Output Regulation for Saturated Systems with Nonlinear Exosystem

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Abstract

In this brief, we investigate the output regulation problem of saturated linear system under the action of nonlinear exosystem. Particularly, for saturated systems with periodically time-dependent exosytem, the K-step asymptotically regulatable region is given, which is a set of all initial states of the plant and the ecosystem. Improved internal model principles are constructed and the feedback controller is designed to ensure exponential output regulation in the regulatable region with disturbance rejection. Simulation examples are given to illustrate the effectiveness of proposed method, the systems go into stable rapidly and periodically.

Keywords: saturation constraint, output regulation, internal model principles, feedback controller

1. Introduction

Saturation constraint is a kind of nonlinear constraint in many practical conditions. In this paper we consider the regulation problem of linear system subject to actuator saturation under the action of nonlinear exosystem. This addresses the problem of designing a feedback controller for an uncertain plant so that the closed-loop system is internally stable and the output of the closed-loop system can asymptotically track a class of reference inputs in the presence of a class of disturbances. Francis and Wonham [1-3] proposed the internal mode principle, which aims to convert the output regulation problem of a given plant into a stabilization problem of an augmented system composed of the given plant and a well defined dynamic compensator.

For the cases where the exogenous signals are constant, Francis [3] designed a linear robust regulator based on the linear approximation of the plant can solve the local structurally stable output regulation problem for the nonlinear plant. Huang and Rugh [4] made a further work and put the solution to nonlinear plant under normal disturbance. Self-Adaptive method and optimal feedback control [5-7] were used in solving the problem of globe robust output regulation for nonlinear system disturbed by uncertain exogenous signals. Disturbance suppression of a class of nonlinear systems was studied in [8-10].

However, it should be pointed that most of the studies are carried with semi-stable exosystem, the problem of output regulation for saturated systems under the action of nonlinear exosystem has received relatively less attention. The few works motivate our current research are [11, 12]. In [11], robust adaptive constrained motion tracking control methodology was derived for bounded nonlinear effects and external disturbance within the closed-loop system. Output regulation for periodic signal of constrained MIMO system subject to actuators saturated is studied in [12-14] proposed adaptive fuzzy sliding mode control approach to solve the control problem of X-Z inverted pendulum in the presents of system uncertainties and external disturbances.

The objective of this paper is to study the problem of output regulation for saturated systems under the action of nonlinear exosystem. Based on our earlier results in [15], a simple feedback controller based on a stabilizing law was achieved for output regulation of linear system with input constrains. Under the action of a nonlinear exosystem action, the problem to be addressed in this paper is the following: (1) Characterize of the regulatable region. The first task of this paper is to characterize the set of initial conditions fro which there exist admissible controls to keep the state bounded and to drive the tracking error to 0 asymptotically. (2) Design

of constrained state feedback controller. Find a state feedback law and construct the state controller.

2. Problem Statement and Preliminaries

Consider the system:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) + P\omega(k) \\ e(k) = Cx(k) + Q\omega(k) \\ \omega(k+1) = S\omega(k) \end{cases} \qquad (1)$$

Where $A \in R^{n \times n}$, $B \in R^{n \times m}$, $P \in R^{n \times r}$, $C \in R^{p \times n}$, $Q \in R^{p \times r}$. The first plant describes a plant, with state $x \in R^n$, input $u \in R^m$ and $\|u\|_\infty \leq 1$, subject to the effect of disturbance represented by $P\omega(k)$. The error between the actual output $Cx(k)$ and a reference signal $Q\omega(k)$ is defined as $e(k)$ by the second equation. The third equation describes the exosystem with state $\omega \in R^r$ and $S \in R^{r \times r}$.

Due to the constraint input, it's well known that the initial state of the plant and exosystem can not be in the whole space. We should characterize the set of all initial states (x_0, $\omega_0) \in R^{n+r}$, on which the problem of constrained output regulation is solvable. This set is called regulatable region. If we can construct a state feedback law, $u = \phi(x, \omega)$, $|\phi(x, \omega)|_\infty \leq 1$ and $\phi(0, 0) = 0$, by which following conditions are satisfied:

A. Plant $x(k+1) = Ax(k) + B\phi(x, 0)$ is asymptotically stable on the equilibrium point $x = 0$.

B. For all initial states (x_0, $\omega_0) \in R^{n+r}$ in regulatable region, the close-loop system has $\lim_{k \to \infty} e(k) = 0$.

