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Total Variation Differential Equation with Wavelet Transform for Image Restoration

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Abstract

To overcome the staircasing effects and simultaneously avoid edge blurring, we present an adaptive partial differential equation combining the total variation with wavelet transform for image restoration. A noise removal algorithm based on variation method and partial difference equations is proposed in this paper. The combining algorithm takes the advantage of both filters since it is able to preserve edges while avoiding the staircase effect in smooth regions. The TV method provides fast adaptive wavelet-based solvers for the TV model. Our approach not only employs a wavelet collocation method applied to the TV model using two-dimensional anisotropic tensor product of wavelets, but also proposes the differential equation for image restoration. Most conventional image processors consider little the influence of human vision psychology. The algorithm inherently not only combines the restoration property of wavelet compression algorithms with that of the TV model, but also gives a relative new TV functional which considers the influence of human vision psychology. We present a detailed description of our method which indicates that a combination of wavelet based restoration techniques with the TV model produces superior results. Experimental results illustrate the effectiveness of the model image restoration.

Keywords: *isotropic diffusion, TV model, wavelet transform, image restoration, Weber's law, Vision, psychophysics*

1. Introduction

The noises always exist through image acquisition and transmission. The presence of noise seriously affects the visual effects of the image and follow-up image processing. Wavelet transformation and the total variation (TV) equation are the most popular image denoising methods in recent years. In this paper, we present an adaptive multilevel total variation method for image denoising which utilizes TV partial differential equation model and exploits the properties of wavelets.In this paper, we use the terminology wavelet restoration and proposed to use the nonlocal total variation for this application. Our main contribution in this paper is to extend the total the total variation based wavelet restoration to the nonlocal total variation based model, in order to recover textures and geometry structures simultaneously.

The presence of noise in image is unavoidable. It may be introduced by the image formation process, image recording, image transmission, etc. These random distortions make it difficult to perform any required picture processing. The conventional image restoration model uses Partial Differential Equation (PDE). Based on Bayesian theory and variation problem, the restoration model is deemed as an energy function of image, and by minimizing the energy function the model restores the target region. It restores the image by minimizing the length energy function of image. The TV model diffuses only cross the isophote, and it restores the unknown region as straight lines. So the inpainted image is not a smooth image, and the target region contour is left. Bertalmio, et al. [1] introduced another inpainting PDE which diffuses only along the isophote. This equation smoothly restores the target region while preserving the isophotes in image. It adopts the PDEs which diffuse only along the isophote direction to inpaint. The inpainting result of this model is not good, and some linear structures in the inpainted image are not preserved. Lysaker, [4] also proposed a fourth order PDE inpainting approach to image interpolation model according to the axiomatic Total variation (TV) in order to remove noise while retaining important features, such as edges. This has been studied

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extensively in [2, 3] The TV restoration model was first proposed by Rudin. The model is to minimize the following energy:

$$E(f \mid f_0) = \int_\Omega |\nabla f| dx + \frac{\lambda}{2}\int_\Omega (f - f_0)^2 dx.$$

However, the diffusion resulting from minimizing the TV norm is strictly orthogonal to the gradient of the image, and tangent to the edges. That is to say, both from theoretical and experimental point of view, it has been shown that the TV-norm transforms smooth signal into piecewise constants, the so-called staircase effect. In order to resolve this problem, there are almost two solutions. One is this model [7, 8] which used a combination of TV diffusion where there are likely edges $(|\nabla f| > \varepsilon)$ and isotropic diffusion in more homogeneous regions $(|\nabla f| < \varepsilon)$. This minimization problem is:

$$\min_{f \in BV(\Omega)} \frac{1}{2\varepsilon} \int_{|\nabla f| \le \varepsilon} |\nabla f|^2 dx + \int_{|\nabla f| > \varepsilon} |\nabla f| dx + \frac{\lambda}{2}\int_\Omega (f - f_0)^2 dx.$$

Our ideas indicate that using wavelets to compress TV denoised images results in a higher compression ratio than the regular wavelet methods [9]. Superior denoised images are obtained from the adaptive TV method when compared to those obtained from wavelet or TV denoising alone. In addition, we note that solving the PDE in the wavelet domain [10] is less expensive than solving the PDE in the image domain on the full grid. We note that the denoised images obtained from using the method of nonlocal means are superior, but this algorithm is vastly more computationally intensive. The paper is organized as follows: Section two introduces the total variation model and discusses the numerical technique used to solve the associated PDE. Section three reviews the background behind Daubechies-type wavelets and indicates how wavelet coefficients may be used to generate sparse grids for use in numerical PDE computations. Section four presents results from several numerical experiments involving the TV model, wavelet-based image denoising, and the wavelet solution of the TV model.

