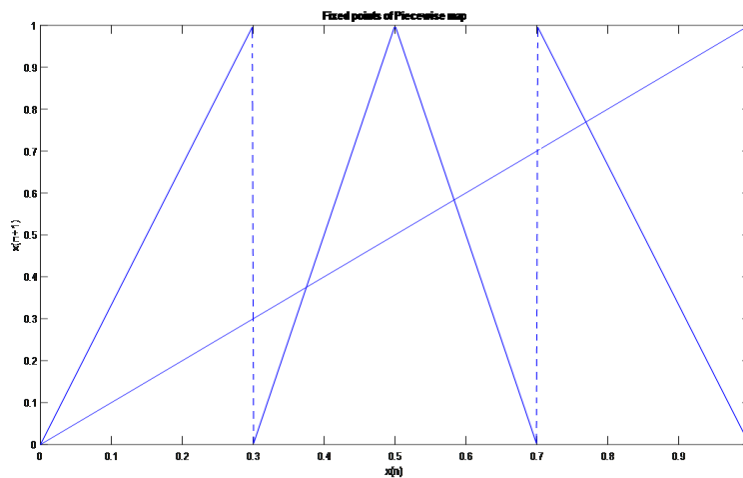

Figure 16. Fixed points of the piece-wise map with $d = 0.5$

3.15. Singer map

The Singer map [23] is a one-dimensional chaotic map and is defined formally as:

$$x_{n+1} = \mu(7.86x_n - 23.31x_n^2 + 28.75\ x_n^3 - 13.3\ x_n^4),\ x_n \epsilon (0,1)$$

Even though the parameter μ value should lie in the interval [0.9, 1.08], it was found that chaotic nature is highest when $\mu = 1.073$. The fixed points for this case are 0 and 0.3772 as described in Figure 17.
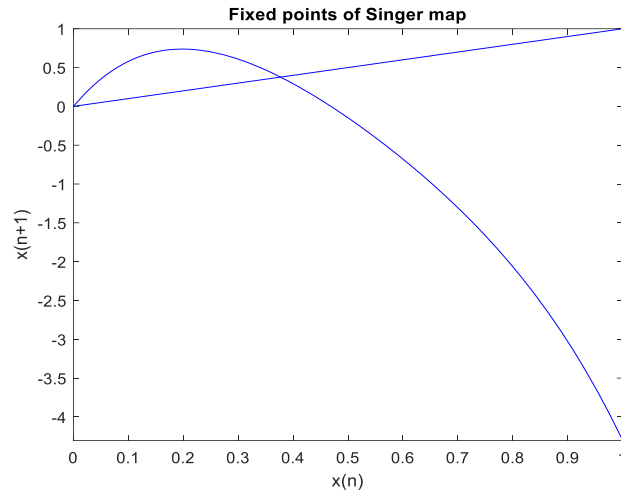

Figure 17. Fixed points of the Singer map with $\mu = 1.073$

3.16. Kent map

The Kent map [23], [34] is a chaotic map that can be used for several applications, including cryptography, to generate pseudo-random numbers and is defined as:

$$x_{n+1} = \begin{cases} \frac{x_n}{m} & 0 < x_n \le m \\ \frac{(1-x_n)}{1-m} & m < x_n \le 1 \end{cases} \qquad x_n \in (0,1)$$

for the parameter m=0.8, the map shows high chaotic behaviour and has two fixed points which are 0 and 0.8333, as can be seen in Figure 18. Kent map also has an eventually fixed point at 0.5. While using the Kent map code for the algorithm, one should ensure to replace a fixed point x^*, if it appears in between the sequence to $x^* + \varepsilon$.
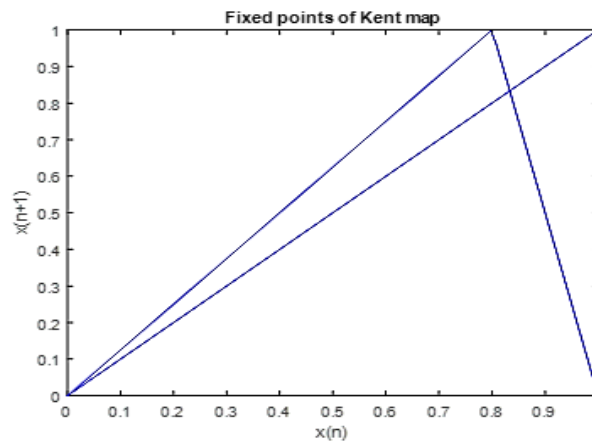

