**Indonesian Journal of Electrical Engineering and Computer Science**
Vol. 33, No. 1, January 2024, pp. 425~432
ISSN: 2502-4752, DOI: 10.11591/ijeecs.v33.i1.pp425-432 

# A modified type of Fletcher-Reeves conjugate gradient method with its global convergence

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**Article Info**

***Article history:***

Received Nov 8, 2022
Revised Nov 7, 2023
Accepted Nov 13, 2023

***Keywords:***

Conjugate gradient method
Exact line search
Fletcher-Reeves method
Global convergence
Unconstrained optimization

**ABSTRACT**

The conjugate gradient methods are one of the most important techniques used to address problems involving minimization or maximization, especially nonlinear optimization problems with no constraints at all. That is because of their simplicity and low memory needed. They can be applied in many areas, such as economics, engineering, neural networks, image restoration, machine learning, and deep learning. The convergence of Fletcher-Reeves (FR) conjugate gradient method has been established under both exact and strong Wolfe line searches. However, it is performance in practice is poor. In this paper, to get good numerical performance from the FR method, a little modification is done. The global convergence of the modified version has been established for general nonlinear functions. Preliminary numerical results show that the modified method is very efficient in terms of number of iterations and CPU time.



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## 1. INTRODUCTION

To determine a function's minimum, one nonlinear optimization technique is the Fletcher-Reeves conjugate gradient (FRCG) method. The FRCG technique builds a series of search directions that are conjugate to each other by using the function's gradients, which is based on the concept of conjugate gradients. The fast convergence of the FRCG approach to the function's minimum is made possible by this characteristic of conjugate gradients. Considering the following problem of unconstrained optimization;

$$\min f(x)\ , x \in R^n, \tag{1}$$

where $f: R^n \rightarrow R$ is a smooth function with gradient $g$. Conjugate gradient methods are well qualified for solving (1) even if it is of large scale. They are use;

$$x_{k+1} = x_k + \alpha_k d_k, k = 0,1,2, \ldots, \tag{2}$$

where $\alpha_k$ is the step-length taken in the direction of search $d_k$. The search direction is given by;

$$d_k = \begin{cases} -g_k, \text{ if } k = 0 \\ -g_k + \beta_k d_{k-1}, \text{ if } k \geq 1, \end{cases} \tag{3}$$

where $g_{k,} = \nabla f(x_k)$ and $\beta_k$ is a real number, called conjugate gradient coefficient. Different formulas for $\beta_k$ determine different conjugate gradient methods such as of Fletcher-Reeves (FR), Dai-Yuan (DY), Conjugate Descent (CD), Polak-Rebiere and Polyak (PRP), Hestenes-Stiefel (HS), Liu-Storey (LS). Fletcher [1], Dai and Yuan [2], Fletcher [3], Polyak [4] and Polak and Rebiere [5], Hestenes and Stiefel [6] and Liu and Storey [7] whose coefficients are respectively given by;

$$\beta_k^{FR} = \frac{\|g_k\|^2}{\|g_{k-1}\|^2}$$

$$\beta_k^{DY} = \frac{\|g_k\|^2}{d_{k-1}^T(g_k - g_{k-1})}$$

$$\beta_k^{CD} = -\frac{\|g_k\|^2}{d_{k-1}^T g_{k-1}}$$

$$\beta_k^{PRP} = \frac{g_k^T(g_k - g_{k-1})}{\|g_{k-1}\|^2}$$

$$\beta_k^{HS} = \frac{g_k^T(g_k - g_{k-1})}{d_{k-1}^T(g_k - g_{k-1})}$$

$$\beta_k^{LS} = -\frac{g_k^T(g_k - g_{k-1})}{d_{k-1}^T g_{k-1}}$$

the step-length $\alpha_k$ can be evaluated using the exact search, in which;

$$f(x_k + \alpha_k d_k) = \min f(x_k + \alpha d_k),\ \alpha \geq 0. \tag{4}$$

Since the computation of $\alpha_k$ using (4) is so hard in practice, other methods for computing the step length $\alpha_k$ are defind. These methods are called inexact line searches. An example of the inexact method which is used widely in practice is the Wolfe, in which the step length satisfies the conditions;

$$f(x_k + \alpha_k d_k) - f(x_k) \leq \delta \alpha_k\, {g_k}^T d_k \tag{5}$$

$$g_k(x_k + \alpha_k d_k)^T \geq \sigma\, {g_k}^T d_k \tag{6}$$

where $0 < \delta < \sigma < 1$.

