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A new approximate solution of the fractional trigonometric functions of commensurate order to a regular linear system

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ABSTRACT

This paper introduces a novel approach to the approximate solution of linear differential equations associated with principal fractional trigonometry and the R function. This method proposes a solution that is expressed by adding appropriate fractional linear fundamental functions. Laplace transforms of these functions are irrational. Therefore, we rounded these functions to obtain rational functions in the form of damped cosine, damped sine, cosine, sine and exponential functions. This transformation was achieved by utilizing the concept of fractional commensurate order and, as a result, has direct practical relevance to real-world physics. The precision and effectiveness of the approach are demonstrated through illustrative examples of solving fractional linear systems.

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1. INTRODUCTION

The field of fractional calculus focuses on derivatives and integrals with non-integer orders in mathematics. In recent years, it has gained increasing attention in multiple areas of scientific and engineering, including control engineering, physical wave phenomena, heat and mass transfer, electromagnetic theory, neural networks, biological and medical treatment, mechanics, and more [1], [2].

Many researchers have found that using fractional calculus can provide a more accurate description of complex systems and phenomena that cannot be fully explained by classical integer-order calculus. Therefore, the statement you provided is consistent with the current understanding of the applications of fractional calculus [3]–[6]. Hence, researchers are currently working to develop precise and efficient methods for solving linear fractional-order differential equations and establish a clear linear fractional systems theory that can be readily understood by engineers. In recent years, there has been a notable focus on developing analytical techniques to solve these equations, with the aim of obtaining a mathematical expression that provides an explicit solution to fractional-order linear differential equations. See previous works [7]–[10].

Although the study of Hopf bifurcation in dynamical models of integer order is well-established, Hopf bifurcation in dynamical models of fractional order has received relatively little attention, as indicated by the limited number of prior studies [11]. In delayed neural networks, the emergence of Hopf bifurcation as a result of delays is a notable dynamic phenomenon. Despite its importance, research on delay-induced Hopf bifurcation in neural networks of fractional-order is relatively scarce, as indicated by the limited number of previous studies [12]. Furthermore, the scarcity of research on Hopf bifurcation in models of prey-predator is evident from the limited number of previous studies [13], [14].

Numerous works have been published on fractional-order models, covering a diverse range of topics. For example, in their study, Xu *et al.* [1] investigated a stage-structured of fractional-order in predator-prey system with both distributed and discrete time delays. They obtained an isovalent version of the model, which provided a more accurate description of the system's dynamics compared to the classical integer-order model. Albadarneh *et al.* [15] introduced new and powerful formulas for sequential solutions with residual errors to approximate the Riemann-Liouville fractional derivatives operator. These formulas were derived using explicit calculations and using weighted mean value theorem (WMVT), a theory used to develop new approaches that allow the solution of many fractional order differential equations, both linear and nonlinear. These formulas are an important improvement and provide accurate and powerful solutions to Riemann-Liouville fractional derivatives. A study by Shihab *et al.* [16] showed the use of the method of variational iteration in solving several types of partial differential equations, whether linear or nonlinear. In this study, it was shown that the Lagrange multiplier can be used to determine an ideal value of parameters in a functional form, and then use these values to construct an iterative series solution. This option is viable and effective in solving partial differential equations. Jameel *et al.* [17] the method of Bezier's curves was introduced and modified to solve fuzzy delay problems while taking advantage of the properties of fuzzy set theory. The approximate solution was compared to different degrees with the exact solution to ensure that the process of differential equations of the fuzzy linear delay is accurate and efficient. The authors presented insightful numerical findings in the form of graphical representations, elucidating the influence of both the recycle ratio and the fractional order on the model's dynamic evolution. Interested readers can refer to the bibliography on these topics for more information [10], [18]–[21].

The aim of this research is to find new solutions for linear fractional systems with commensurate order specifically, the focus is on systems related to fractional trigonometry and real fractional exponential functions in the form of the R function. This is done by using appropriate fundamental functions that correspond to the range of commensurate order, and by using Laplace transforms for these functions. The solution can be expressed as a linear combination of generalized exponential functions, which fulfill a similar role as classical exponential functions.

