

The Existence and Simulations of Periodic Solution for Panda-Bamboo-Arbor System

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Abstract

In this paper, we consider a kind of three-dimensional biological model of food system including Panda, bamboo and arbor with delays. The stabilities of positive equilibrium points are considered. Some theories of the existence of positive periodic solutions via Hopf bifurcation with respect to both delays are established. The paper ends with some numerical simulations.

Keywords: giant panda, time delays, hopf bifurcation, simulation

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1. Introduction

The giant panda is a bear native to central western and south western China. The giant panda lives in a few mountains ranges in central China, mainly in Sichuan province, but also in the Shanxi and Gansu provinces [1]. Though it belongs to the order Carnivora, the panda's diet is 99% bamboo. The giant panda is an endangered species, threatened by continued habitat loss and by a very low birthrate, both in the wild and in captivity [2].

The protection of giant pandas and their habitat is an important step for the conservation of the integrity and stability of biological diversity and ecosystem function [3]. Many scholars have been committed to researching how to protect giant pandas [4-8]. Some of them have proposed related dynamic models, established theoretical foundations through qualitative analysis and computer numerical simulations.

In [9], according to the relationship among arbor, bamboo and giant panda, Song proposed three assumptions to explain a kind of source habitat plates:

(1) Bamboo is the main source of food for giant pandas and giant pandas mainly take the one year-growth bamboo;

(2) The update and growth of bamboo depends on the shade of arbor. The growth of bamboo competes with the update of arbor seedlings. Bamboo competes nutritional resources with arbor and its seedlings;

(3) Arbor provides a habitat for the giant pandas and the necessary shade conditions for bamboo. At the same time, arbor can inhibit bamboo growing too fast, slow the rate of lignification.

Based on the assumptions above, Song developed the following differential equations to reflect the relationship among arbor, bamboo and giant panda.

$$\begin{cases} \dot{x}(t) = x(t)[a_1 - b_1x(t) + a_{12}y(t-\tau) + a_{13}z(t)], \\ \dot{y}(t) = y(t)[a_2 - b_2y(t) - a_{21}x(t) + a_{23}z(t)], \\ \dot{z}(t) = z(t)[a_3 - b_3z(t) - a_{32}y(t)]. \end{cases}$$

Here $x(t)$, $y(t)$, $z(t)$ denote the population density of giant panda, bamboo and arbor at time t , respectively. The parameters a_j ($j=1,2,3$) and a_{ik} ($i=1,2,3, k=1,2,3$) are positive constants. The parameter a_j ($j=1,2,3$) denote the intrinsic growth rate of each population; furthermore, each population is density dependent. The value $\tau > 0$ is a constant representing a time delay due to the gestation of bamboo, $a_{12}y(t-\tau)$ indicates that the growth rate of Giant panda at time t is directly proportional to the number of feeding bamboo at time $t-\tau$.

Though arbor provides the necessary shade conditions for bamboo, arbor and bamboo do exist competitions. Furthermore, Panda's habitats are mostly located in the hollow tree base or tree stumps, so the arbor which can provides a habitat for the giant pandas must big enough. Taking into account above points, we modify Song's model as follows:

$$\begin{cases} \dot{x}(t) = x(t)[a_1 - b_1x(t) + a_{12}y(t - \tau_1) + a_{13}z(t - \tau_2)], \\ \dot{y}(t) = y(t)[a_2 - b_2y(t) - a_{21}x(t) - a_{23}z(t)], \\ \dot{z}(t) = z(t)[a_3 - b_3z(t) - a_{32}y(t)], \end{cases} \quad (1)$$

Where $a_{13}z(t - \tau_2)$ indicates that the growth rate of giant panda at time t is directly proportional to the number of arbor at time $t - \tau_2$.

In next section, according to the Hopf bifurcation theory in [10-13], we will discuss the existence of Hopf bifurcation in system (1). In section 3, we will present some numerical simulations to demonstrate our main results given in section 2.

2. Stability of the Positive Equilibrium and Existence and Existence of Hopf Bifurcations

System (1) has a unique positive equilibrium point $E^* = (x^*, y^*, z^*) = \left(\frac{\Delta_1}{\Delta}, \frac{\Delta_2}{\Delta}, \frac{\Delta_3}{\Delta}\right)$ under the condition:

(H1) $\frac{\Delta_1}{\Delta} > 0; \frac{\Delta_2}{\Delta} > 0; \frac{\Delta_3}{\Delta} > 0$, where:

$$\Delta = \begin{vmatrix} b_1 & -a_{12} & a_{13} \\ a_{21} & b_2 & a_{23} \\ 0 & a_{32} & b_3 \end{vmatrix}, \quad \Delta_1 = \begin{vmatrix} a_1 & -a_{12} & a_{13} \\ a_2 & b_2 & a_{23} \\ a_3 & a_{32} & b_3 \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} b_1 & a_1 & a_{13} \\ a_{21} & a_2 & a_{23} \\ 0 & a_3 & b_3 \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} b_1 & -a_{12} & a_1 \\ a_{21} & b_2 & a_2 \\ 0 & a_{32} & a_3 \end{vmatrix}$$

