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Oscillation Criteria for Even-order Half-linear Functional Differential Equations with Damping

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Abstract

In this paper, a class of even-order half-linear functional differential equations with damping is studied. By using the generalized Riccati transformation and the integral averaging technique, six new oscillation criterias are obtained for all solutions of the equations. The results obtained generalize and improve some known results.

Keywords: *oscillation criteria, damping, half-linear, functional differential equation, integral averaging method*

1. Introduction

In this paper, we consider the oscillatory behavior of solutions for the n-th order half-linear functional differential equation with damping of the form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[r(t)\Phi(x^{(n-1)}(t))\right]+p(t)\Phi(x^{(n-1)}(t))+f(t,x(g(t)))=0,\quad t\ge t_0\,. \tag{1}$$

Where n is even, $\Phi(u)=|u|^{\alpha-1}u$, α is a real number and $\alpha>0$. For simplicity, we note :

$$\mathrm{I}=[t_0,\infty),$$
$$\mathrm{R}^+=(0,\infty),\mathrm{R}^0=[0,\infty)\,.$$

Throughout this paper, we assume that:

(H_1) $r(t)\in C^1(\mathrm{I},\mathrm{R}^+), r'(t)\ge 0, p(t)\in C(\mathrm{I},\mathrm{R}^0)$.

(H_2) $g(t)\in C(\mathrm{I},\mathrm{R}^0), g(t)\ge t, g'(t)\ge 0\,, \lim_{t\to\infty} g(t)=\infty$.

(H_3) $f(t,x)\in C(\mathrm{I}\times\mathrm{R},\mathrm{R})$.

In order to discuss conveniently in the following context, several definitions will firstly be given.

Definition 1. The function $x(t)\in C^{n-1}([T_x,\infty),\mathrm{R}), T_x\ge t_0$ is called a solution of (1), if $r(t)\Phi(x^{(n-1)}(t))\in C^1([T_x,\infty),\mathrm{R})$ and $x(t)$ satisfy (1) on an interval $[T_x,\infty)$.

Definition 2. A nontrivial solution of (1) is said to be oscillatory if it has arbitrarily large zeros; otherwise, it is called nonoscillatory. (1) is said to be oscillatory if all its solutions are oscillatory.

Very few people have studied the oscillatory behavior of even-order half-linear functional differential equations with damping. So, much research, especially some on the Philos oscillation criteria of (1) and the other related results, will be done in this paper by referring to [1-8]. Moreover, functional inequalities in this paper hold for all sufficient large t if there is no special explanation.

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2. Main Results

The following lemma is a well-known result; let us see [1, Lemma 2.2.1] and [2].

Lemma 1. Let u be a positive and n-times differentiable function on an interval $[T,\infty)$ with its n-th derivative $u^{(n)}$ non-positive on $[T,\infty)$ and not identically zero on any interval of the form $[T',\infty)$, $T'\geq T$. Then there exists an integer l, $0\leq l\leq n-1$, with $n+l$ odd and such that for some $T^*\geq T'$:

$$(-1)^{l+j}u^{(j)}>0, t\in[T^*,\infty), \quad (j=l,l+1,\cdots,n-1);$$
$$u^{(i)}>0, t\in[T^*,\infty), \quad (i=1,2,\cdots,l-1) \text{ , when } l>1.$$

Lemma 2 [8**]**. Assume that $x(t)$ satisfies all the conditions in Lemma 1 and $x^{(n-1)}(t)x^{(n)}(t)\leq 0, t\geq t_x$; then there exists constants $\theta\in(0,1)$, and $M>0$ such that:

$$x'(\theta\ t)\geq Mt^{n-2}x^{(n-1)}(t),$$

For all sufficient large *t*.