Our planned approach for this work is outlined as follows: In section 2, we begin by introducing the fractional generalization of the exponential function, which serves as the foundation for deriving the specialized Laplace transforms and R-function from the principal meta-trigonometric fractional order functions. Moving on to section 3, we showcase the accuracy and practicality of our proposed analytical approach by presenting and analyzing the results obtained thus far. Finally, in section 4, we conclude this article by summarizing the key findings and implications of our work.

2. METHOD

2.1. Fractional meta-trigonometry: a generalization of classical trigonometry for fractional calculus applications

The applications of traditional (integer-order) trigonometry extend far beyond the calculation of triangles and triangulation and are widely used in analysis, engineering, and science. Trigonometric functions hold tremendous significance in various mathematical domains, including spectral analysis, fourier analysis, and the solutions to both ordinary and partial differential equations, and many other mathematical fields, making them an essential part of modern mathematics.

Fractional meta-trigonometry [22]–[25] provides a generalization of classical trigonometry and encompasses an infinite set of fractional trigonometric based on the R-function [26], which is:

$$R_{q,\upsilon}\left(ai^{\alpha}, i^{\beta}t\right), t \succ 0, i = \sqrt{-1}, q, \upsilon, \alpha, \beta \in \Re \tag{1}$$

and are based on the definition:

$$R_{q,\upsilon}(a,t) = \sum_{n=0}^{\infty} \frac{a^n t^{(n+1)q-1-\upsilon}}{\Gamma((n+1)q-\upsilon)}, t \succ 0 \tag{2}$$

the meta-trigonometric functions are derived based on the intricacy of (1), specifically considering its real and imaginary components, are defined as follows for t > 0:

$$\sin_{q,\upsilon}(a,\alpha,\beta,\kappa,t) \equiv \sum_{n=0}^{\infty} \frac{a^n t^{(n+1)q-1-\upsilon}}{\Gamma((n+1)q-\upsilon)} \times \sin\left((n(\alpha+\beta q)+\beta[q-1-\upsilon])\left(\frac{\pi}{2}+2\pi\kappa\right)\right) \tag{3}$$

$$cos_{q,v}(a,\alpha,\beta,\kappa,t) \equiv \sum_{n=0}^{\infty} \frac{a^n t^{(n+1)q-1-v}}{\Gamma((n+1)q-v)} \times cos\left((n(\alpha+\beta q)+\beta[q-1-v])\left(\frac{\pi}{2}+2\pi\kappa\right)\right) \quad (4)$$

these following functions corresponding to Laplace transforms can be expressed as [24]:

$$L\{sin_{q,v}(a,\alpha,\beta,t)\} = s^v \left[\frac{s^q\, sin(\lambda) - a\, sin(\lambda-\sigma)}{s^{2q} - 2a\, cos(\sigma) s^q + a^2}\right], q > v \quad (5)$$

$$L\{cos_{q,v}(a,\alpha,\beta,t)\} = s^v \left[\frac{s^q\, cos(\lambda) - a\, cos(\lambda-\sigma)}{s^{2q} - 2a\, cos(\sigma) s^q + a^2}\right], q > v \quad (6)$$

with $\lambda = \beta\,[q-1-v](\pi/2+2\pi k)$ and $\sigma = (\alpha+\beta q)(\pi/2+2\pi k), q, v, a, \alpha, \beta \in \Re$

$$L\{sin_{q,v}(a,\alpha,\beta,t)\} = s^v \left[\frac{s^q\, sin(\lambda) - a\, sin(\lambda-\sigma)}{s^{2q} - 2a\, cos(\sigma) s^q + a^2}\right] = [s^v] \times \left[\begin{array}{l} sin(\lambda)\left(\frac{s^q + \|\lambda'\|\zeta}{(s)^{2q} - 2a\, cos(\sigma) s^q + a^2}\right) - \\ \left(\frac{a\, sin(\lambda-\sigma)}{\|\lambda'\|^2} + \frac{\zeta\, sin(\lambda)}{\|\lambda'\|}\right)\left(\frac{\|\lambda'\|^2}{(s)^{2q} - 2a\, cos(\sigma) s^q + a^2}\right) \end{array}\right] \quad (7)$$