To begin with, some necessary assumptions are made:

A1. The pair (A, B) is stabilizable.

A2. S has all its eigenvalues on the unit circle and diagonalizable.

A3. $\left(\begin{bmatrix} C & Q \end{bmatrix}, \begin{bmatrix} A & P \\ 0 & S \end{bmatrix} \right)$ is measurable.

A4. There exist matrices Π and Γ solve the linear matrix equation

$$\begin{cases} \Pi S = A\Pi + B\Gamma + P \\ 0 = C\Pi + Q \end{cases} \qquad (2)$$

In this paper, we consider two kinds of nonlinear external disturbance: the square wave and triangle wave. The square wave is discontinuous and underivable, can be described as $\omega(k+1) = S\omega(k)$, S is an unit matrix. Let $\omega(0) = [m \; m]'$, when $k = nT/2$ ($n = 0, 1, 2 \ldots$), $\omega(k) = (-1)^n \omega(0)$. There're two step signals of different amplitude in one cycle, and the step signal is linear. As the period T is long enough, the action of exosystem can be viewed as tow constant disturbance that works alternatively. Review our earler works in [15], it is possible to design an easily implementable state controller to make the close loop system stable asymptotically, simulation results are shown in section 5. Detailed study on output regulation problem is fouces on the influence of periodic triangle wave.

3. The Regulatable Ragion

The triangle wave is continuous but underivable. Triangle with period T and amplitude m is described as follows, where $\omega(0) = 0$:

$$\omega(k+1) = \begin{cases} \omega(k) + a & nT \leq k < nT + T/2 \\ \omega(k) - a & nT + T/2 \leq k < (n+1)T \end{cases} \qquad n = 0, 1, 2, 3 \cdots \qquad (3)$$

At the equilibrium point, let $u(k) = \Gamma\omega(k) + Ga$, $x(k) = \Pi\omega(k)$, by (1) then:

$$e(k) = Cx(k) + Q\omega(k) = C\Pi\omega(k) + Q\omega(k) = 0$$

$$\begin{cases} \Pi\omega(k) - A\Pi\omega(k) - B\Gamma\omega(k) - P\omega(k) = BGa - \Pi a & nT \le k < nT + T/2 \\ \Pi\omega(k) - A\Pi\omega(k) - B\Gamma\omega(k) - P\omega(k) = BGa + \Pi a & nT + T/2 \le k < (n+1)T \end{cases} \quad n = 0, 1, 2, 3 \cdots \qquad (4)$$

If *B* has full row rank, then G exists made:

$$\begin{cases} \Pi = BG & nT \le k < nT + T/2 \\ \Pi = -BG & nT + T/2 \le k < (n+1)T \end{cases} \quad n = 0, 1, 2, 3 \cdots$$

$$\begin{cases} \Pi\omega(k) = A\Pi\omega(k) + B\Gamma\omega(k) + P\omega(k) & nT \le k < nT + T/2 \\ \Pi\omega(k) = A\Pi\omega(k) + B\Gamma\omega(k) + P\omega(k) & nT + T/2 \le k < (n+1)T \end{cases} \quad n = 0, 1, 2, 3 \cdots \qquad (5)$$

Due to $\omega(k) \ne 0$, by (4), (5), the internal mode of triangle wave action is represents by (6):

$$\begin{cases} \Pi = A\Pi + B\Gamma + P \\ C\Pi + Q = 0 \end{cases} \qquad (6)$$

Consider system (1), a control signal *u* is said to be admissible if $\|u(k)\|_\infty \le 1$.

Definition 3.1: For some $K>0$,$(x_0, \omega_0)\in R^n \times R^r$ is said to be *K*-step regulatable if there exists an admissible u makes (1) satisfy $e(K)=0$. The set of all regulatable pair (x_0, ω_0) is *K*-step regulatable region, denoted by $R_g(K)$.