Theorem 1. If the following conditions are true:

(H4) $\int_{t_0}^{\infty}[E(t)r(t)]^{-1/\alpha}\mathrm{d}t=\infty$, where $E(t)=\exp\int_{t_0}^{t}p(u)r^{-1}(u)\mathrm{d}u$;

(H5) Suppose that there exists $q(t)\in C(\mathrm{I},\mathrm{R}^0)$, $F(x)\in C(\mathrm{R},\mathrm{R})$ such that:

$$f(t,x)\mathrm{sgn}(x)\geq q(t)F(x)\mathrm{sgn}(x),\ -F(-x)\geq F(x)\geq k|x|^{\alpha-1}x,$$

Then $\alpha>0, x>0, k>0$;

(H6) $\limsup_{t\to\infty}\int_{t_0}^{t}E(s)q(s)\mathrm{d}s=\infty$.

Then (1) is oscillatory.

Proof. Assume that $x(t)$ is an eventually positive solution of (1), then there exists $t_1\geq t_0$, such that:

$$x(t)>0, x(g(t))>0, \text{ for all } t\geq t_1.$$

From (1) and (H5), we obtain:

$$[E(t)r(t)\Phi(x^{(n-1)}(t))]'+kE(t)q(t)x^{\alpha}(g(t))\leq 0,\ t\geq t_1. \tag{2}$$

Hence,

$$[E(t)r(t)\Phi(x^{(n-1)}(t))]'\leq 0,\ t\geq t_1. \tag{3}$$

From (3) and (H4), then there exists $t_2\geq t_1$ such that:

$$x^{(n-1)}(t)>0, \quad t\geq t_2. \tag{4}$$

From (4) and (1), we obtain:

$$[r(t)(x^{(n-1)}(t))^{\alpha}]'\leq 0, \quad t\geq t_2. \tag{5}$$

It follows from $r'(t)>0$ that:

$$x^{(n)}(t)\le 0, \quad t\ge t_2 . \tag{6}$$

From Lemma 1, we obtain:

$$x'(t)>0, \quad t\ge t_2 . \tag{7}$$

From (3) and (4), we obtain:

$$[E(t)r(t)(x^{(n-1)}(t))^{\alpha}]'+kE(t)q(t)x^{\alpha}(g(t))\le 0,\ t\ge t_2 . \tag{8}$$

In view of $x(t)>0, x'(t)>0,$ then there exists $T\ge t_2$ and $\delta>0$, for all $t\ge T$, we have $x(g(t))\ge\delta$. Hence,

$$[E(t)r(t)(x^{(n-1)}(t))^{\alpha}]'+kE(t)q(t)\delta^{\alpha}\le 0,\ t\ge T . \tag{9}$$

We get that:

$$E(t)r(t)(x^{(n-1)}(t))^{\alpha}+k\delta^{\alpha}\int_T^t E(s)q(s)\mathrm{d}s\le E(T)r(T)(x^{(n-1)}(T))^{\alpha} .$$

Hence, we have a contradiction to the condition (H_6). The proof is complete.

Theorem 2. Assume conditions (H_4) and (H_5) hold, and the following condition is true (H_7) Suppose that there exists $\rho(t)\in C^1(\mathrm{I},\mathrm{R}^{+}),\rho'(t)\ge 0$, and $\lambda>0$ such that:

$$\limsup_{t\to\infty}\int_{t_0}^t[k\rho(s)E(s)q(s)-\lambda\rho'(s)]\mathrm{d}s=\infty .$$

Then (1) is oscillatory.

Proof. Assume that $x(t)$ is an eventually positive solution of (1), proceeding as the proof of Theorem 1, we obtain (8) holds. Consider the function:

$$W(t)=\rho(t)E(t)r(t)\left(\frac{x^{(n-1)}(t)}{x(g(t))}\right)^{\alpha}, \quad t\ge t_0 . \tag{10}$$

Then $W(t)>0$, and:

$$\begin{aligned}W'(t)&=\rho'(t)E(t)r(t)\left(\frac{x^{(n-1)}(t)}{x(g(t))}\right)^{\alpha}+\frac{\rho(t)[E(t)r(t)(x^{(n-1)}(t))^{\alpha}]'}{x^{\alpha}(g(t))}-\frac{\alpha\rho(t)E(t)r(t)(x^{(n-1)}(t))^{\alpha}x'(g(t))g'(t)}{x^{\alpha+1}(g(t))}\\&\le\rho'(t)E(t)r(t)\left(\frac{x^{(n-1)}(t)}{x(g(t))}\right)^{\alpha}+\frac{\rho(t)[E(t)r(t)(x^{(n-1)}(t))^{\alpha}]'}{x^{\alpha}(g(t))} .\end{aligned} \tag{11}$$