$$L\{cos_{q,v}(a,\alpha,\beta,t)\} = s^v \left[\frac{s^q\, cos(\lambda) - a\, cos(\lambda-\sigma)}{s^{2q} - 2a\, cos(\sigma) s^q + a^2}\right] = [s^v] \times \left[\begin{array}{l} cos(\lambda)\left(\frac{s^q + \|\lambda'\|\zeta}{(s)^{2q} - 2a\, cos(\sigma) s^q + a^2}\right) - \\ \left(\frac{a\, cos(\lambda-\sigma)}{\|\lambda'\|^2} + \frac{\zeta\, cos(\lambda)}{\|\lambda'\|}\right)\left(\frac{\|\lambda'\|^2}{(s)^{2q} - 2a\, cos(\sigma) s^q + a^2}\right) \end{array}\right] \quad (8)$$

with $\|\lambda'\| = a$.

To effectively solve the differential equation for fractional-order linear systems, it is crucial to approximate the irrational functions F1(s) and F2(s) of $\left(\frac{s^q + \|\lambda'\|\zeta}{(s)^{2q} - 2a\, cos(\sigma) s^q + a^2}\right)$ and $\left(\frac{\|\lambda'\|^2}{(s)^{2q} - 2a\, cos(\sigma) s^q + a^2}\right)$ with rational functions. In (7) and (8) provide the relevant context for this approximation.

2.2. Fundamental functions approximations

In previous works [27]–[32] have approximated the elementary fundamental functions using rational functions. This approach facilitated the representation of these functions through linear time-invariant system models. Consequently, their closed-form impulse and step responses could be derived, and their performance characteristics could be examined. Moreover, these approximations played a crucial role in enabling the derivation of straightforward analogue circuits capable of accurately representing the complex irrational functions associated with the fractional-order system.

Note that all these fundamental functions are irrational functions. Then, to establish the explicit expressions for solving linear differential equations of fractional order and to study of the dynamic behavior of fractional-order systems, similar to the analysis of regular linear systems, the irrational functions $H_k(s)$ must be approximated by rational functions. We followed the methods of [27]–[32], the fundamental functions have been approximated by rational functions of practical interest in a frequency band. This approximation allowed for the representation of these functions by time-invariant linear system models.

3. RESULTS AND DISCUSSION

In order to highlight the effectiveness of the proposed method. We will present a numerical example that has been implemented in MATLAB on a PC. The numerical example will showcase the results obtained through the implementation, providing evidence of the efficacy and applicability of the proposed method.

3.1. q = 0.23 (0 < q < 0.5)

We followed the methods of [28], [30]-[32], the fundamental functions fdsin $(t, \lambda_3, 0.23)$, fdcos $(t, \lambda_3, 0.23)$, which can be summarized as follows:

$$\text{fdsin}(t,\lambda_3,0.23) = L^{-1}\left\{\frac{5}{[s^{0.46}+(4.47)\zeta s^{0.23}+5]}\right\} = L^{-1}\left\{\sum_{i=1}^{N_3} k_{3i} \frac{a_{3i}s+\sqrt{1-\zeta^2}}{s^2+2\beta\omega_i s+\omega_i^2}\right\} = \sum_{i=1}^{N_3} \omega_i k_{3i}\, exp(-\beta\omega_i t)\, sin\left(\omega_i\left(\sqrt{1-\beta^2}\right)t - (3.35)\phi_3\right)$$

$$\text{fdcos}(t,\lambda_3,0.23) = L^{-1}\left\{\frac{s^{0.23}+(2.24)\zeta}{[s^{0.46}+(4.47)\zeta s^{0.23}+5]}\right\} = L^{-1}\left\{\sum_{i=1}^{N_4} k_{4i} \frac{a_{4i}s+\zeta}{s^2+2\beta\omega_i s+\omega_i^2}\right\} = \sum_{i=1}^{N_4} \omega_i k_{4i}\, exp(-\beta\omega_i t)\, cos\left(\omega_i\left(\sqrt{1-\beta^2}\right)t - (3.35)\phi_4\right)$$

with $\|\lambda_3\| = 2.24$, $\zeta = 0.97$, $\beta = 0.48$, $\varphi_3 = \varphi_4 = 0.25$, $N_3 = N_4 = 821$ and the parameters ω_i, a_{3i}, $a_{4i}$and $k_{3i} = k_{4i}$ are given, for $1 \le i \le 821$, as:

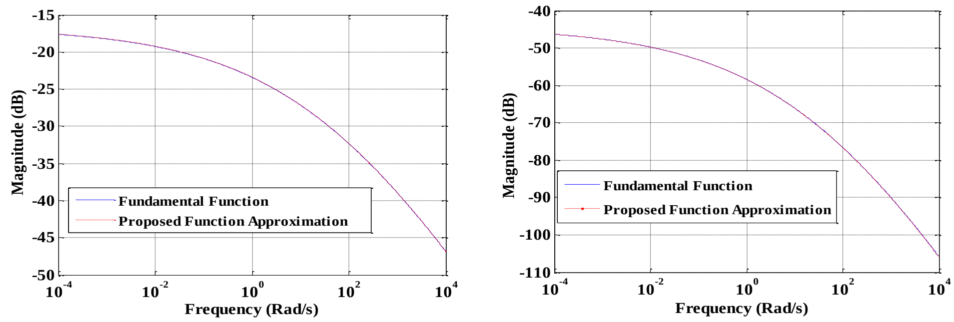
$$\omega_i = \frac{1}{(0.03)*(1.10)^{(411-i)}},\ a_{3i} = (-0.022)*(1.10)^{(411-i)},\ a_{4i} = (0.022)*(1.10)^{(411-i)},$$

$$k_{3i} = k_{4i} = \frac{1}{(0.002)*(1.10)^{2(411-i)}}\left[\frac{sin[0.77\pi]}{cosh[(0.23)\,log((1.10)^{(411-i)})]-cos[0.77\pi]}\right]$$

$$L\{sin_{q,v}(a,\alpha,\beta,t)\} = [s^{v}] \times \left[\left(\frac{s^{0.23}+(2.24)\zeta}{[s^{0.46}+(4.47)\zeta s^{0.23}+5]}\right) + 5.43*10^{-6}\left(\frac{5}{[s^{0.46}+(4.47)\zeta s^{0.23}+5]}\right)\right]$$

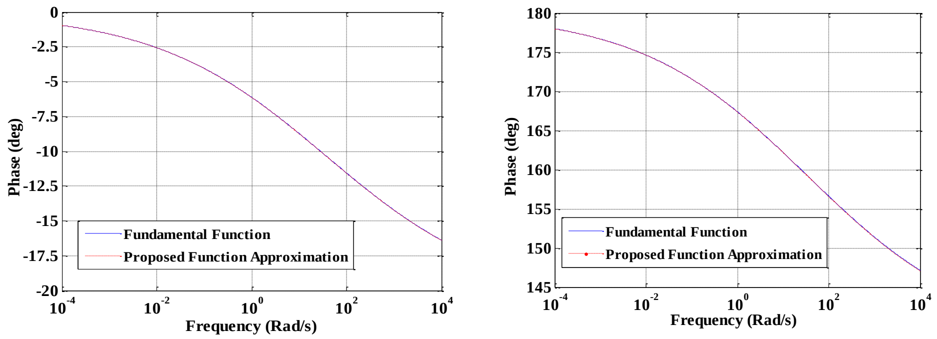
$$L\{cos_{q,v}(a,\alpha,\beta,t)\} = [s^{v}] \times \left[1.57*10^{-6}\left(\frac{s^{0.23}+(2.24)\zeta}{[s^{0.46}+(4.47)\zeta s^{0.23}+5]}\right) - 0.11\left(\frac{5}{[s^{0.46}+(4.47)\zeta s^{0.23}+5]}\right)\right].$$

Figure 1 and Figure 2 illustrate the bode plots representing the transfer function of the fundamental fractional-order system, as well as the corresponding rational function approximations proposed by our method. The bode plots of L {sinq, υ (a, α, β, t)} Figure 1(a), Figure 2(a) (magnitude) and L {cosq, υ (a, α, β, t)} Figure 1(b), Figure 2(b) (phase) were generated, accompanied by their respective rational function approximations obtained using the proposed method. Remarkably, both curves exhibit an exact match, confirming the accuracy and effectiveness of the proposed method.