According to classical regulation theory, there exists matrix $\Pi \in R^{n \times r}$ and matrix $\Gamma \in R^{m \times r}$ makes the equation (6) solvable, and (6) is a zero state equation which describes the equilibrium point as $u(k) = \Gamma\omega(k) + Ga$, $x(k) = \Pi\omega(k)$ and where *e*=0. Due to the restriction that $\|u(k)\|_\infty \le 1$, $e(k)$ will go to zero asymptotically at the equilibrium point only if:

$$\sup_{k \ge 0} |\Gamma\omega(k) + Ga|_\infty \le 1 \qquad (7)$$

So, the exosystem initial conditions corresponding to this equilibrium point are restricted in the compact set $W_0 = \{\omega_0 \in R^r : |\Gamma aT/2 + Ga|_\infty \le 1, \forall k \ge 0\}$.

Definition 3.2: For some $K>0$, a state x_0 is said to be null controllable if there exists an admissible *u* makes the system state transforms from $x(0) = x_0$ and satisfies $\lim_{k \to \infty} x(k) = 0$. The set of all the null controllable region x_0 is null controllable region, denoted by $C(A, B)$. Specially, the set of null controllable region is called *K*-step null controllable region when $x(K) = 0$, denoted by $C_K(A, B)$.

By similarity transformation, we may assume:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \in R^{(n_1 + n_2) \times (n_1 + n_2)}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \in R^{(n_1 + n_2) \times m} \qquad (8)$$

Where A_1 has all eigenvalues inside or on the unit circle and A_2 has all eigenvalues outside the unit circle. So, the null controllable region $C(A, B) = R^{n_1} \times C(A_2, B_2)$. We consider the condition about all the eigenvalues of *A* are outside the unit circle. Generally, if K is large enough (i.e. K=10~30), $C_K(A, B)$ is fairly approximate to $C(A, B)$.

Correspondingly, let:

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}, \quad Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \qquad (9)$$

Now, we will describe the regulatable region R_g in terms of $C_K(A,B)$ and W_0.

Lemma 1 [16]. Let $V_0 \in R^{n_2 \times r}$ be the unique solution to the linear matrix equation $V_0 S - A_2 V_0 = P_2$. Then the K-step regulatable region $R_g(K)$ is given by:

$$R_g(K) = \left\{ (x_{10}, x_{20}, \omega_0) \in R^{n_1} \times R^{n_2} \times W_0 : x_{20} - V\omega_0 \in C_K(A_2, B_2) \right\} \tag{10}$$

For the first semi-cycle of triangle wave, let $T_1 = T/2$, by carrying out a similarity transformation.

$$x(T_1) = A^{T_1} x_0 + \sum_{i=0}^{T_1-1} A^{T_1-i-1} B u(i) + \sum_{i=0}^{T_1-1} A^{T_1-i-1} P\omega(i)$$

We get:

$$\begin{bmatrix} e_1(T_1) \\ e_2(T_1) \end{bmatrix} = \begin{bmatrix} Cx_1(T_1) - Q_1\omega(T_1) \\ Cx_2(T_1) - Q_2\omega(T_1) \end{bmatrix}$$

Since $Q_2\omega(T_1)$ is bounded for all k and $A_2^K \to \infty$ when $k \to T_1$, $\lim_{k-T_1} e(k) = 0$ stands on:

$$x_{20} + \sum_{i=0}^{T_1} A_2^{-i-1} B_2 u(i) + \sum_{i=0}^{T_1} A_2^{-i-1} P_2 ia = 0$$

Denote $V_0 = -\sum_{i=0}^{T_1} A_2^{-i-1} P_2 i$, V_0 satisfies $V_0 - A_2 V_0 = (A-I)^{-1} P_2$. Let $(A-I) = D$, then $D(V_0 - A_2 V_0) = P_2$.

For the second semi-cycle of triangle wave, which can be viewed as the result of half a cycle parallel translation towards the right direction on the time axis.

$$\omega(k+1) = \omega(k) - a, \quad \omega(0) = aT_1$$

The regulator equation:

$$\begin{cases} \Pi = A\Pi + B\Gamma + P \\ C\Pi + Q = 0 \\ \Pi = -BG \end{cases}$$

Similarly, let $V_0 = -\sum_{i=0}^{T_1} A_2^{-i-1} P_2 (T_1 - i)$, $-(A-I) = D$, we get $D(V_0 - A_2 V_0) = P_2$.