Figure 18. Fixed points of the Kent map with m=0.8

3.17. Iterative map

The following iterative function [34] introduces the iterative map:

$$x_{n+1} = sin\left(\frac{a\pi}{x_n}\right), x_n \in (0,1),$$

Even though , $a \in (0,1)$ choosing a as 0.7, a larger LE is achieved. The iterative map is similar to the ICMIC map in terms of fixed points. As can be seen in Figure 19, an infinite number of fixed points exists for the iterative map in the interval $(0, 0.05)$. Also, there are a few more fixed points that are 0.3321, 0.2395, 0.1726, 0.1413, 0.1160, 0.1005, 0.0872, 0.0780, 0.0698, and 0.0582 as can be seen in Figure 19.
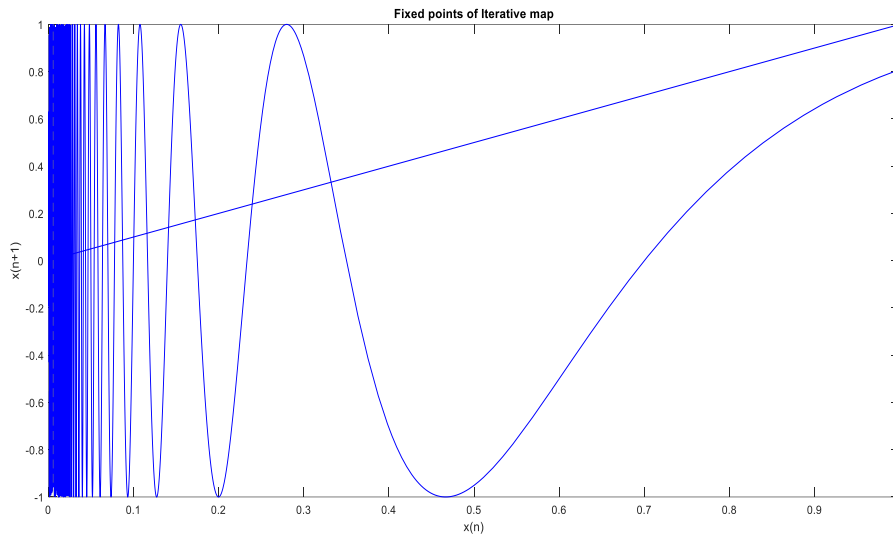

Figure 19. Fixed points of Iterative map with a as 0.7

3.18. Sine powered chaotic map

The mathematical definition of Sine powered chaotic map [40] is as follows.

$$x_{n+1} = (x_n(\alpha + 1))^{\sin(\beta\pi + x_n)}, x_n \in (0,1)$$

Here $\alpha > 0$ and $\beta \epsilon [0,1]$ are the control parameters. With a larger LE, the parameters α and β are set as α=4.4926 and $\beta = 0.3306$. The sine powered chaotic map has only one fixed point which is at 0, as can be viewed in Figure 20.
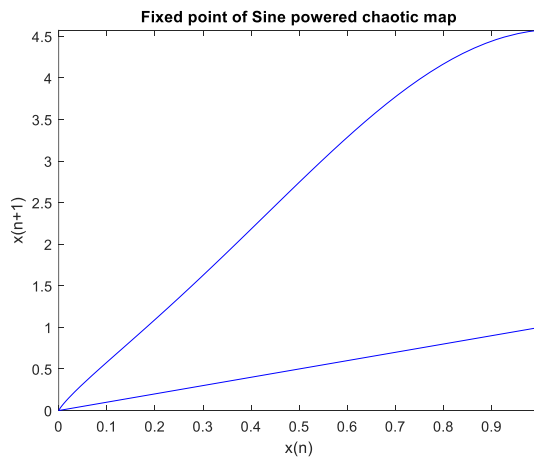

Figure 20. Fixed point of Sine powered chaotic map with α=4.4926 and $\beta = 0.3306$