(a) (b)

Figure 1. Magnitude bode plot of (a) L {$sin_{q,\upsilon}$ (a, α, β, t)} and (b) L {$cos_{q,\upsilon}$ (a, α, β, t)}, respectively, along with their corresponding rational function approximations proposed by the method



(a) (b)

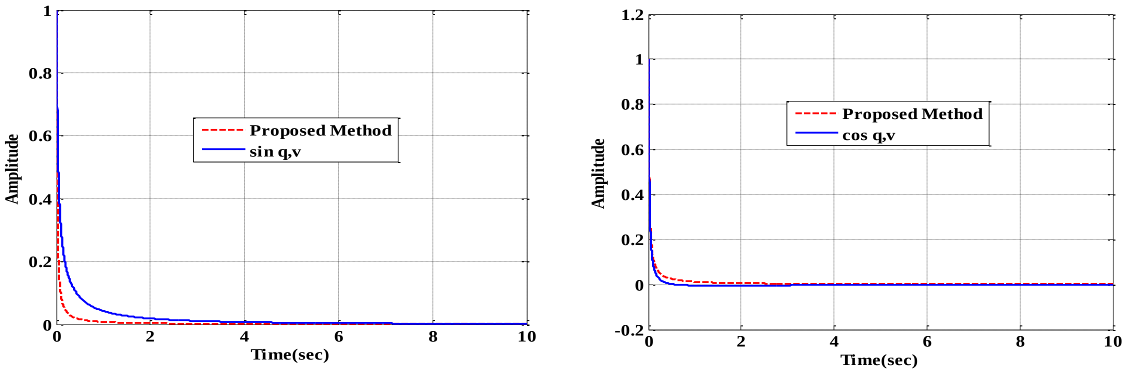
Figure 2. Phase bode plot of (a) L {$sin_{q,\upsilon}$ (a, α, β, t)} and (b) L {$cos_{q,\upsilon}$ (a, α, β, t)}, respectively, along with their corresponding rational function approximations proposed by the method

Upon examination, it is evident that the functions significantly overlap within the frequency band of interest. The significant overlap observed between the bode plots indicates that the proposed rational function approximation provides a good approximation of the original fractional-order system within the frequency band of interest. Nevertheless, it is important to note that the accuracy of the approximation may vary outside this frequency band and may depend on the type and range of the fractional order q. Furthermore, the choice of rational function approximation may also depend on the nature of the poles and their impact on the decay or growth rate of the model.

The inverse laplace transforms of the functions, $\left(\frac{s^{0.23}+(2.24)\zeta}{[s^{0.46}+(4.47)\zeta s^{0.23}+5]}\right)$ and $\left(\frac{5}{[s^{0.46}+(4.47)\zeta s^{0.23}+5]}\right)$ are, respectively, given by the expressions fdcos $(t, \lambda_3, 0.23)$ and $fd\sin$ $(t, \lambda_3, 0.23)$. So, the inverse laplace transform of (7) and (8), $\sin_{q,\upsilon}$ (a, α, β, t) and $\cos_{q,\upsilon}$ (a, α, β, t) are given by [28], [30]–[32].

$$\begin{cases} sin_{q,v}(a,\alpha,\beta,t) = \text{fdcos}\ (t, \lambda_3, 0.23) + 5.43*10^{-6}\text{fdsin}\ (t, \lambda_3, 0.23) \\ cos_{q,v}(a,\alpha,\beta,t) = 1.57*10^{-6}\text{fdcos}\ (t, \lambda_3, 0.23)\text{-}0.11\text{fdsin}\ (t, \lambda_3, 0.23) \end{cases}$$

Figure 3 illustrates the time-domain plots. The Figure 3(a) showed the function $\sin_{q,\upsilon}$ (a, α, β, t) and the Figure 3(b) illustrate the function $\cos_{q,\upsilon}$ (a, α, β, t). The solutions obtained are presented as a linear combination of basic functions, which have been meticulously chosen and are composed of a linear combination of trigonometric and regular exponential functions, just as is the case with regular linear systems.