Another strong version is of wolfe is the strong wolfe, given by (5) and (7);

$$|g(x_k + \alpha_k d_k)^T d_k| \leq \delta |{g_k}^T d_k| \tag{7}$$

to ensure that the search for the solution of the problem in (1) is in the correct direction, the direction is imposed to be descent. So, the descent of $d_k$ (downhill condition) has a great role in the conjugate gradient methods. The (8) of the descent of $d_k$ occurs from the inequality;

$${g_k}^T d_k < 0, \tag{8}$$

which can be extended to the so-called the sufficient descent criterion;

$$-\,{g_k}^T d_k \geq c\|g_k\|^2,\ c > 0. \tag{9}$$

Conjugate gradient methods are still the best choice for solving (1). Over the years, for better performance, many efforts have been devoted to define new methods and to modify others such as the studies in [8]–[17]. The FR conjugate gradient method has rich convergence properties. However, it is numerical performance is much slower than that of many others. Zoutendijk [18] reported that, the FR method via exact line search converges globally on general functions. Later, Al-Baali [19] has proven this result via strong wolfe.

Recently, to establish the convergence properties and to obtain good numerical performance in practice, remarkable efforts have been dedicated to upgrade new versions and to modify well-known methods. For example, the modification of the PRP and the HS methods via exact and strong wolfe line searches [20]–[22]

and the proof of the global convergence of Rivaie-Mamat-Ismail-Leong modified method (RMIL+) via the strong wolfe [23]. In this article, for better numerical results of FR method in practice, we present a modified FR method which is again globally convergent and of better performance in practice than the FR.

## 2. THE PROPOSED METHOD AND ALGORITHM

Motivated by the global convergence of the FR, we made a little change to its formula to obtain a modified version with global convergence and better numerical results. The modified formula is given by;

$$\beta_k^{WFR} = \frac{\|g_k\|^2}{\|g_{k-1}\|\|d_{k-1}\|} \tag{10}$$

where the letter W stands for Weis, the family name of the first author.

Having a new formula in (10), we get a new conjugate gradient method called WFR and can be described in the following algorithm 1;

Algorithm 1. WFR algorithm
Step 1. Initialization, choose $x_0 \in R^n, \varepsilon \geq 0$, let $d_0 = -g_0$, and $k = 0$.
Step 2. If $\|g_k\| \leq \varepsilon$ , then terminate.
Step 3. Compute $\alpha_k$ using (4).
Step 4. Set $x_{k+1} = x_k + \alpha_k d_k$, and $g_{k+1} = g(x_{k+1})$. If $\|g_{k+1}\| \leq \varepsilon$, terminate.
Step 5. Evaluate $\beta_k$ using (10) and $d_k$ using (3).
Step 6. Put $k = k + 1$ and go to Step 3.

## 3. GLOBAL CONVERGENCE OF THE PROPOSED ALGORITHM

The convergence analysis plays an important role when studing conjugate gradient method. In this section, we establish the convergence of Algorithm 1. Firstly, we assume the following assumptin on f.

### 3.1. Assumption

- f is bounded below on the set $\{x \in R^n: f(x) \leq f(x_0)\}$.
- In some neighborhing $\mathcal{N}$ of the set in (i), f(x) is differentiable and its g is Lipschitz continuous, namely, there exists a constant $\mu > 0$ such that;

$$\|g(x) - g(y)\| \leq \mu\|x - y\| \ \forall x, y \in \mathcal{N}$$

By considering assumption 3.1, the following Zoutendijk [18] condition refer to Zoutendijk [18] holds;

$$\sum_{k=0} \cos^2 \theta_k \|g_k\|^2 < \infty$$

where $\theta_k$ is the angle between $d_k$ and the steepest descent direction $-g_k$.

Observe that the Zoutendijk [18] condition implies;

$$\lim_{k\to\infty} \cos^2 \theta_k \|g_k\|^2 = 0$$

to establish the global convergence, we prove;

$$\lim_{k\to\infty} \|g_k\| = 0 \tag{11}$$

hence, if $\cos^2\theta_k \geq \delta > 0$, then (11) is hold true.