3.19. Sine-circle map

A sine-circle map is an iconic one-dimensional representation of oscillation dynamics of two oscillators of natural frequencies coupled together with strength couplings [40], [41]. The sequence generated by this map lies between 0 and 1. Sin-circle map is defined as:

$$x_{n+1} = x_n + \alpha - \frac{\mu}{2\pi} \sin(2\pi x_n) \bmod 1$$

where $\alpha = 0.606661$, $\mu = 2.1$ are the best parameter values. Figure 21 shows the sine-circle map. It can be seen that the curves of this map do not intersect with the line $y = x$ and hence it does not have any fixed point. The Sine-Circle map is also known as the standard circle map.
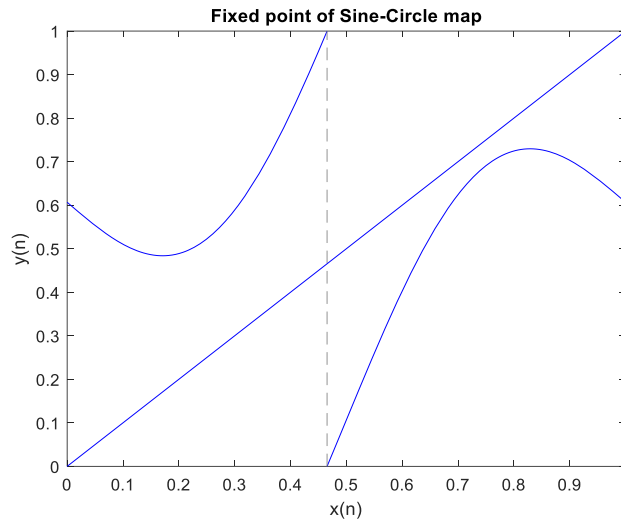

Figure 21. Sine-circle map and the line *y=x*

3.20. Sinus map

Sinus map [38], [42] generates a chaotic sequence in (0, 1) and is defined as:

$$x_{n+1} = 2.3(x_n)^{2\sin(\pi x_n)}$$

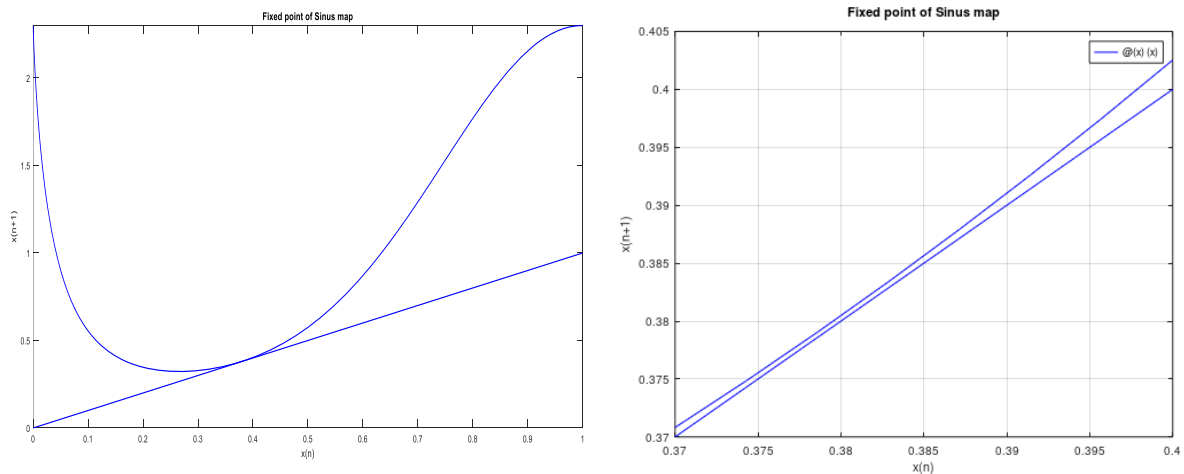The sinus map does not have any fixed points as can be seen in Figure 22.

Figure 22. Sinus map does not intersect the line y=x

3.21. Dyadic map/saw tooth map
A one-dimensional saw tooth map [42] is well-defined as follows:

$$x_{n+1} = 2x_n \bmod 1$$

$$\text{or } x_{n+1} = \begin{cases} 2x_n & 0 \le x_n < \frac{1}{2} \\ 2x_n - 1 & \frac{1}{2} \le x_n \le 1 \end{cases}$$

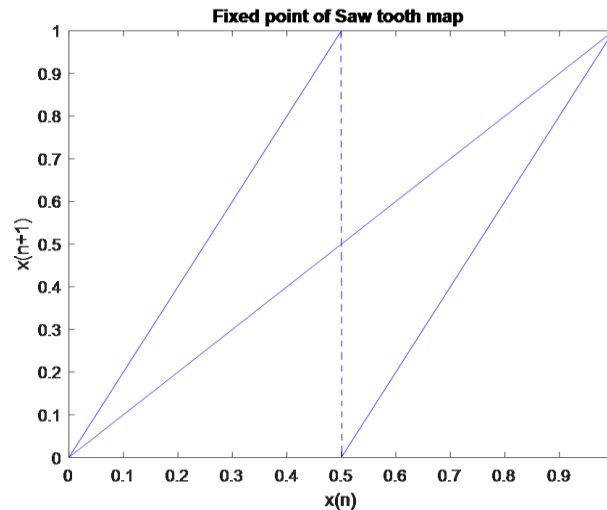The fixed points of this map are 0 and 1 as can be seen in Figure 23.