(a) (b)

Figure 3. Shows the time-domain plots of the functions $\sin_{q,\upsilon}$ (a, α, β, t) (a) and $\cos_{q,\upsilon}$ (a, α, β, t) and (b) with respect to time (t), using the parameters a=2.24, α =2.1424, β =-1.2987, q=0.23 and υ =0

3.2. q = 0.77 (0.5 < q < 1)

We followed the methods of [28, 30-32], the fundamental functions fdsin $(t, \lambda_3, 0.77)$, fdcos $(t, \lambda_3, 0.77)$, which can be summarized as follows:

$$\text{fdsin}\ (t, \lambda_3, 0.77) = L^{-1}\left\{\frac{5}{[s^{1.54}+(4.47)\zeta s^{0.77}+5]}\right\} = L^{-1}\left\{\sum_{i=1}^{N_3} k_{3i}\frac{a_{3i}s+\sqrt{1-\zeta^2}}{s^2+2\beta\omega_i s+\omega_i^2}\right\} = \sum_{i=1}^{N_3}\omega_i k_{3i}\, exp(-\beta\omega_i t)\, sin\ \left(\omega_i\left(\sqrt{1-\beta^2}\right)t-(0.30)\phi_3\right).$$

$$\text{fdcos}\ (t, \lambda_3, 0.77) = L^{-1}\left\{\frac{s^{0.77}+(2.24)\zeta}{[s^{1.54}+(4.47)\zeta s^{0.77}+5]}\right\} = L^{-1}\left\{\sum_{i=1}^{N_4} k_{4i}\frac{a_{4i}s+\zeta}{s^2+2\beta\omega_i s+\omega_i^2}\right\} = \sum_{i=1}^{N_4}\omega_i k_{4i}\, exp(-\beta\omega_i t)\, cos\ \left(\omega_i\left(\sqrt{1-\beta^2}\right)t-(0.30)\phi_4\right).$$

with $\|\lambda_3\| = 2.24$, ζ = 0.71, β = 0.52, $\varphi_3 = \varphi_4 = 0.79$, $N_3 = N_4 = 63$ and the parameters ω_i, a_{3i}, $a_{4i}$and $k_{3i} = k_{4i}$ are given, for $1 \le i \le 63$, as:

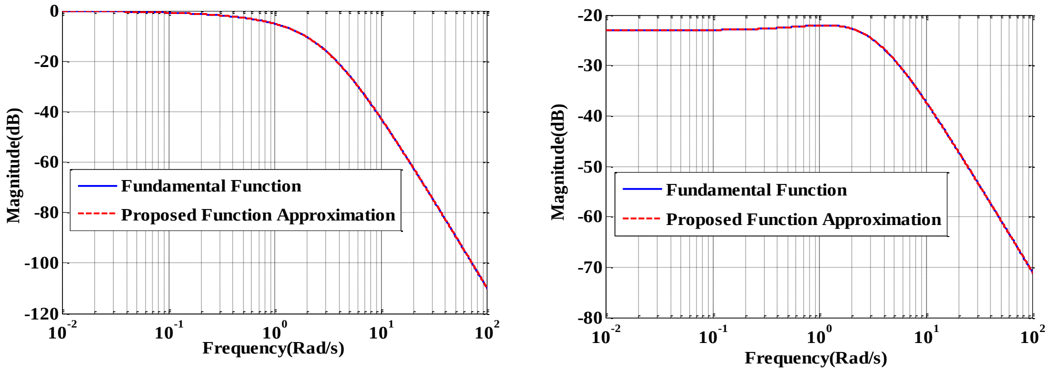
$$\omega_i = \frac{1}{(0.35)*(1.50)^{(32-i)}},\ a_{3i} = (-0.082)*(1.50)^{(32-i)},\ a_{4i} = (0.34)*(1.50)^{(32-i)},$$

$$k_{3i} = k_{4i} = \frac{1}{(0.78)*(1.50)^{2(32-i)}}\left[\frac{sin[0.23\pi]}{cosh[(0.77)\,log((1.50)^{(32-i)})]-cos[0.23\pi]}\right]$$

$$L\{sin_{q,v}(a,\alpha,\beta,t)\}[s^v] \times \left[0.45\left(\frac{s^{0.77}+(2.24)\zeta}{[s^{1.54}+(4.47)\zeta s^{0.77}+5]}\right)+0.28\left(\frac{5}{[s^{1.54}+(4.47)\zeta s^{0.77}+5]}\right)\right]$$