In exact line search, the orthogonality condition;

$$g_k^T d_{k-1} = 0 \tag{12}$$

holds for all k. from (12) and (3), we get;

$$g_k^T d_k = -\|g_k\|^2 + \beta_k g_k^T d_{k-1} = -\|g_k\|^2 \tag{13}$$

hence, since $g_k^T d_k = \|g_{k-1}\| \, \|d_{k-1}\| \cos\theta_{k-1}$, we get;

$$\cos\theta_{k-1} = \frac{\|g_{k-1}\|^2}{\|g_{k-1}\| \, \|d_{k-1}\|} \tag{14}$$

substituting (14) into the formula (10), we get;

$$\beta_k{}^{WFR} = \frac{\|g_k\|^2}{\|g_{k-1}\|^2}\cos\theta_{k-1} = \beta_k^{FR}\cos\theta_{k-1}$$

since $-1 \le \cos\theta_{k-1} \le 1$, we deduce that;

$$-\beta_k^{FR} \le \beta_k^{WFR} \le \beta_k^{FR}$$

also, substituting (14) into Zoutendijk's condition, we get;

$$\sum_{k=1}^{\infty}\frac{\|g_k\|^4}{\|d_k\|^2} < \infty \tag{15}$$

next, we set the convergence of Algorithm 2.1.

### 3.2. Theorem

Suppose that Assumption 3.1 is satisfied. Let $\{x_k\}$ be produced by Algorithm 1. Then Algorithm 1 is globally convergent, that is, holds true (11).

#### 3.2.1. Proof

To prove, we use contradiction, that is, by supposing (11) is not true. Thus, there exists ε > 0 that;

$$\|g_k\| \ge \varepsilon, \text{ for all } k \tag{16}$$

from (3), indeed, $d_{k+1}$ can be displayed as;

$$d_{k+1} + g_{k+1} = \beta_{k+1}{}^{WFR}\, d_k$$

squaring the both sides, we come to;

$$\|d_{k+1}\|^2 = \left(\beta_{k+1}{}^{WFR}\right)^2\|d_k\|^2 - 2g_{k+1}{}^T\, d_{k+1} - \|g_{k+1}\|^2 \tag{17}$$

due to (10) and (17), we have;

$$\|d_{k+1}\|^2 = \left(\frac{\|g_{k+1}\|^2}{\|g_k\|\|d_k\|}\right)^2\|d_k\|^2 - 2g_{k+1}{}^T\, d_{k+1} - \|g_{k+1}\|^2 \tag{18}$$

using (12), we obtain;

$$\|d_{k+1}\|^2 = \frac{\|g_{k+1}\|^4}{\|g_k\|^2\|d_k\|^2}\,\|d_k\|^2 + 2\|g_{k+1}\|^2 - \|g_{k+1}\|^2$$

which implies;

$$\|d_{k+1}\|^2 = \frac{\|g_{k+1}\|^4}{\|g_k\|^2} + \frac{\|g_{k+1}\|^4}{\|g_{k+1}\|^2}$$

so that;

$$\|d_{k+1}\|^2 = \|g_{k+1}\|^4\left(\frac{1}{\|g_k\|^2} + \frac{1}{\|g_{k+1}\|^2}\right)$$

consequently;

$$\frac{\|d_{k+1}\|^2}{\|g_{k+1}\|^4} = \left(\frac{1}{\|g_k\|^2} + \frac{1}{\|g_{k+1}\|^2}\right) \tag{19}$$

from (16) which means that $\frac{1}{\|g_k\|^2} \leq \frac{1}{\varepsilon^2}$ together with (19), we come to;

$$\sum_{k=1}^{\infty} \frac{\|d_{k+1}\|^2}{\|g_{k+1}\|^4} \leq \sum_{k=1}^{\infty} \frac{2}{\varepsilon^2}$$

that is,

$$\sum_{k=1}^{\infty} \frac{\|g_{k+1}\|^4}{\|d_{k+1}\|^2} \geq \sum_{k=1}^{\infty} \frac{\varepsilon^2}{2} \tag{20}$$

inequality (20), leads to;

$$\sum_{k=1}^{\infty} \frac{\|g_{k+1}\|^4}{\|d_{k+1}\|^2} \geq \infty,$$

which contradicts (15). Therefore, the proof is completed. After proving the global convergence, it remains to show the performance in practical computations.

## 4. RESULTS AND DISCUSSION

In this part, we compare FR and WFR via exact line search and show the results Table 1 in Appendix. Table 1 in Appendix, most of the test problems are taken from Andrei [24]. Also, problem is the test problem, Dim is the dimension taken of the problem, $X_0$ is the initial points from the literature, NI is the number of iterations, FR is the FR method by exact line search, WFR is the modified FR method shown in Algorithm 2.1, and CT(s) is the CPU time. Now to show the improvement in WFR when it is compared with FR, we take FR as 100% with respect to number of iterations (NOI) and CPU time to obtain the following table.