Figure 23. Fixed points of the saw tooth map

4. DISCUSSION
This research determines optimal parameters for chaotic maps, identifies fixed points for numerous chaotic maps to prevent algorithm collapses, and proposes a correction method for avoiding fixed points in chaotic sequences. The significant findings are:
a) A methodology for finding the best value for the parameter in a chaotic map with the help of LE and scatter diagrams is illustrated. This methodology is explained in detail for the logistic map with supporting figures (Figures 1(a) to 1(d)) in the section 2.1. Using the same methodology, the best parameters for the remaining 19 chaotic maps are found that are tabulated in Table 1.
b) Using Newton's iterative method that is the most popular one for solving transcendental equations, $f(x) = x$ is numerically solved and found the fixed points of various chaotic maps. These are explained in sections 3.2 to 3.21 and the fixed points are tabulated in Table 2. The fixed points can also be visualized in the figures (Figures 2 to 23). In these figures the X coordinate of the point of intersection of the line $y=x$ and the map $y=f(x)$ are the fixed points of each chaotic map. The knowledge of fixed points shall help researchers using chaotic maps to avoid collapses of their chaotic algorithm. A fixed point should not be taken as a starting point of the chaotic sequence.
c) Also, it must be ensured that the fixed point of the corresponding chaotic map does not appear in the sequence at any time in the chaotic algorithm. It will appear in the sequence if the starting point is an eventually fixed point. This paper also explains the eventually fixed points and a method to find them. For example, a few eventually fixed points of the logistic map are obtained in section 3.2. In the same way, eventually fixed points can also be calculated for the other maps. However, this way of finding the eventually fixed points is tedious and impractical, as these points also may go into an infinite sequence for some maps. Hence, it is suggested to the researchers who use chaotic maps, to correct the issue of a fixed point whenever it appears in the sequence. It can be done by replacing the fixed point x^* with $x^* + \varepsilon$, where $0 < \varepsilon \le 0.09$. With this correction incorporated in each iteration, the code for a COA will become flawless.