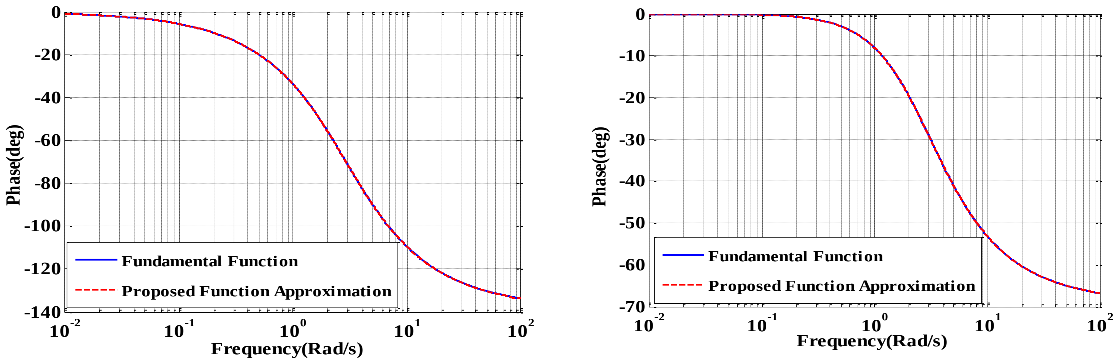
$$L\{cos_{q,\upsilon}(a,\alpha,\beta,t)\} = [s^{\upsilon}] \times \left[0.89\left(\frac{s^{0.77}+(2.24)\zeta}{[s^{1.54}+(4.47)\zeta s^{0.77}+5]}\right) - 0.14\left(\frac{5}{[s^{1.54}+(4.47)\zeta s^{0.77}+5]}\right)\right].$$

Figure 4 and Figure 5 illustrate the bode plots of the transfer function for the fundamental fractional-order system, along with the rational function approximations proposed by our method. The bode plots of L {$\sin_{q,\upsilon}$ (a, α, β, t)} Figure 4(a), Figure 5(a) (magnitude) and L {$\cos_{q,\upsilon}$ (a, α, β, t)} Figure 4(b), Figure 5(b) (phase) were meticulously constructed, alongside their corresponding rational function approximations derived through the proposed method. Notably, both curves demonstrate a remarkable congruence, providing compelling evidence for the exceptional accuracy and effectiveness of the proposed method.



(a) (b)

Figure 4. Magnitude bode plot of (a) L {$\sin_{q,\upsilon}$ (a, α, β, t)} and (b) L {$\cos_{q,\upsilon}$ (a, α, β, t)}, respectively, along with their corresponding rational function approximations proposed by the method



(a) (b)

Figure 5. Phase bode plot of; (a) L {$\sin_{q,\upsilon}$ (a, α, β, t)} and (b) L {$\cos_{q,\upsilon}$ (a, α, β, t)}, respectively, along with their corresponding rational function approximations proposed by the method

Upon examination, it is evident that the functions significantly overlap within the frequency band of interest. The significant overlap observed between the Bode plots indicates that the proposed rational function approximation provides a good approximation of the original fractional-order system within the frequency band of interest. Nevertheless, it is important to note that the accuracy of the approximation may vary outside this frequency band and may depend on the type and range of the fractional order q. Furthermore, the choice of rational function approximation may also depend on the nature of the poles and their impact on the decay or growth rate of the model.