4. State Feedback Controller Design

In this section, we will construct a state feedback controller for above system.

Lemma 2 [17]. Let $\lambda \in (0, 1)$, for any initial condition $\widetilde{x}_0 \in C_\lambda = C(\lambda^{-1} A_2, \lambda^{-1} B_2)$, there exists a state feedback law $u(k) = h[x(k)]$ such the solution of $x(k+1) = A_2 x(k) + B_2 u(k)$ satisfies $x(k) \in \lambda^k \rho c_\lambda(x_0) C_\lambda$ and the control signal $|u(k)|_\infty \leq \lambda^k \rho c_\lambda(x_0) \leq \lambda^k$

Lemma 2 gives a balance between the state convergence rate and the control of all the initial state in $\overline{C}_\lambda$, denoted by λ^k. The construction of this state feedback controller can be fund in [14], based on which, we will construct a revised controller law for regulation problem in this paper.

Theorem 2. Assume there exists a matrix V_0 which satisfies $D(V_0 - AV_0) = P_2$, for every initial pair (x_0, ω_0) in the regulatable region, under the following controller:

$$u(k) = h[x(k) - \lambda^k V_2\omega(k) - (1-\lambda^k)\Pi_2\omega(k)] + (1-\lambda^k)(\Gamma\omega(k) + Ga)$$

The closed-loop system satisfies $\lim_{k\to\infty} e(k)=0$.

    Proof. Corresponding to (8), we can divide system (1) in to two subsystems.

$$x_1(k+1) = A_1 x(k) + B_1 u(k) + P_1 \omega(k)$$
$$x_2(k+1) = A_2 x(k) + B_2 u(k) + P_2 \omega(k)$$

Denote $\tilde{x}_i(k) = x_i(k) - \lambda^k V_i \omega(k) - (1-\lambda^k)\Pi_i \omega(k), \quad i = 1, 2$

By Lemma 1, for i =1, 2, we get:

$$\tilde{x}_i(k+1) = A_i \tilde{x}_i(k) + B_i u(k) + (\lambda^k - 1)B_i \Gamma \omega(k) - \lambda^k (I-D)P_i \omega(k) - \lambda^k V_i a - (1-\lambda^k)\Pi_i a$$

Base on the controller defined in Lemma 2, we construct a controller:

$$u(k) = h[\tilde{x}_2(k)] + (1-\lambda^k)(\Gamma \omega(k) + Ga)$$

Apply it to the two subsystems:

$$\tilde{x}_1(k+1) = A_1 \tilde{x}_1(k) + B_1 h[\tilde{x}_2(k)] - \lambda^k(I-D)P_1 \omega(k) - \lambda^k V_1 a$$
$$\tilde{x}_2(k+1) = A_2 \tilde{x}_2(k) + B_2 h[\tilde{x}_2(k)] - \lambda^k(I-D)P_2 \omega(k) - \lambda^k V_2 a$$

We can get $\lim_{k\to T_1} \tilde{x}_2(k) = 0, \quad |h[\tilde{x}_2(k)]_{T_1} \le \lambda^k$ by Lemma 2. Since A_1 is semi-stable and $|h[\tilde{x}_2(k)]_{T_1} \le \lambda^k$, $\tilde{x}_1(k)$ also convergences to the origin.

$$|u(k)|_{T_1} = \left| h[\tilde{x}_2(k)] + (1-\lambda^k)(\Gamma \omega(k) + Ga) \right|_{T_1} \le 1$$

The closed-loop system satisfies $\lim_{k\to T_1} e(k) = 0$. Similar controller can be constructed for the second semi-cycle of a triangle cycle.