Table 2. Fixed points of chaotic maps

Chaotic map	Governing equation	Fixed points
Logistic	$x_{n+1} = 4x_n(1 - x_n), x_n \epsilon(0,1)$	0, 0.75
Sinusoidal	$x_{n+1} = 2.3x_n{}^2 sin(\pi x_n), x_n \epsilon(0,1)$	0, 0.4421 and 0.8228
Sine	$x_{n+1} = 0.95 \, sin(\pi x_n), x_n \epsilon(0,1)$	0, 0.7241
Neuron	$x_{n+1} = 0.5 - 2 \, tanh \, tanh \, (5) \, e^{-3x_n{}^2}, \quad x_n \epsilon(-1.5, 0.5)$	-0.4856
Tent	$x_{n+1} = \begin{cases} 2x_n & 0 \le x_n < 0.5 \\ 2(1 - x_n) & 0.5 \le x_n \le 1 \end{cases}$	0, 0.6667
Chebyshev	$x_{n+1} = cos(8 \, cos^{-1}(x_n)), x_n \epsilon(-1,1)$	$-0.9397, -0.9010, -0.5,$ $-0.2225, 0.1736, 0.6235,$ 0.7660 and1 1
Circle	$x_{n+1} = x_n + 0.2 - \frac{0.5}{2\pi}(sin(2\pi x_n)) \, mod \, 1, \quad x_n \epsilon(0,1)$	0.7217
Cubic	$x_{n+1} = 2.59 \, x_n(1 - x_n{}^2), x_n \epsilon(0,1)$	0, 0.7835
ICMIC	$x_{n+1} = sin\left(\frac{3}{x_n}\right), x_n \in (-1,1)$	$\pm0.0952, \pm0.1065, \pm0.1188, \pm0.1373,$ and infinitely many fixed points in the interval $(-0.1, 0.1)$.
Bernoulli shift	$x_{n+1} = \begin{cases} 1.4 \, x_n - 1 & x_n \ge 0 \\ 1.4 \, x_n + 1 & x_n < 0 \end{cases} \quad x_n \in [-1,1]$	-
Liebovitch	$x_{n+1} = \begin{cases} 1.08 \, x_n & 0 < x_n \le 0.5 \\ \frac{0.6 - x_n}{0.6 - 0.5} & 0.5 < x_n \le 0.6 \\ 1 - 1.125(1 - x_n) & 0.6 < x_n < 1 \end{cases}, x_n \in (0,1)$	0, 0.5455
Intermittency	$x_{n+1} = \begin{cases} 0.49 + x_n + \frac{1 - 0.49 - 0.5}{1 - 0.5^2}x_n{}^2 & 0 < x_n \le 0.5 \\ \frac{x_n - 0.5}{1 - 0.5} & 0.5 < x_n < 1 \end{cases}$	1
Piecewise	$= \begin{cases} \frac{x_n}{d0.3}, & 0 \le x_n < 0.3 \\ \frac{x_n - 0.3}{0.5 - 0.3}, & 0.3 \le x_n < \frac{1}{2} \\ \frac{1 - 0.3 - x_n}{0.5 - 0.3}, & \frac{1}{2} \le x_n < 1 - 0.3 \\ \frac{1 - x_n}{0.3}, & 1 - 0.3 \le x_n < 1 \end{cases}$	0, 0.3750, 0.5833 and 0.7692
Singer	$x_{n+1} = 1.073(7.86x_n - 23.31x_n{}^2 + 28.75 \, x_n{}^3 - 13.3 \, x_n{}^4), x_n \epsilon(0,1)$	0, 0.3772
Kent	$x_{n+1} = \begin{cases} \frac{x_n}{0.8} & 0 < x_n \le 0.8 \\ \frac{(1 - x_n)}{1 - 0.8} & 0.8 < x_n \le 1 \end{cases} \quad x_n \in (0,1)$	0, 0.8333
Iterative	$x_{n+1} = sin\left(\frac{0.7\pi}{x_n}\right), x_n \in (0,1)$	0.3321, 0.2395, 0.1726, 0.1413, 0.1160, 0.1005, 0.0872, 0.0780, 0.0698 , 0.0582 and an infinite number of fixed points in interval $(0, 0.05)$.
1-D sine powered chaotic	$x_{n+1} = (x_n(4.4926 + 1))^{sin (0.3306\pi + x_n)}$	0
Sine-circle	$x_{n+1} = x_n + 0.606661 - \frac{2.1}{2\pi} sin(2\pi x_n) \, mod \, 1$	-
Sinus	$x_{n+1} = 2.3(x_n)^{2 \, sin(\pi x_n)} \quad x_n \in (0,1)$	-
Dyadic or saw tooth	$x_{n+1} = 2x_n \, mod \, 1$ Or $x_{n+1} = \begin{cases} 2x_n & 0 \le x_n < \frac{1}{2} \\ 2x_n - 1 & \frac{1}{2} \le x_n \le 1 \end{cases} \quad x_n \in (0,1)$	0, 1

5. CONCLUSIONS

Chaotic maps have a remarkable array of applications, like building robust pseudo-random number generators and multimedia encryption schemes for enhanced data protection, generating captivating video-game animations and dynamic digital marketing experiences, accurately simulating chaotic systems, such as chaotic missile systems, for critical research and development, thwarting data mining attempts by unauthorized personnel, safeguarding sensitive information, improving optimization algorithms to avoid the local optima and speed up the convergence. All these applications are based on the chaotic nature that gives a good spread for the chaotic numbers generated by these maps. If the proper parameter is not chosen for the map, the map will not have the expected chaotic behaviour. In this paper, the apt values of the parameters for these chaotic maps are provided based on a study of their Lyapunov exponent. At fixed points, the chaotic behaviour of the map gets lost. This may lead to the failure or delay of the algorithm used for any of the applications. Hence, a proper analysis of the map and an understanding of their fixed points is essential for the researchers using chaotic maps. This paper gives the fixed points of 20 most popular chaotic maps. In addition to these, a method to address the issue of a fixed point whenever it enters the sequence in between simulation is also provided.

The paper emphasizes the importance of chaotic maps in various applications, highlighting the essential roles of parameter selection and comprehension of fixed points for algorithmic reliability. Additionally, it outlines a method for avoiding fixed points within sequences, although it may not comprehensively identify all fixed or eventually fixed points within chaotic maps. The future scope of the paper involves exploring diverse chaotic maps to determine their suitability for specific applications, emphasizing the importance of selecting the most appropriate map for optimal performance in various scenarios.

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