Let $\bar{x}(t) = x(t) - x^*$, $\bar{y}(t) = y(t) - y^*$, $\bar{z}(t) = z(t) - z^*$. In order to simplify the notation, we drop the bars, namely denote $\bar{x}(t)$, $\bar{y}(t)$, $\bar{z}(t)$ by $x(t)$, $y(t)$, $z(t)$, respectively. Then, system (1) becomes:

$$\begin{cases} \dot{x}(t) = (x(t) + x^*)[-b_1x(t) + a_{12}y(t - \tau_1) + a_{13}z(t - \tau_2)], \\ \dot{y}(t) = (y(t) + y^*)[-b_2y(t) - a_{21}x(t) - a_{23}z(t)], \\ \dot{z}(t) = (z(t) + z^*)[-b_3z(t) - a_{32}y(t)]. \end{cases} \quad (2)$$

The linearization of (2.1) at $x(t) = 0$, $y(t) = 0$, $z(t) = 0$ is:

$$\begin{cases} \dot{x}(t) = -b_1x^*x(t) + a_{12}x^*y(t - \tau_1) + a_{13}x^*z(t - \tau_2) \\ \dot{y}(t) = -b_2y^*y(t) - a_{21}y^*x(t) - a_{23}y^*z(t) \\ \dot{z}(t) = -b_3z^*z(t) - a_{32}z^*y(t) \end{cases} \quad (3)$$

Thus, the characteristic equation of system (3) is:

$$\lambda^3 + p_1\lambda^2 + p_2\lambda + p_3 + e^{-\lambda\tau_1}(q_1\lambda + q_2) + e^{-\lambda\tau_2}q_3 = 0, \quad (4)$$

Where:

$$p_1 = b_1x^* + b_2y^* + b_3z^*, \quad p_2 = b_1b_2x^*y^* + b_1b_3x^*z^*, \quad p_3 = b_1b_2b_3x^*y^*z^* - a_{23}a_{32}b_1x^*y^*z^*, \\ q_1 = a_{12}a_{21}x^*y^*, \quad q_2 = a_{12}a_{21}b_3x^*y^*z^*, \quad q_3 = -a_{13}a_{21}a_{32}x^*y^*z^*.$$

In the following, we will consider five cases.

Case (a) $\tau_1 = \tau_2 = 0$, then (4) becomes:

$$\lambda^3 + p_1\lambda^2 + (p_2 + q_1)\lambda + (p_3 + q_2 + q_3) = 0. \quad (5)$$

By the Routh-Hurwitz criterion, we get that all roots of (5) have negative real parts if and only if:

$$(H2): p_1(p_2 + q_1) > p_3 + q_2 + q_3.$$

Thus, the positive equilibrium point E^* is locally asymptotically stable if condition (H2) holds.

Case (b) $\tau_2 = 0$, $\tau_1 > 0$, then (4) becomes:

$$\lambda^3 + p_1\lambda^2 + p_2\lambda + (p_3 + q_3) + e^{-\lambda\tau_1}(q_1\lambda + q_2) = 0. \quad (6)$$

If $\lambda = \omega i$, $\omega > 0$ is a root of (6), then $-p_1\omega^2 + p_3 + q_3 + q_2 \cos \omega\tau_1 + q_1\omega \sin \omega\tau_1 + (-\omega^3 + p_2\omega + q_1\omega \cos \omega\tau_1 - q_2 \sin \omega\tau_1)i = 0$. Separating the real and imaginary parts, we have:

$$\begin{cases} p_1\omega^2 - p_3 - q_3 = q_2 \cos \omega\tau_1 + q_1\omega \sin \omega\tau_1, \\ \omega^3 - p_2\omega = q_1\omega \cos \omega\tau_1 - q_2 \sin \omega\tau_1. \end{cases} \quad (7)$$

By squaring and adding the two equations in (7), we get:

$$\omega^6 + (p_1^2 - 2p_2)\omega^4 + (p_2^2 - 2p_1p_3 - 2p_1q_3 - q_1^2)\omega^2 + (p_3 + q_3)^2 - q_2^2 = 0. \quad (8)$$

Let $z = \omega^2$, (8) becomes:

$$z^3 + (p_1^2 - 2p_2)z^2 + (p_2^2 - 2p_1p_3 - 2p_1q_3 - q_1^2)z + (p_3 + q_3)^2 - q_2^2 = 0. \quad (9)$$