From (3), (7), (8) and (11), we obtain:

$$W'(t)\le -k\rho(t)E(t)q(t)+\frac{\rho'(t)E(T)r(T)(x^{(n-1)}(T))^{\alpha}}{x^{\alpha}(g(T))}, \quad T\ge t_0 . \tag{12}$$

Let $\lambda=\frac{E(T)r(T)\left(x^{(n-1)}(T)\right)^{\alpha}}{x^{\alpha}(g(T))}$, we get:

$$W'(t)\le -k\rho(t)E(t)q(t)+\lambda\rho'(t) . \tag{13}$$

Integrating the above from *T* to *t*, we obtain:

$$W(t) \le W(T) - \int_T^t [k\rho(s)E(s)q(s) - \lambda\rho'(s)]\mathrm{d}s \,. \tag{14}$$

In (14), let $t \to \infty$. Because $W(t) > 0$, we have a contradiction to condition (H_7). The proof is complete.

Theorem 3. Assume condition (H_4) and (H_5) hold, and the following condition is true (H_8) Suppose that there exists $\rho(t) \in C^1(\mathrm{I}, \mathrm{R}^+)$ such that:

$$\limsup_{t\to\infty} \int_T^t [k\rho(s)E(s)q(s) - \frac{(\rho'(s))^{\alpha+1}E(s)r(s)}{(\alpha+1)^{\alpha+1}(\rho(s)G(s))^{\alpha}}]\mathrm{d}s = \infty \,, \quad T \ge t_0 \,, \tag{15}$$

Wherein $G(s) = \theta M g'(s) g^{n-2}(s)$, θ, M in Lemma 2, $E(s)$ in (H_4). Then (1) is oscillatory.

Proof. Assume that $x(t)$ is an eventually positive solution of (1), proceeding as the proof of Theorem 1, we obtain (8) holds. Consider the function:

$$W(t) = \rho(t)E(t)r(t)\left(\frac{x^{(n-1)}(t)}{x(\theta\, g(t))}\right)^{\alpha}, \qquad t \ge t_0 \,. \tag{16}$$

Then $W(t) > 0$. From (8) and Lemma 2, we obtain:

$$W'(t) \le -k\rho(t)E(t)q(t) + \frac{\rho'(t)}{\rho(t)}W(t) - \frac{\alpha G(t)}{[\rho(t)E(t)r(t)]^{1/\alpha}}W^{(\alpha+1)/\alpha}(t), \qquad t \ge T \,. \tag{17}$$

By using the inequality:

$$Au - Bu^{(\alpha+1)/\alpha} \le \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{A^{\alpha+1}}{B^{\alpha}} \,. \tag{18}$$

Then $A \ge 0, B > 0, u \ge 0$, we have:

$$W'(t) \le -k\rho(t)E(t)q(t) + \frac{(\rho'(t))^{\alpha+1}E(t)r(t)}{(\alpha+1)^{\alpha+1}(\rho(t)G(t))^{\alpha}}, \qquad t \ge T \,. \tag{19}$$

Integrating the above from *T* to *t*, we get:

$$W(t) \le W(T) - \int_T^t [k\rho(s)E(s)q(s) - \frac{(\rho'(s))^{\alpha+1}E(s)r(s)}{(\alpha+1)^{\alpha+1}(\rho(s)G(s))^{\alpha}}]\,\mathrm{d}s \,.$$

Because $W(t) > 0$, we have:

$$\limsup_{t\to\infty} \int_T^t [k\rho(s)E(s)q(s) - \frac{(\rho'(s))^{\alpha+1}E(s)r(s)}{(\alpha+1)^{\alpha+1}(\rho(s)G(s))^{\alpha}}]\mathrm{d}s \le W(T) \,. \tag{20}$$

Hence, we have a contradiction to the condition (H_8). The proof is complete.