Table 2. Performance of Table 1 based on NOI and CPU time

| TOOLS | FR | WFR |
|---|---|---|
| NOI | 100% | 39% |
| CPU | 100% | 21% |

From Table 2, we see that there is an improvement of about 61% in NOI and about 79% in CPU time. Therefore, in practical computation, WFR is much better than FR method. Also, based on Table 1, we can show the performance by considering $t_{p,s}$ is the result when a solver s is applied to solve problem p (here our solvers are FR and WFR method) and $P_s(t)$ is the ratio $\frac{t_{p,s}}{\min\{t_{p,s}:s\in S\}}$ as in Dolan and More [25] profile, we show the efficiency of Algorithm 2.1 in Figures 1 and 2. Dolan and More [25] performance profile a method of high curve is the best, so it is clear from Figures 1 and 2 that, the WFR method which is described in Algorithm 1 is much better than the FR method.

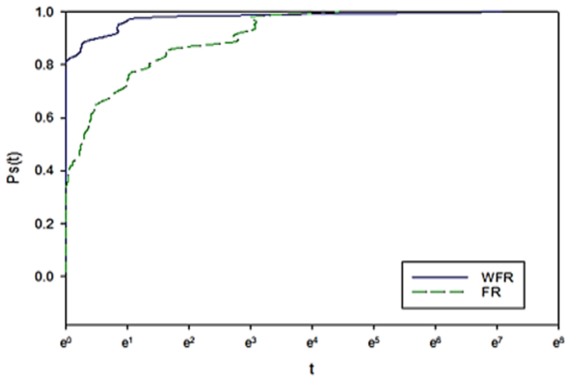


Figure 1. Performance results in case of NI

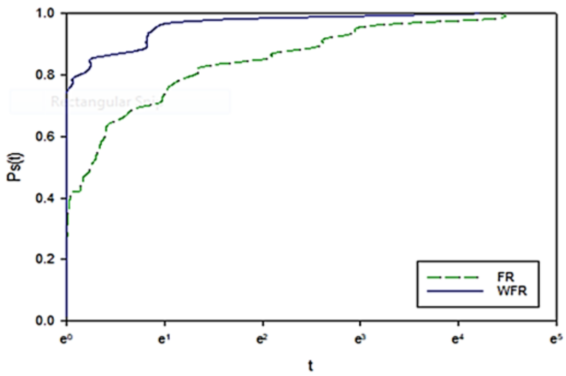


Figure 2. Performance results in case of CPU time

**APPENDIX**

Table 1. Numerical results

| Problem | Dim | $X_0$ | FR | | WFR | |
|---|---|---|---|---|---|---|
| | | | NI | CT(s) | NI | CT(s) |
| Extended white and holst | 10 | -2 | 882 | 5.0080 | 41 | 0.2632 |
| | 1,000 | | 632 | 30.8033 | 41 | 2.1916 |
| | 10 | 6 | 598 | 3.8213 | 772 | 4.9758 |
| Extended freudenstein and roth | 100 | 2 | 16 | 0.1422 | 13 | 0.3207 |
| | 1,000 | | 18 | 0.5242 | 13 | 0.3835 |
| | 100 | 4 | 8 | 0.2460 | 6 | 0.1642 |
| Extended beale | 10 | 6 | 277 | 3.2659 | 13 | 0.1753 |
| | 100 | | 277 | 3.2565 | 13 | 0.2748 |
| Raydan 1 | 10 | 2 | 408 | 3.6955 | 137 | 1.2304 |
| | 100 | | 327 | 2.9331 | 120 | 1.1057 |
| | 100 | -3 | 197 | 1.7995 | 130 | 1.1991 |
| Extended tridiagonal 1 | 10 | 3 | 324 | 2.3845 | 83 | 0.6257 |
| | 100 | | 511 | 6.2461 | 99 | 1.2157 |
| | 1,000 | | 843 | 50.7911 | 107 | 6.4142 |
| | 10 | 9 | 45 | 0.5610 | 57 | 0.7072 |
| | 100 | | 33 | 0.4116 | 77 | 0.9463 |
| | 1,000 | | 46 | 2.7753 | 81 | 4.8669 |
| Diagonal 4 function | 10 | 2 | 5 | 0.0598 | 5 | 0.0637 |
| Extended himmelblau | 10 | 2 | 9 | 0.0748 | 8 | 0.0737 |
| | 1,000 | | 10 | 0.2371 | 9 | 0.1996 |
| | 10 | -2 | 19 | 0.4247 | 20 | 0.4534 |
| | 100 | | 19 | 0.4171 | 19 | 0.4286 |
| | 10 | 6 | 15 | 0.3315 | 15 | 0.3291 |
| | 500 | | 15 | 0.3355 | 15 | 0.3263 |
| | 1,000 | | 15 | 0.3378 | 15 | 0.3280 |
| FLETCHCR function | 100 | 4 | 27 | 0.2303 | 26 | 0.2273 |
| Extended DENSCHNB | 100 | -2 | 14 | 0.1519 | 13 | 0.1502 |