If (H3): $(p_3 + q_3)^2 - q_2^2 < 0$, system (9) has at least a positive real root. Without loss of generality, let z_1, z_2, z_3 are three positive real roots of (2.8), then $\omega_i = \sqrt{z_i}$, $(i = 1, 2, 3)$. From (7), we have:

$$\tau_{1n}^i = \frac{1}{\omega_i} \arccos \left[\frac{q_1\omega_i^4 - q_1p_2\omega_i^2 + p_1q_2\omega_i^2 - p_3q_2 - q_3q_2}{q_1^2\omega_i^2 + q_2^2} \right] + \frac{2n\pi}{\omega_i} \quad i = 1, 2, 3, \quad n = 0, 1, 2, \dots$$

Moreover, let $\tau_{10} = \min_{i \in \{1, 2, 3\}} \{\tau_{10}^i\}$.

From (6), we have:

$$\begin{aligned} \left[\frac{d\lambda}{d\tau_1} \right]^{-1} &= \frac{3\lambda^2 + 2p_1\lambda + p_2}{\lambda(q_1\lambda + q_2)e^{-\lambda\tau_1}} + \frac{q_1}{\lambda(q_1\lambda + q_2)} - \frac{\tau_1}{\lambda}, \\ &= -\frac{3\lambda^2 + 2p_1\lambda + p_2}{\lambda(\lambda^3 + p_1\lambda^2 + p_2\lambda + p_3 + q_3)} + \frac{q_1}{\lambda(q_1\lambda + q_2)} - \frac{\tau_1}{\lambda}, \end{aligned}$$

Which, together with (7), leads to:

$$\operatorname{Re} \left[\frac{d\lambda}{d\tau_1} \right]_{\tau_1 = \tau_{10}}^{-1} = -\frac{q_1^2}{q_1^2\omega_0^2 + q_2^2} + \frac{(2p_1\omega_0 - 3\omega_0^2)[(p_3 + q_3)\omega_0^3] - p_2(\omega_0^4 - p_2\omega_0^2)}{(\omega_0^4 - p_2\omega_0^2)^2 + [(p_3 + q_3)\omega_0 - 3p_1\omega_0^3]^2}$$

According to the above analysis, we have the following results.

Theorem 2.1 For $\tau_2 = 0$, suppose conditions (H1), (H2), (H3) hold, and $\operatorname{Re}\left[\frac{d\lambda}{d\tau_1}\right]_{\tau_1=\tau_{10}}^{-1} \neq 0$, then (1) when $0 < \tau_1 < \tau_{10}$, the equilibrium point E^* is locally asymptotically stable; (2) when $\tau_1 > \tau_{10}$, the equilibrium point E^* is unstable; (3) when $\tau_1 = \tau_{10}$ the equilibrium point E^* undergoes Hopf bifurcation at $\tau_1 = \tau_{10}$.

Case (c) $\tau_1 = 0$, $\tau_2 > 0$, the situation is similar to case (b).

Case (d) τ_1 is fixed in the interval $(0, \tau_{10})$ and $\tau_2 > 0$.

Let $\lambda = \omega i$ ($\omega > 0$) be a root of (4). Separating real and imaginary parts, leads to:

$$\begin{cases} -\omega^3 + p_2\omega + q_1\omega\cos\omega\tau_1 - q_2\sin\omega\tau_1 = q_3\sin\omega\tau_2, \\ -p_1\omega^2 + p_3 + q_2\cos\omega\tau_1 + q_1\omega\sin\omega\tau_1 = -q_3\cos\omega\tau_2. \end{cases} \quad (10)$$

Squaring and adding the two equations in (10) gives:

$$\omega^6 + A\omega^4 + B\omega^3 + C\omega^2 + D\omega + E = 0, \quad (11)$$

Where $A = p_1^2 - 2p_2 - 2q_1\cos\omega\tau_1$, $B = 2q_2\sin\omega\tau_1 - 2p_1q_1\sin\omega\tau_1$,

$C = p_2^2 - 2p_1p_3 + q_1^2 + 2p_1q_2\cos\omega\tau_1 + 2p_2q_1\cos\omega\tau_1$, $D = 2(p_3q_1 - p_2q_2)\sin\omega\tau_1$,

$E = p_3^2 + q_2^2 - q_3^2 + 2p_3q_2\cos\omega\tau_1$.

If system (11) has positive root, we know that (2.10) has finite positive roots mostly. Without loss of generality, let $\omega_1, \omega_2, \dots, \omega_N$ are N positive real roots of (11). From (10), we have:

$$\tau_{2n}^i = \frac{1}{\omega_i} \arcsin\left[\frac{-\omega_i^3 + p_2\omega_i + q_1\omega_i\cos\omega_i\tau_1 - q_2\sin\omega_i\tau_1}{q_3}\right] + \frac{2n\pi}{\omega_i}, \quad i=1,2,\dots,N, \quad n=0,1,2,\dots$$

Moreover, let $\tau'_{20} = \min_{i \in \{1,2,\dots,N\}} \{\tau_{20}^i\}$.