For simplicity, we introduce the notation $D^2 u = \left(\nabla u_x \cdot \nabla u_x + \nabla u_y \cdot \nabla u_y\right)^{\frac{1}{2}}$

$$\|u\|^2_{L_2(\Omega)} = \left|\langle u, 1\rangle\right|^2 + \sum_{j,k,\psi} \left|\langle u, \psi_{j,k}\rangle\right|^2 \tag{1}$$

If the wavelet Ψ in space $B_p^\beta\left(L_p(\Omega)\right)$, then there is an equivalence relation mmm

$$|u|_{B_p^\alpha\left(L_p(\Omega)\right)} \approx \left(\sum_{j,k,\psi} 2^{\alpha k p} 2^{k(p-2)} \left|\langle u, \psi_{j,k}\rangle\right|^p\right)^{1/p}$$

Here only consider the special circumstances $p = 1$, $\alpha \le 1$ that space $B_1^\omega\left(L_1(\Omega)\right)$.For convenience, write $u_\omega = \langle u, \psi_{j,k}\rangle$, $\omega \in \Gamma$, then:

$$|u|_{B_1^\alpha\left(L_1(\Omega)\right)} \approx 2^{k(\alpha-1)} \left\|(u_\omega)\right\|_{\ell_1(\Gamma)} \tag{2}$$

In which,

$$S = \left\{i, j, k \,\middle|\, i = 1, 2, 3 \;\; k \ge 0,\; j \in \left\{0, 1, \ldots 2^k - 1\right\}\right\}$$

Furthermore, to improve the edge-preserving capability, Strong and Chan [7, 8] presented the adaptive TV approach to image restoration:

$$\min_{u\in BV(\Omega)\cap L^2(\Omega)}\int_\Omega \alpha(x)|\nabla u|dx+\frac{\lambda}{2}\int_\Omega (u-u_0)^2dx.$$

Where $\alpha(x)$ is a spatially and scale adaptive function, $\alpha(x)=\frac{1}{1+k|\nabla G_\sigma * u_0|}$ as an edge stopping function used for controlling the speed of the diffusion, where k represents a threshold parameter, and $G_\sigma(x)=\frac{1}{2\pi\sigma^2}\exp\left(-\frac{|x|^2}{2\sigma^2}\right)$ denotes the Gaussian filter with parameter σ. The corresponding theory of viscosity solutions was investigated there in detail.

The paper is organized as follows: In section 2, we present the adaptive total variation model and the associated partial differential equations. In section 3, we introduce the unified model and general algorithm framework based on variation and wavelet transform denoising algorithm. In section 4, further numerical examples highlight the remarkable restoration qualities of non local total variation regularization for natural images.

2. Models and Related Algorithms

We all know that all images are eventually perceived and interpreted by the human visual system. As a result, vision psychology and psychophysics play an important role in the successful communication of image information. This fact implies that any ideal image processor should take into account the consequences of vision psychology and psychophysics. The current paper makes an attempt in this direction. We develop an image restoration model that intends to incorporate one of the most well known and influential psychological results—Weber's law for sound and light perception.We study the computational strategy for the associated nonlinear PDE.

Most conventional image processors [9, 10] consider little the influence of human vision psychology and many conventional restoration models don't take into account that our visual sensitivity to the local fluctuation depends on the ambient intensity level. That is, models such as (2) assume that a local variation should be treated equally independent of the background intensity level. So in this paper the minimization problem:

$$\Psi(u)=\int_\Omega \alpha(x)\frac{f(|Du|)}{u}dxdy++\frac{\lambda}{2}\int_\Omega (u-u_0)^2dxdy. \tag{3}$$

To simplify the complication of the problem, we propose the much simpler model:e manuscript with other papers, that it is innovative, it are used in the chapter "Research Method" to describe the step of research and used in the chapter "Results and Discussion" to support the analysis of the results [5]. If the manuscript was written really have high originality, which proposed a new:

$$\Psi(u)=\int_\Omega (\alpha(x))\frac{|Du|}{u}dxdy++\frac{\lambda}{2}\int_\Omega (u-u_0)^2dxdy. \tag{4}$$

The following will put forward the differential equation from the above model. Where $\alpha(x)$ is a spatially and scale adaptive function, $\alpha(x)=\frac{1}{1+k|\nabla G_\sigma * u_0|}$ as an edge stopping function used for controlling the speed of the diffusion, where k represents a threshold parameter, and $G_\sigma(x)=\frac{1}{2\pi\sigma^2}\exp\left(-\frac{|x|^2}{2\sigma^2}\right)$ denotes the Gaussian filter with parameter σ.