Theorem 4. Assume the condition (H_4) and (H_5) hold, and the following condition is true (H_9) Suppose that there exists $\rho(t) \in C^1(\mathrm{I}, \mathrm{R}^+)$ such that:

$$\limsup_{t\to\infty} \frac{1}{t^n}\int_T^t (t-s)^n [k\rho(s)E(s)q(s) - \frac{(\rho'(s))^{\alpha+1}E(s)r(s)}{(\alpha+1)^{\alpha+1}(\rho(s)G(s))^{\alpha}}]\mathrm{d}s = \infty \,, \tag{21}$$

Where *n* > 1 and function $E(s), G(s)$ is given by (H4) and (H8). Then (1) is oscillatory.

Proof. Assume that $x(t)$ is an eventually positive solution of (1), proceeding as the proof of Theorem 3, and function $W(t)$ is given by (16), we get (19) holds.
From (19), we obtain:

$$k\rho(s)E(s)q(s)-\frac{(\rho'(s))^{(\alpha+1)}E(s)r(s)}{(\alpha+1)^{(\alpha+1)}(\rho(s)G(s))^{\alpha}}\le -W'(s), \qquad s\ge T .$$

Multiplying two sides by $(t-s)^n$ and integrating the above from *T* to *t* (*t* >*T*) , we get:

$$\int_T^t (t-s)^n[k\rho(s)E(s)q(s)-\frac{(\rho'(s))^{\alpha+1}E(s)r(s)}{(\alpha+1)^{\alpha+1}(\rho(s)G(s))^{\alpha}}]\mathrm{d}s\le -\int_T^t (t-s)^n W'(s)\mathrm{d}s .$$

Since:

$$\int_T^t (t-s)^n W'(s)\mathrm{d}s = n\int_T^t (t-s)^{n-1}W(s)\mathrm{d}s - W(T)(t-T)^n ,$$

We get:

$$\frac{1}{t^n}\int_T^t (t-s)^n[k\rho(s)E(s)q(s)-\frac{(\rho'(s))^{\alpha+1}E(s)r(s)}{(\alpha+1)^{\alpha+1}(\rho(s)G(s))^{\alpha}}]\mathrm{d}s\le W(T)(\frac{t-T}{t})^n-\frac{n}{t^n}\int_T^t (t-s)^{n-1}W(s)\mathrm{d}s .$$

Therefore:

$$\frac{1}{t^n}\int_T^t (t-s)^n[k\rho(s)E(s)q(s)-\frac{(\rho'(s))^{\alpha+1}E(s)r(s)}{(\alpha+1)^{\alpha+1}(\rho(s)G(s))^{\alpha}}]\mathrm{d}s\le W(T)(\frac{t-T}{t})^n .$$

Then:

$$\limsup_{t\to\infty}\frac{1}{t^n}\int_T^t (t-s)^n[k\rho(s)E(s)q(s)-\frac{(\rho'(s))^{\alpha+1}E(s)r(s)}{(\alpha+1)^{\alpha+1}(\rho(s)G(s))^{\alpha}}]\mathrm{d}s<\infty .$$

Hence, we have a contradiction to the condition (H9). The proof is complete.

By Philos integral average conditions, the new oscillation theorems are given for Equation (1). Consider the sets:

$$\mathrm{D}_0=\{(t,s):t>s\ge t_0\},\quad \mathrm{D}=\{(t,s):t\ge s\ge t_0\} .$$

Assume that $H\in C(\mathrm{D},\mathrm{R})$ satisfies the following conditions:

(i) $H(t,t)=0, t\ge t_0$; $H(t,s)>0, (t,s)\in \mathrm{D}_0$;

(ii) H has a non-positive continuous partial derivative with respect to the second variable in D_0.

Then the function H has the property P (Denoted as $H(t,s)\in P$).