5. Numerical Examples

    Example 1. A semi-stable system as follows under the action of square signal (T/2=1000)

$$x(k+1) = \begin{bmatrix} 1.4 & 0 \\ 0.2 & 1.2 \end{bmatrix} x(k) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u(k) + \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} \omega(k)$$

$$\omega(k+1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \omega(k)$$

$$e(k) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(k) - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \omega(k)$$

With $x_0 = [-1.5\ -0.8]^\mathsf{T}$, $\omega(0) = [1.5\ 1.5]^\mathsf{T}$, the regulation equation has solutions.

$$\Pi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \Gamma = S - A - P = \begin{bmatrix} -0.5 & 0 \\ -0.2 & -0.3 \end{bmatrix}, \ V = \begin{bmatrix} -0.25 & 0 \\ 0.25 & -0.5 \end{bmatrix}$$

Applying the controller provided in [13], we get:

$u(k)=h[x(k)-0.97^k V\omega(k)-(1-0.97^k)\Pi\omega(k)]+ (1-0.97^k)\Gamma\omega(k)$

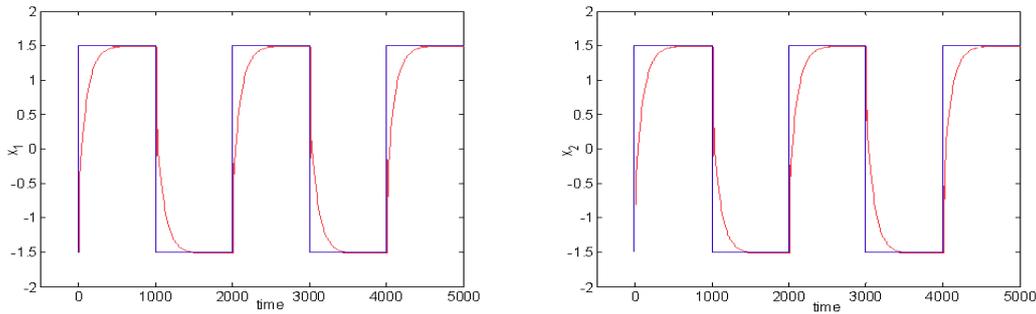The closed-loop state tracking shown in Figure 1.

Figure 1. Closed-loop State Tracking under the Square Signal Disturbance in Example 1

Example 2. The following system under the action of triangle signal (T=1000)

$$x(k+1) = \begin{bmatrix} 1.4 & 0 \\ 0.2 & 1.2 \end{bmatrix} x(k) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u(k) + \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} \omega(k)$$

$$e(k) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(k) - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \omega(k)$$

In the fist semi-cycle, x_0=[-0.1 -0.01]^T, ω_0=[0 0]^T, a=[0.003 0.004]^T. The regulation equation has solutions

$$\Pi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \Gamma = S - A - P = \begin{bmatrix} -0.5 & 0 \\ -0.2 & -0.3 \end{bmatrix}, \ G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$D(V-AV)=P$ has a unique solution:

$$V = \begin{bmatrix} -0.625 & 0 \\ 1.875 & -2.5 \end{bmatrix}$$

We get the state feedback controller $u(k)=h[x(k)-0.95^kV\omega(k)-(1-0.95^k)\Pi\omega(k)]+$ (1-0.95^k)($\Gamma\omega(k)+Ga$)
The closed-loop state tracking are plotted in Figure 2.

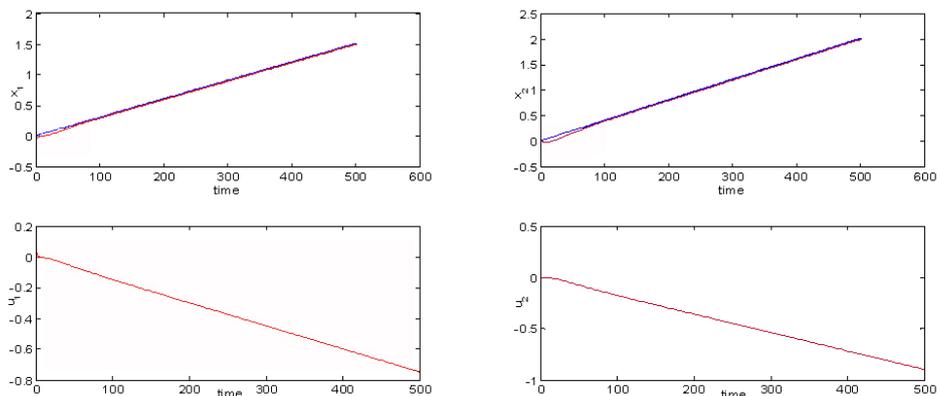
Figure 2. Closed-loop State Tracking in First Semi-cycle in Example 2