The corresponding theory of viscosity solutions was investigated there in detail.

2.1. The Eular-Laglange Equation with the Model

Most conventional restoration models(6,7) do not take account that our visual sensitivity to the local fluctuation depends on the ambient intensity level. Therefore, technically we should stay away from the black hole and assume that $u > 0$. The black hole similarly imposes some natural restrictions on the noise model: $u_0 = u + \alpha$. Since u_0 also represents the intensity value, we must have $u_0 > 0$, which implies that $u > -\alpha$. The condition $u > -\alpha$ is equivalent to $u > |\alpha|$. Therefore $u_0 = u + \alpha \le u + u = 2u$. The combination of all the elements discussed above leads to the following natural admissible space for the new Weber TV restoration (4):

$$S = \left\{ u > 0 \middle| u \in BV(\Omega), \int_\Omega \alpha(x) \frac{|Du|}{u} dxdy < \infty, u \ge \frac{1}{2} u_0 \right\}$$

This is the space that we shall work with from now on.

Theorem 1. Suppose $D^2u = (\nabla u_x \cdot \nabla u_x + \nabla u_y \cdot \nabla u_y)^{\frac{1}{2}}$, Then the variation functional will become:

$$\Psi(u) = \iint_\Omega \alpha(x) \frac{(\nabla u_x \cdot \nabla u_x + \nabla u_y \cdot \nabla u_y)}{u} dxdy + \frac{\lambda}{2} \iint_\Omega |u - u_0|^2 dxdy$$

Then the formal Euler-Lagrange differential of $\Psi(u)$ is:

$$-\frac{1}{u} \nabla \cdot \left(\alpha(x) \frac{\nabla u_x}{|Du|} \right) - \lambda(u - u_0) = 0$$

2.2. Total Variation and Wavelet Inform Image Denoising

In [12], the author proposed one TV regularized wavelet restoration models depending on whether or not noise is considered. The idea is to combine a regularization term in the image domain. For a two-dimensional image u , let us denote the standard wavelet representation as:

$$u_\omega = u(\omega, x) = \sum_{j \in Z, k \in Z} \omega_{j,k} \phi_{j,k}(x),$$

Where $x \in \Omega = [1, M] \times [1, N]$, and $\omega = (\omega_{j,k})$ denotes wavelet coefficients of u at level j and location *k.* And for simplicity $\phi_{j,k}$ denotes a given orthogonal or biorthogonal wavelet basis function. If we use an orthogonal basis, the coefficient $\omega_{j,k} = \langle \omega, \phi_{j,k} \rangle$. In discrete case, let $I \subset \Omega$ be the uncorrupted known index set, $\alpha = (\alpha_{j,k})$ for $(j, k) \in I$ denotes measured coefficients, the following two models, respectively for the noiseless case and the noisy one,are considered in the paper [12]:

$$\omega^* = \arg\min_\omega TV(u(\omega, x)) = \arg\min_\omega \sum_{x \in \Omega} |\nabla_x u(\omega, x)|$$
$$s.t \quad \omega_{j,k} = \alpha_{j,k}, \quad j, k \in I$$

Lemma: Let $\{\psi_\omega\}_{\omega\in\Gamma}$ be a Hilbert space a set of orthogonal wavelet bases, f_ω, u_ω are the basis Functions f and u in this group under the wavelet coefficients, the smallest solution of functional is:

$$F(u_\omega)=\alpha(x)\|u_\omega - f_\omega\|^2_{\ell_2(\Gamma)}+2\lambda\|u_\omega\|_{\ell_1(\Gamma)}$$

$$u_\omega = W_\lambda(f_\omega)=\begin{cases} f_\omega-\phi, & f_\omega\ge\phi \\ 0, & -\phi\le f_\omega\le\phi \\ f_\omega+\phi, & f_\omega\le-\phi \end{cases}$$