Theorem 5. Assume the condition (H_4) and (H_5) hold, and the following condition is true (H_{10}) $H(t,s)\in P$, and that there exists functions $h(t,s)\in C(\mathrm{D}_0,\mathrm{R})$ and $\rho(t)\in C^1(\mathrm{I},\mathrm{R}^+)$ such that:

$$\frac{\partial H(t,s)}{\partial s}+\frac{\rho'(s)}{\rho(s)}H(t,s)=-h(t,s)H^{\alpha/(\alpha+1)}(t,s),\quad (t,s)\in \mathrm{D}_0 , \tag{22}$$

And,

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}[H(t,s)k\rho(s)E(s)q(s)-\frac{\rho(s)E(s)r(s)\,|\,h(t,s)\,|^{\alpha+1}}{(\alpha+1)^{\alpha+1}G^{\alpha}(s)}]\mathrm{d}s=\infty\,, \tag{23}$$

Where functions $E(s), G(s)$ are given by (H4) and (H8). Then (1) is oscillatory.

Proof. Assume that $x(t)$ is an eventually positive solution of (1), proceeding as the proof of Theorem 3, and function $W(t)$ is given by (16), we get (17) holds.
From (17), we obtain:

$$k\rho(t)E(t)q(t)\le -W'(t)+\frac{\rho'(t)}{\rho(t)}W(t)-\frac{\alpha G(t)}{[\rho(t)E(t)r(t)]^{1/\alpha}}W^{(\alpha+1)/\alpha}(t),\qquad t\ge T\,. \tag{24}$$

Replacing *t* by *s*, multiplying two sides by $H(t,s)$ and integrating the above from *T* to *t* (*t* >*T*), we get:

$$\begin{aligned}\int_{T}^{t}H(t,s)k\rho(s)E(s)q(s)\mathrm{d}s &\le \int_{T}^{t}H(t,s)[-W'(s)+\frac{\rho'(s)}{\rho(s)}W(s)-\frac{\alpha G(s)}{[\rho(s)E(s)r(s)]^{1/\alpha}}W^{(\alpha+1)/\alpha}(s)]\mathrm{d}s\\ &\le H(t,T)W(T)+\int_{T}^{t}\{[\frac{\partial H(t,s)}{\partial s}+H(t,s)\frac{\rho'(s)}{\rho(s)}]W(s)-\frac{\alpha G(s)H(t,s)}{[\rho(s)E(s)r(s)]^{1/\alpha}}W^{(\alpha+1)/\alpha}(s)\}\mathrm{d}s\\ &\le H(t,T)W(T)+\int_{T}^{t}\{[|\,h(t,s)\,|\,H^{\alpha/(\alpha+1)}(t,s)]W(s)-\frac{\alpha G(s)H(t,s)}{[\rho(s)E(s)r(s)]^{1/\alpha}}W^{(\alpha+1)/\alpha}(s)\}\mathrm{d}s\end{aligned} \tag{25}$$

The right end of (25) is integrable functions for using the inequality (18), then for $t>s\ge T$, we have:

$$|\,h(t,s)\,|\,H^{\alpha/(\alpha+1)}(t,s)W(s)-\frac{\alpha G(s)H(t,s)}{[\rho(s)E(s)r(s)]^{1/\alpha}}W^{(\alpha+1)/\alpha}(s)\le\frac{\rho(s)E(s)r(s)\,|\,h(t,s)\,|^{\alpha+1}}{(\alpha+1)^{\alpha+1}G^{\alpha}(s)}\,. \tag{26}$$

Form (25) and (26), we have:

$$\int_{T}^{t}[H(t,s)k\rho(s)E(s)q(s)-\frac{\rho(s)E(s)r(s)\,|\,h(t,s)\,|^{\alpha+1}}{(\alpha+1)^{\alpha+1}G^{\alpha}(s)}]\mathrm{d}s\le H(t,T)W(T)\le H(t,t_0)W(T)\,. \tag{27}$$