| Extended quadratic penalty QP1 | 10 | 3 | 18 | 0.1515 | 14 | 0.1281 |
| | 100 | | 314 | 2.6221 | 19 | 0.1913 |
| | 100 | -2 | 1,466 | 12.4513 | 17 | 0.1587 |
| Extended penalty | 10 | 3 | 19 | 0.1937 | 15 | 0.1683 |
| | 100 | | 33 | 0.3281 | 16 | 0.1745 |
| | 100 | -2 | 46 | 0.4181 | 12 | 0.1257 |
| Hager function | 10 | -2 | 24 | 0.2270 | 18 | 0.1680 |
| | 100 | | 45 | 0.5284 | 31 | 0.3776 |
| | 10 | -3 | 89 | 1.0508 | 32 | 0.3829 |
| | 100 | | 59 | 0.7012 | 33 | 0.3964 |
| Extended maratos | 100 | 2 | 39 | 0.4379 | 38 | 0.3335 |
| | 100 | -2 | 272 | 4.6560 | 53 | 0.5677 |
| Shallow function | 100 | -2 | 41 | 0.3843 | 28 | 0.2703 |
| | 1,000 | | 45 | 0.9068 | 28 | 0.5784 |
| | 1,000 | 2 | 30 | 0.7150 | 12 | 0.2538 |
| Generalized quartic function | 100 | 5 | 10 | 0.2524 | 10 | 0.2222 |
| | 500 | | 10 | 0.2526 | 10 | 0.2512 |
| | 1,000 | | 10 | 0.2602 | 10 | 0.2587 |
| Quadratic QF2 function | 100 | 2 | 400 | 3.7661 | 147 | 1.4687 |
| | 1,000 | -3 | 24 | 0.5763 | 16 | 0.3916 |
| Generlized tridiagonal 1 | 100 | 3 | 43 | 0.5983 | 27 | 0.4777 |
| | 10 | -2 | 36 | 0.4908 | 29 | 0.3859 |
| Generlized tridiagonal | 10 | 2 | 22 | 0.2232 | 30 | 0.2862 |
| POWER function | 10 | 3 | 22 | 0.1971 | 158 | 1.2606 |
| Quadratic QF1 | 10 | 2 | 10 | 0.1063 | 27 | 0.2598 |
| Extended QP2 | 10 | 5 | 24 | 0.2363 | 29 | 0.2817 |
| Extended quadratic penalty QP1 | 100 | -2 | 283 | 4.3781 | 61 | 1.1500 |
| | 100 | 3 | 1,086 | 54.3123 | 49 | 0.6430 |
| | 500 | 9 | 1,268 | 62.4915 | 82 | 2.1236 |
| Dixon and price function | 10 | -2 | 73 | 0.7407 | 71 | 0.6997 |
| Sphere function | 100 | -2 | 1 | 0.1828 | 1,202 | 12.3998 |
| Sum square’s function | 10 | -2 | 59 | 0.5832 | 135 | 1.3047 |
| | 100 | | 59 | 0.5789 | 135 | 1.3204 |
| Strait function | 10 | 5 | 81 | 0.8645 | 37 | 0.4187 |
| ARWHEAD function | 10 | 3 | 10 | 0.1039 | 10 | 0.2854 |

## 5. CONCLUSION

In this article, a modified version of FR is proposed. Based on the proposed formula, a new modified method is presented. The global convergence is established, provided the line search is exact. To show the efficiency of the modified method in practice, it has been compared with the FR method. It has been reported that the new modified one is much better than FR method.

## ACKNOWLEDGEMENTS

Researchers would like to thank the Deanship of Scientific Research, Qassim University for funding publication of this project.

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