In which, W_λ is for the wavelet soft threshold operator. Basov space is the space of bounded variation near the minimum [11] and space images is not allowed in the border. So with a smooth of order, Basov space $C_1^\alpha\left(L_2(\Omega)\right)$. Space to describe the image of the regularity, Get a new variation functional, then by minimizing the variation function restored image. Using Basov norm of wavelet coefficients can describe the nature of the equivalent norms will solve the problem is transformed to the wavelet domain minimum by iterative calculation. High frequency components of wavelet transform has a wealth of details of the edge information, so it can reconstruct high quality images, and the introduction of wavelet algorithm makes the text of the new running time is short, and fast speed. This model overcomes slow Chambolle image restoration, the shortcomings of a long time, and have significantly improved the quality of the image. Not only that, the paper taking into account the psychology of the visual effect of image restoration, proposed a new variation functional, thus noise model:

$$\min_{u\in X}\|u-u_0\|^2_{L_2(\Omega)}+2\lambda\alpha(x)\left|\frac{\nabla u}{u}\right|_{B_1^\alpha(L_1(\Omega))}$$

Among them, Under the equivalence relation (1) and (2), there:

$$\|u-u_0\|^2_{L_2(\Omega)}\approx\|u_\omega-u_{0\omega}\|^2_{\ell_2(\Gamma)}$$

$$\left|\frac{\nabla u}{u}\right|_{B_1^\alpha(L_1(\Omega))}\approx 2^{k(\alpha-1)}\left\|\frac{\nabla u_\omega}{u_\omega}\right\|_{\ell_1(\Gamma)}$$

The first two β wavelet coefficients function, resulting in the following sequence of convex functional equivalen.

$$Q_g(u_\omega,w_\omega)=\|u_\omega-u_{0\omega}\|^2_{\ell_2(\Gamma)}+2\lambda 2^{k(\alpha-1)}\alpha(x)\left\|\frac{\nabla u_\omega}{u_\omega}\right\|_{\ell_1(\Gamma)} \quad (5)$$

About to u_ω take the smallest $u_{0\omega}\in Z_\omega$, where the wavelet coefficients Z_ω so that low-frequency part of the zero in the space $l_2(\Gamma)$. Equivalent to consider the following two minimization problems: $\min_{u_\omega\in l_2}\|u_\omega-u_{0\omega}\|^2_{\ell_2(\Gamma)}+2\lambda 2^{k(\alpha-1)}\alpha(x)\left\|\frac{\nabla u_\omega}{u_\omega}\right\|_{\ell_1(\Gamma)}$

Fixed u_ω, find the functional (5) on the minimum solution u_ω, According to Lemma,

$$u_\omega = Q_{2^{k(\alpha-1)}\lambda}(u_{0\omega})$$

Namely: $\min_{u \in Z_\omega} \|u_\omega - u_{0\omega}\|^2_{\ell_2(\Gamma)}$, solution of the minimum: $u_{0\omega} = T_L(u_\omega)$

Where the wavelet coefficients of the function T_L, that the low-frequency part of the threshold value is zero, to sum up functional (5) minimization of the solution can be obtained by iteration, the algorithm is Algorithm:

1. Initialization: $(u_\omega)_0 = 0$;
2. Iteration: $(u_\omega)_n = W_{2^{k(\alpha-1)}\lambda}\left((u_{0\omega})_n\right)$;
3. Stop condition: $u_n = \sum_\omega (u_\omega)_n \psi_\omega$, $u_{n+1} = \sum_\omega (u_\omega)_{n+1} \psi_\omega$;

Assumption ϕ is that a pre-given small positive number, If you meet the conditions $\max(|u_{n+1} - u_n|) \le \phi$, to stop the iterative.

3. Test and Results

The following new algorithm will be the text with a single Weber total variation denoising model comparison. In order to reduce blurring the edges of the image and block effects, the paper uses translation invariance wavelet transform, only a layer of wavelet transform. Use db4 wavelet, the experimental results shown in Figure 1 and Figure 2. Figure 1 is a total variation model using the method of (256 × 256) image denoising results. It can be seen, the original model are more ambiguous edge. Figure 2 is a city (256 × 256) image using the original method of Culture and the denoising results. Objectively, you can use the peak signal to noise ratio (PSNR) and mean square error (MSE) to evaluate the effects of good or bad image denoising. Close to the peak signal to noise ratio is a more effective evaluation of the human visual content. Mean square error of the restored image with the standard used to measure the closeness of the image. In addition, taking into account the practical feasibility of the time (Time) is also used as an evaluation index. Using these three indicators of the two methods of quantitative analysis of image restoration results, the experimental data shown in Table 1, we can see that the proposed method is not only fast, with time short and get better quality restored image.