Therefore:

$$\begin{aligned}&\int_{t_0}^{t}[H(t,s)k\rho(s)E(s)q(s)-\frac{\rho(s)E(s)r(s)\,|\,h(t,s)\,|^{\alpha+1}}{(\alpha+1)^{\alpha+1}G^{\alpha}(s)}]\mathrm{d}s\\ &=\{\int_{t_0}^{T}+\int_{T}^{t}\}[H(t,s)k\rho(s)E(s)q(s)-\frac{\rho(s)E(s)r(s)\,|\,h(t,s)\,|^{\alpha+1}}{(\alpha+1)^{\alpha+1}G^{\alpha}(s)}]\mathrm{d}s\\ &\le H(t,t_0)\int_{t_0}^{T}k\rho(s)E(s)q(s)\mathrm{d}s+H(t,t_0)W(T),\quad t>t_0\,,\end{aligned}$$

Which implies:

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}[H(t,s)k\rho(s)E(s)q(s)-\frac{\rho(s)E(s)r(s)\,|\,h(t,s)\,|^{\alpha+1}}{(\alpha+1)^{\alpha+1}G^{\alpha}(s)}]\mathrm{d}s\le\int_{t_0}^{T}k\rho(s)E(s)q(s)\mathrm{d}s+W(T)\,.$$

Hence, we have a contradiction to the condition (H10). The proof is complete.

If condition (H10) does not hold, then we can use the following oscillatory theorem to Equation (1).

Theorem 6. Assume the condition (H4) and (H5) hold, $H(t,s) \in P$, and the following conditions is true.

(H11) $0 < \inf_{s \geq t_0} \left\{ \liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)} \right\} \leq \infty$;

(H12) $\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \frac{\rho(s)E(s)r(s)\,|\,h(t,s)\,|^{\alpha+1}}{(\alpha+1)^{\alpha+1}G^{\alpha}(s)} \mathrm{d}s < \infty$, where $E(s), G(s)$ is given by (H4) and (H8);

(H13) That there exists $\varphi(t) \in C(\mathrm{I},\mathrm{R})$ such that:

$$\int_{T}^{\infty} \frac{G(s)\varphi_{+}^{(\alpha+1)/\alpha}(s)}{[\rho(s)E(s)r(s)]^{1/\alpha}} \mathrm{d}s = \infty, \quad T \in [t_0, \infty) ,$$

Then $\varphi_{+}(s) = \max\{\varphi(s), 0\}$; and:

(H14) $\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} [H(t,s)k\rho(s)E(s)q(s) - \frac{\rho(s)E(s)r(s)\,|\,h(t,s)\,|^{\alpha+1}}{(\alpha+1)^{\alpha+1}G^{\alpha}(s)}] \mathrm{d}s \geq \varphi(T)$.

Then (1) is oscillatory.

Proof. Assume that $x(t)$ is an eventually positive solution of (1), proceeding as the proof of Theorem 3, and function $W(t)$ is given by (16), we get (25) holds.

From (25), we obtain:

$$\frac{1}{H(t,T)} \int_{T}^{t} H(t,s)k\rho(s)E(s)q(s)\mathrm{d}s$$
$$\leq W(T) + \frac{1}{H(t,T)} \int_{T}^{t} |\,h(t,s)\,|\, H^{\alpha/(\alpha+1)}(t,s)W(s)\mathrm{d}s - \frac{\alpha}{H(t,T)} \int_{T}^{t} \frac{G(s)H(t,s)}{[\rho(s)E(s)r(s)]^{1/\alpha}} W^{(\alpha+1)/\alpha}(s)\mathrm{d}s .$$

Let,

$$A(t) = \frac{1}{H(t,T)} \int_{T}^{t} |\,h(t,s)\,|\, H^{\alpha/(\alpha+1)}(t,s)W(s)\mathrm{d}s , \tag{28}$$

$$B(t) = \frac{\alpha}{H(t,T)} \int_{T}^{t} R(s)H(t,s)W^{(\alpha+1)/\alpha}(s)\mathrm{d}s , \tag{29}$$

Which,

$$R(s) = \frac{G(s)}{[\rho(s)E(s)r(s)]^{1/\alpha}} . \tag{30}$$

Then,

$$\frac{1}{H(t,T)} \int_{T}^{t} H(t,s)k\rho(s)E(s)q(s)\mathrm{d}s \leq W(T) + A(t) - B(t) . \tag{31}$$

From (27), we have:

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t [H(t,s)k\rho(s)E(s)q(s) - \frac{|h(t,s)|^{\alpha+1}}{(\alpha+1)^{\alpha+1}R^{\alpha}(s)}]\mathrm{d}s \le W(T) . \quad (32)$$

From (32) and (H14), we have:

$$W(T) \ge \varphi(T), \quad T \in [t_0, \infty) , \quad (33)$$

And,

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t H(t,s)k\rho(s)E(s)q(s)\mathrm{d}s \ge \varphi(T) . \quad (34)$$

Joint (31) and (34) to produce:

$$\liminf_{t\to\infty}[B(t)-A(t)] \le W(T) - \limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t H(t,s)k\rho(s)E(s)q(s)\mathrm{d}s \le W(T)-\varphi(T) < \infty . \quad (35)$$

We claim that:

$$\int_T^\infty R(s)W^{(\alpha+1)/\alpha}(s)\mathrm{d}s < \infty . \quad (36)$$

Otherwise, if:

$$\int_T^\infty R(s)W^{(\alpha+1)/\alpha}(s)\mathrm{d}s = \infty . \quad (37)$$

From (H11), then there exists $\eta > 0$ can be used in:

$$\inf_{s\ge t_0}\left\{\liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)}\right\} > \eta > 0 . \quad (38)$$

Let $\mu > 0$ be an arbitrary constant from (37), then there exists $T_1 > T$ can be used in:

$$\int_T^t R(s)W^{(\alpha+1)/\alpha}(s)\mathrm{d}s \ge \frac{\mu}{\alpha\eta}, \quad t > T_1 . \quad (39)$$

Thus,

$$\begin{aligned} B(t) &= \frac{\alpha}{H(t,T)}\int_T^t R(s)H(t,s)W^{(\alpha+1)/\alpha}(s)\mathrm{d}s = \frac{\alpha}{H(t,T)}\int_T^t H(t,s)\mathrm{d}[\int_T^s R(u)W^{(\alpha+1)/\alpha}(u)\mathrm{d}u] \\ &= \frac{\alpha}{H(t,T)}\int_T^t[\int_T^s R(u)W^{(\alpha+1)/\alpha}(u)\mathrm{d}u](-\frac{\partial H(t,s)}{\partial s})\mathrm{d}s \ge \frac{\alpha}{H(t,T)}\int_{T_1}^t[\int_T^s R(u)W^{(\alpha+1)/\alpha}(u)\mathrm{d}u](-\frac{\partial H(t,s)}{\partial s})\mathrm{d}s \\ &\ge \frac{\mu}{\eta H(t,T)}\int_{T_1}^t(-\frac{\partial H(t,s)}{\partial s})\mathrm{d}s = \frac{\mu H(t,T_1)}{\eta H(t,T)}, \quad t > T_1 . \end{aligned} \quad (40)$$

From (38), then there exist $T_2 > T_1$, can be used:

$$\frac{H(t,T_1)}{H(t,t_0)} \ge \eta, \quad t > T_2 . \quad (41)$$

Joint (40) and (41) to produce:

$$B(t) \geq \mu, \quad t > T_2 .$$

For $\mu > 0$ is arbitrary then,

$$\lim_{t\to\infty} B(t) = \infty . \tag{42}$$

Consider next sequence $\{t_n\}_{n=1}^{\infty} \subset [t_0, \infty), \lim_{n\to\infty} t_n = \infty$, can be used in:

$$\lim_{n\to\infty}[B(t_n) - A(t_n)] = \liminf_{t\to\infty}[B(t) - A(t)] .$$

From (35), then there exists *M* can be used in:

$$B(t_n) - A(t_n) \leq M , \quad n = 1,2,\cdots . \tag{43}$$

From (42),

$$\lim_{n\to\infty} B(t_n) = \infty . \tag{44}$$

From (43), we have:

$$\lim_{n\to\infty} A(t_n) = \infty . \tag{45}$$

From (43) and (44), when *n* is sufficiently large, we have:

$$\frac{A(t_n)}{B(t_n)} - 1 \geq -\frac{M}{B(t_n)} > -\frac{1}{2} .$$

Therefore when *n* is sufficiently large,

$$\frac{A(t_n)}{B(t_n)} > \frac{1}{2} .$$

From (45), we have:

$$\lim_{n\to\infty} \frac{A^{\alpha+1}(t_n)}{B^{\alpha}(t_n)} = \infty . \tag{46}$$