From (4), we have:

$$\left[\frac{d\lambda}{d\tau_2}\right]^{-1} = \frac{3\lambda^2 + 2p_1\lambda + p_2 - \tau_1(q_1\lambda + q_2)e^{-\lambda\tau_1} + q_1e^{-\lambda\tau_1} - q_3e^{-\lambda\tau_2}\tau_2}{\lambda q_3 e^{-\lambda\tau_2}},$$

Which, together with (4), lead to $\operatorname{Re}\left[\frac{d\lambda}{d\tau_2}\right]_{\tau_2=\tau'_{20}}^{-1} = \operatorname{Re}\left[\frac{K+iL}{M+iN}\right]$, where:

$$K = p_2 + \tau_1 p_3 + q_1 \cos \tau_1 - \tau_1 p_1 - 3, \quad L = 2p_1 + \tau_1 p_2 - \tau_1 \omega^3 - q_1 \sin \omega \tau_1, \quad M = \omega q_3 \sin \omega \tau_2, \quad N = \omega q_3 \cos \omega \tau_2.$$

According to the above analysis, we have the following results.

Theorem 2.2. For τ_1 is fixed in the interval $[0, \tau_{10})$, if condition (H1), (H2) holds, system (11) has positive roots, and $\operatorname{Re}\left[\frac{d\lambda}{d\tau_2}\right]_{\tau_2=\tau'_{20}}^{-1} \neq 0$, then (1) when $0 < \tau_2 < \tau'_{20}$, the equilibrium point E^* is locally asymptotically stable; (2) when $\tau_2 > \tau'_{20}$, the equilibrium point E^* is unstable; (3) when $\tau_2 = \tau'_{20}$ the equilibrium point E^* undergoes Hopf bifurcation at $\tau_2 = \tau'_{20}$.

Case (e) τ_2 is fixed in the interval $(0, \tau_{20})$ and $\tau_1 > 0$.

The situation is similar to Case (d).

3. Numerical Examples

In this section, we present some numerical examples to demonstrate our main results given in section 2. We consider the following system:

$$\begin{cases} \dot{x}(t) = x(t)[0.6 - 0.4x(t) + 0.3y(t - \tau_1) + 0.3z(t - \tau_2)], \\ \dot{y}(t) = y(t)[0.8 - 0.25y(t) - 0.25x(t) - 0.1z(t)], \\ \dot{z}(t) = z(t)[0.4 - 0.3z(t) - 0.3y(t)]. \end{cases} \quad (12)$$

It is easy to verify that the conditions of Theorem (3) hold and compute $\tau_{10} \approx 33.2777$, $\tau'_{20} \approx 126.147$. Now, let $\tau_1 = 30 \in (0, \tau_{10})$ fixed. From Theorem (3), we know that the positive equilibrium point E^* is locally asymptotically stable when $\tau_2 = 55 < \tau'_{20}$ (see Figure 1). The positive equilibrium point E^* lose its stability and a family of periodic solutions bifurcate from the positive equilibrium point E^* when $\tau_2 = 127 > \tau'_{20}$ (see Figure 2).

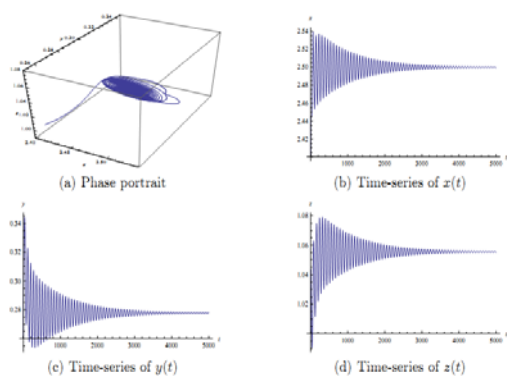


Fig. 1. Numerical solution (x, y, z) of system (3.1) at the initial value $(2.4, 0.26, 1)$

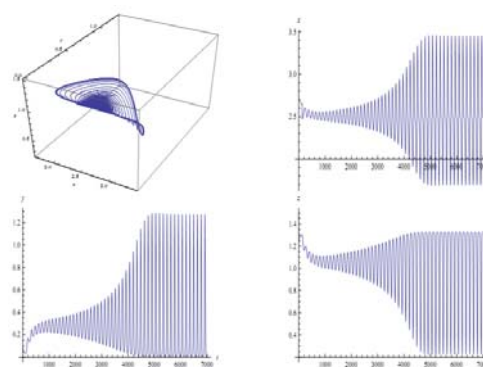


Fig. 2. Numerical solution (x, y, z) of system (3.1) at the initial value $(3, 0.1, 1.5)$.

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