In this section, we present a series of numerical experiments which show that image denoising performed with the adaptive TV method with wavelet denoising .Now we present some numerical results. We use the standard Peak Signal to Noise(PSNR) to quantify the performance of wavelet coefficient filling:

$$PSNR(u, \overline{u}) = 10 \log_{10}^{\left(\frac{1}{\|u-\overline{u}\|^2}\right)}$$

(a) (b) (c)

Figure 1. Woman Image

A is corrupted with additive white noise at a rate of 17.6174 PSNR; B is processed by TV algorithm for 50 times with λ=0.028， Δt =0.2, h=1,and the result is at a rate of 22.0630 PSNR; C is processed by our new algorithm for 20 times with wavelet'db6',Δt =0.2, h=2,and the result is at a rate of 22.7335 PSNR;

Table 1. Image Woman

Image Woman	TV method	Our new algorithm
Iterations	50	20
Wavelet	----	db4
PSNR	22.1530	22.8450
Time	4.072016s	0.948516s

(a) (b) (c)

Figure 2. City Image

The image was first corrupted with additive white noise at a rate of 29.6 PSNR. We then executed 200time steps of the TV algorithm with $\Delta t = 0.2, \lambda = 3,$ and $h = 1$. A is the original image; B is restored with TV model at a rate of 17.2929 PSNR; C is processed by our new algorithm for 20 times, $\Delta t = 0.2, h = 2$,and the result is at a rate of 22.8999 PSNR;

TV model and the new algorithm compared the results of image restoration:

Table 2 Image City

Image	TV method	Our new algorithm
Iterations	50	20
Wavelet	----	Db4
PSNR	22.3318	22.7888
Time	7.445699s	3.322172s

Furthermore,we present the other numerical experiments to illustrate the efficiency and feasibility of our novel method. We have tested the proposed method. This text will compare improved TV model with traditional model on image restoration effect. The below is the brief procedure of repairing algorithm:

Details: For each pixel in the region to be repaired

(i) Input noise image to be processed, compute the vertical/horizontal smoothness variation using Laplace;

(ii) Computing the isophotes direction;

(iii) Propagating smoothness variations along to the isophotes;

(iv) Updating the value of the pixel through the combination of(i)–(iii);

Iterate step: (i)–(iv)

During the restoration, diffusion procedure was interleaved with one per tenth repairing loop to ensure noise insensitivity and preserve the sharpness of edges. We performed the restoration algorithm until the pixel values in the regions to be restored did not change. We use Gauss-Jacobi iterate methods, this completes the last conclusion.

The below two experiment is based on traditional TV model and TV model proposed by this paper for image restoration. In the following, the size of Lena image is 256×256; grey degree: 256; depth degree: 8. The contrast on the effect of image restoration:

(a) (b) (c)

Figure 3. Lena Image

Table 3. Time and Square on Traditional TV Model

T(time/second)	MSE	MSE1
98.13767	246.60331	2.4761e+004

Table 4. Time and Square on Improved TV Model

T(time/second)	MSE	MSE1
23.326316	6.5603e+003	2.2880e+004

The other example is the size of toys image is 640×640; grey degree: 256; depth degree: 8. The contrast on the effect of image restoration:

(a) (b) (c)

Figure 4. Plane Image

Table 5 Time and square on traditional TV model

T(time/second)	MSE	MSE1
422.21138	4.0568e+003	3.1086e+004

Table 6. Time and Square on Improved TV Model

T(time/second)	MSE	MSE1
278.535649	8.1456e+003	3.1056e+004

From the above two test effect of two restoration algorithm, we can know the restoration time on improved TV model is less than the traditional model. The restoration efficiency is higher, consuming time is shorter. In a word, the restoration effect of the improved algorithm is much better than the traditional algorithm.

4. Conclusion

This paper presents a adaptive differential equation combining the total variation with Wavelet transform for image denoising is presented. The presented adaptive TV method provides fast adaptive wavelet-based solvers for the TV model. This approach employs a wavelet collocation method applied to the TV model using two-dimensional anisotropic tensor product of wavelets. This algorithm inherently not only combines the denoising property of wavelet compression algorithms with that of the TV model, but also gives a relative new TV functional, and produces results superior to each method when implemented alone. Furthermore this paper considers the human psychology system. Of course, this point adds the difficult extent of the proposed problem, because this paper add the influence of human vision psychology for the regularity item. It exploits the edge preservation property of the TV model to reduce the oscillations that may be generated around the edges in wavelet compression. We present a detailed description of our method and results which indicate that a combination of wavelet based denoising techniques with the TV model produces superior results.

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