In the last semi-cycle, $x_0=[1.5\ 2.0]^T$, $\omega_0=[1.5\ 2.0]^T$, $a=[0.003\ 0.004]^T$

$$\Pi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \Gamma = S - A - P = \begin{bmatrix} -0.5 & 0 \\ -0.2 & -0.3 \end{bmatrix}, \ G = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

There exists the unique solution to $D(V\text{-}AV)=P$

$$V = \begin{bmatrix} 0.625 & 0 \\ -1.875 & 2.5 \end{bmatrix}$$

The state feedback controller $u(k)=h[x(k)-0.95^k V\omega(k)-(1-0.95^k)\Pi\omega(k)]+ (1-0.95^k)(\Gamma\omega(k)+Ga)$. The closed-loop state trackings are plotted in Figure 3.
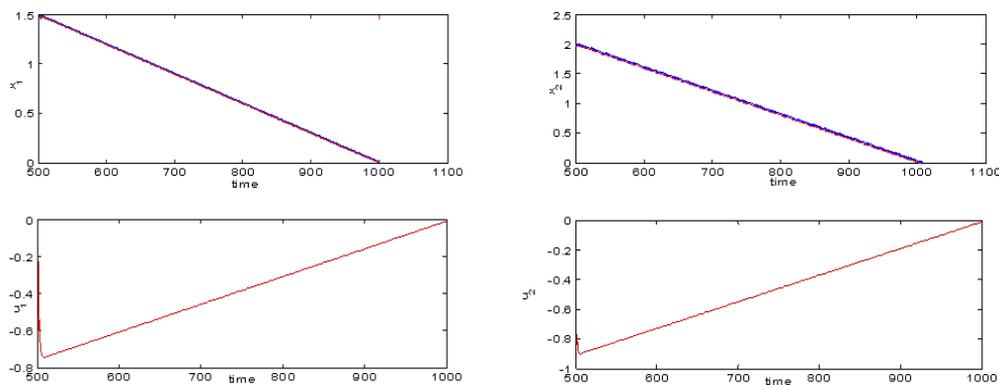

Figure 3. Closed-loop State Tracking in Last Semi-cycle in Example 2

In each cycle period, two different internal mode principles are applied for a semi-cycle respectively, thus G and V are got and the state-feedback controller $u(k)$ are constructed. State tracking in two cycles are shown in Figure 4, with $x_0=[-0.1\ -0.01]^T$, $\omega_0=[0\ 0]^T$, $a=[0.003\ 0.004]^T$.
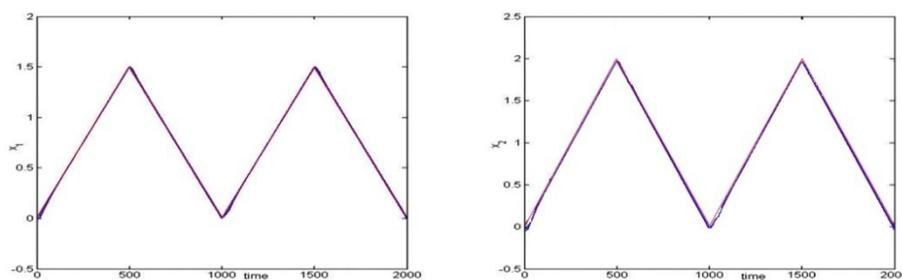

Figure 4. State Tracking in Two Cycles in Example 2

6. Conclusion

In this paper, we studied the output regulation problem of sutured line system under the action of nonlinear excosystem. At the equilibrium point, initial state of the plant and the exosystem are restricted in a compact set W_0. The *K-Step* asymptotically regualatable region $R_g(K)$ is described by W_0 and *K*-Step null controllable region $C_K(A,B)$. Segmented control strategies are applied to external disturbances like the square signal and triangle signal. The internal principles for each semi-cycle of the exosystem are given. Based on the state feedback

laws proposed, the controller is constructed respectively and examples show that it can suppress some external nonlinear disturbance effectively.

Acknowledgements

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