On the other hand using the Holder inequality, we have:

$$A(t_n) = \frac{1}{H(t_n,T)} \int_T^{t_n} \frac{|h(t_n,s)|}{R^{\alpha/(\alpha+1)}(s)} R^{\alpha/(\alpha+1)}(s) H^{\alpha/(\alpha+1)}(t_n,s) W(s) \mathrm{d}s$$
$$\leq [\frac{1}{H(t_n,T)} \int_T^{t_n} \frac{|h(t_n,s)|^{\alpha+1}}{R^{\alpha}(s)} \mathrm{d}s]^{1/(\alpha+1)} \cdot [\frac{1}{H(t_n,T)} \int_T^{t_n} R(s) H(t_n,s) W^{(\alpha+1)/\alpha}(s) \mathrm{d}s]^{\alpha/(\alpha+1)} .$$

Therefore:

$$A^{(\alpha+1)/\alpha}(t_n) \leq \frac{1}{\alpha} [\frac{1}{H(t_n,T)} \int_T^{t_n} \frac{|h(t_n,s)|^{\alpha+1}}{R^{\alpha}(s)} \mathrm{d}s]^{1/\alpha} [\frac{\alpha}{H(t_n,T)} \int_T^{t_n} R(s) H(t_n,s) W^{(\alpha+1)/\alpha}(s) \mathrm{d}s] .$$

Noted that $B(t)$ is definedby the above equation was when *n* is sufficiently large.

$$\frac{A^{(\alpha+1)/\alpha}(t_n)}{B(t_n)} \le \frac{1}{\alpha}\Big[\frac{1}{H(t_n,T)}\int_T^{t_n}\frac{|h(t_n,s)|^{\alpha+1}}{R^{\alpha}(s)}\mathrm{d}s\Big]^{1/\alpha}.$$

That is,

$$\frac{A^{\alpha+1}(t_n)}{B^{\alpha}(t_n)} \le \frac{1}{\alpha^{\alpha}H(t_n,T)}\int_T^{t_n}\frac{|h(t_n,s)|^{\alpha+1}}{R^{\alpha}(s)}\mathrm{d}s. \tag{47}$$

From (38),

$$\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)} > \eta.$$

Then there exists $T_3 > T$, can be used in:

$$\frac{H(t,T)}{H(t,t_0)} \ge \eta, \quad t > T_3.$$

Therefore when *n* is sufficiently large,

$$\frac{H(t_n,T)}{H(t_n,t_0)} \ge \eta. \tag{48}$$

From (47) and (48), we get:

$$\frac{A^{\alpha+1}(t_n)}{B^{\alpha}(t_n)} \le \frac{1}{\alpha^{\alpha}\eta H(t_n,t_0)}\int_{t_0}^{t_n}\frac{|h(t_n,s)|^{\alpha+1}}{R^{\alpha}(s)}\mathrm{d}s. \tag{49}$$

From (46) and (49), we get:

$$\lim_{n\to\infty}\frac{1}{H(t_n,t_0)}\int_{t_0}^{t_n}\frac{|h(t_n,s)|^{\alpha+1}}{R^{\alpha}(s)}\mathrm{d}s = \infty,$$

Which implies:

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\frac{|h(t,s)|^{\alpha+1}}{R^{\alpha}(s)}\mathrm{d}s = \infty.$$

Notice (30), this is contrary to condition (H_{12}). Therefore, our assertion (36) is established. However, by (36) and (33):

$$\int_T^{\infty}R(s)\varphi_+^{(\alpha+1)/\alpha}(s)\mathrm{d}s \le \int_T^{\infty}R(s)W^{(\alpha+1)/\alpha}(s)\mathrm{d}s < \infty.$$

Notice (30), this is contrary to condition (H_{13}).The proof is complete.

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