The inverse laplace transforms of the functions, $\left(\frac{s^{0.77}+(2.24)\zeta}{[s^{1.54}+(4.47)\zeta s^{0.77}+5]}\right)$ and $\left(\frac{5}{[s^{1.54}+(4.47)\zeta s^{0.77}+5]}\right)$ are, respectively, given by the expressions fdcos $(t, \lambda_3, 0.77)$ and $fd\sin(t, \lambda_3, 0.77)$. So, the inverse Laplace transform of (7) and (8), $\sin_{q,\upsilon}$ (a, α, β, t) and $\cos_{q,\upsilon}$ (a, α, β, t) are given by [28], [30]-[32].

$$\begin{cases} sin_{q,\upsilon}(a,\alpha,\beta,t) = 0.45\text{fdcos}(t,\lambda_3,0.77) + 0.28\text{fdsin}(t,\lambda_3,0.77) \\ cos_{q,\upsilon}(a,\alpha,\beta,t) = 0.89\text{fdcos}(t,\lambda_3,0.77) - 0.14\text{fdsin}(t,\lambda_3,0.77) \end{cases}$$

Figure 6 illustrates the time-domain plots. The Figure 6(a) showed the function $\sin_{q,\upsilon}(a, \alpha, \beta, t)$ and the Figure 6(b) illustrate the function $\cos_{q,\upsilon}(a, \alpha, \beta, t)$. The obtained solutions are formulated as a meticulous selection of fundamental functions, which themselves consist of a linear combination of trigonometric and exponential regular functions. This parallels the structure found in regular linear systems.

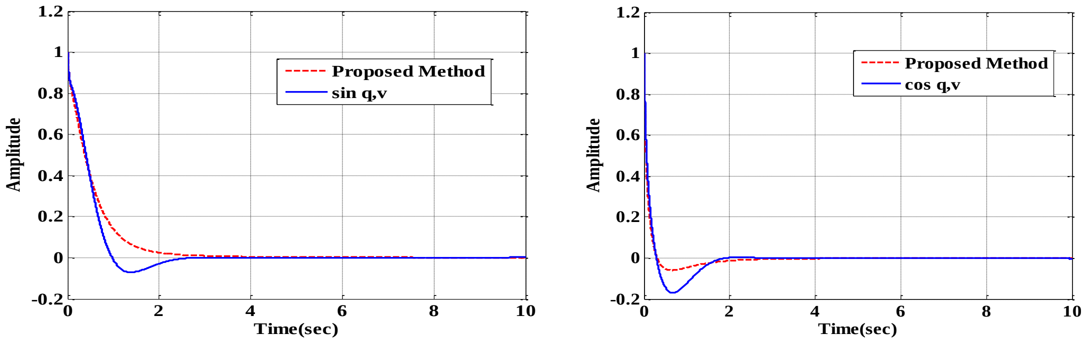


Figure 6. Shows the time-domain plots of the functions (a) $\sin_{q,\upsilon}(a, \alpha, \beta, t)$ and (b) $\cos_{q,\upsilon}(a, \alpha, \beta, t)$ with respect to time (t), using the parameters a=2.24, α =2.1424, β =-1.2987, q=0.77 and υ =0

4. CONCLUSION

This paper presents a novel method for solving such equations using Laplace transforms of fractional meta-trigonometric functions in conjunction with the R-function. The method offers several benefits, including the ability to express solutions in the form of exponentials, sine, and cosine fractional order functions. This approach presents a highly promising analytical tool for effectively solving fractional differential equations with commensurate order. To illustrate the effectiveness and usefulness of the proposed method, a numerical example is solved. As a perspective, we suggest considering extending the research work carried out to solve fractional ordinary differential equations fully online distance education symposium (FODEs) using an Adomian decomposition or the solution by the green's function or bifurcation analysis, they are promising techniques for solving and analyzing FODEs. Further research in this area can lead to the development of new and improved methods for solving these equations and gaining insights into the behavior of complex systems modeled by FODEs.

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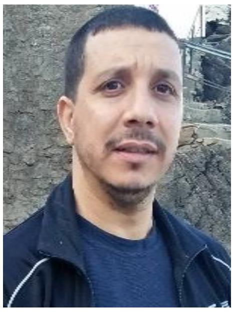

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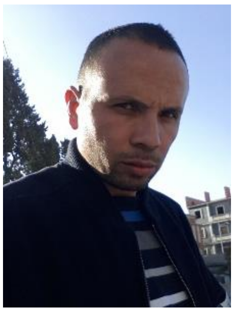

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