

# Non-Equidistant Multivariable Model MGRM (1,n) Based on Vector Valued Continued Fraction and Reciprocal Accumulated Generating Operation

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## Abstract

Grey system theory is a scientific theory possessed with wide adaptability to study poor information. The construction method of the background value in multivariable grey model was analyzed. The trapezoid formula and extrapolation method using rational interpolation and numerical integration was proposed based on the theory of vector valued continued fractions. And a non-equidistant multivariable grey model MGRM (1,n) was built through applying reciprocal accumulated generating operation. The model is suitable for building both equidistant and non-equidistant models, and it broadens the application range of the grey model and effectively increases both the fitting and the prediction precisions of the model. The applicability and the reliability of the model built were proven by real cases.

**Keywords:** Multivariable, Background Value, Reciprocal Accumulated Generating Operation, Non-Equidistant, Continued Fraction, Trapezoid Formula, MGRM (1,N) Model, Least Square Method

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## 1. Introduction

Grey model is an important element of the grey system theory, GM(1,1), GM(1,N) and MGM(1,N) are among the grey models most commonly used [1-3]. Grey system models are mostly based on equidistant sequences while a big part of the original data obtained in practical work is constituted of non-equidistant sequences. Therefore, it is of practical and theoretical significances to build models based on non-equidistant sequences. Since the construction method of the background value is a major factor that influences the prediction precision and the adaptability, optimizing the background values of the model would be an important way to improve the model. References [4-7] and etc proposed multiple construction methods of backgrounds values and multiple non-equidistant GM(1,1) models in order to improve both the fitting and the prediction precision of GM(1,1) model. However, systems in society, economics and engineering usually contain multiple variables with inner connections. Model MGM(1,N) is the extended form of model GM(1,1) with n-element variables, but it's neither a simple combination of n GM(1,1) models nor a single n-element first-order differential equation like model GM(1,n). MGM(1,N) builds n n-element differential equations and then solves them to obtain a series of parameters that could reflect the correlations among multiple variables [8]. Reference [8] took the first component of sequence  $\mathbf{x}^{(1)}$  as the initial condition of the grey differential equations, and an optimized MGM(1,N) model was built after some modification. Reference [9] built a multivariable new-information MGM(1,N) model with the n'th component of  $\mathbf{x}^{(1)}$  being the initial condition of the grey differential equations based on the principle that new information should be considered in priority. And reference [10] built a multivariable new-information MGM(1,N) model with the n'th component of  $\mathbf{x}^{(1)}$  being the initial condition of the grey differential equations and the initial values as well as the background value coefficient  $q$  optimized and modified. However, these MGM(1,N) models are all equidistant. Reference [11] built non-equidistant multivariable MGM(1,N) models through fitting background values with homogeneous exponential functions. But there were inherent defects in the modeling mechanism of this model because non-homogeneous exponential functions are of greater universality than homogeneous ones. Reference [12] built a multivariable non-equidistant

MGM(1,N) model while its model precision requires further improvement for the background values were generated using the average value. Reference [13] built a non-equidistant multivariable GM(1,N) model through fitting background values with non-homogeneous exponential functions, and the precision was therefore improved. Reference [14] analyzed the construction method of background values in multivariable grey model MGM(1,m), and proposed the trapezoid formula and extrapolation method using rational interpolation and numerical integration to reconstruct background values based on vector valued continued fractions. Reference [15] built the non-equidistant multivariable new information optimization NMGRM(1,n) based on new information background value constructing. So both the simulation and the prediction precisions were effectively increased while the model would usually be multivariable equidistant MGM(1,m) model. For the original sequence  $x^{(0)}$  which is monotonically decreasing, reference [16] proposed the inverse accumulated generating operation and built the grey model GOM(1,1) which is based on inverse accumulated generating operation, reference [17] proposed the reciprocal accumulated generating operation and built the grey model GRM(1,1) which is based on reciprocal generating operation, and reference [18] improved the grey model GRM(1,1) and built the improved grey model CGRM(1,1) which based on reciprocal accumulated generating operation. The models based on both reciprocal accumulated generating operation and inverse accumulated generating operation make the generated sequence  $x^{(1)}$  decrease monotonically. Fit  $x^{(1)}$  with a monotonically decreasing curve, and  $\hat{x}^{(1)}$  which is the model value of  $x^{(1)}$  could be obtained. And there won't be any illogical errors like those in traditional accumulated generating operation and inverse accumulated generating operation modeling methods when  $\hat{x}^{(0)}$  which is the predicted value of  $x^{(0)}$ . So the modeling precision is increased. This paper built a multivariable non-equidistant grey model MGRM(1,n) using reciprocal accumulated generating operation based on the idea of constructing background values, while the model GRM(1,1) built in reference [17] and [18] is equidistant and univariate. The model MGRM(1,n), with high precision as well as great theoretical and applicable values, is suitable for both equidistant and non-equidistant modeling, and it extends the applied range of grey models.

## 2. Multivariable Non-Equidistant Grey Model MGRM(1,N) Based on Reciprocal Accumulated Generating Operation Method

Suppose sequence  $\mathbf{X}_i^{(00)} = [x_i^{(00)}(t_1), \dots, x_i^{(00)}(t_j), \dots, x_i^{(00)}(t_m)]$ . When  $\Delta t_j = t_j - t_{j-1} \neq \text{const}$  and  $i = 1, 2, \dots, n, j = 2, \dots, m$  where  $n$  is the number of variables and  $m$  is the number of sequences of each variable, we call  $\mathbf{X}_i^{(00)}$  a non-equidistant distance sequence.

Make  $x_i^{(0)}(t_j) = \frac{1}{x_i^{(00)}(t_j)} (j=1, 2, \dots, m)$ , and we call  $\mathbf{x}_i^{(0)} = (x_i^{(0)}(t_1), \dots, x_i^{(0)}(t_m))$  the reciprocal sequence of  $\mathbf{X}_i^{(00)}$ .

Suppose sequence  $\mathbf{X}_i^{(1)} = \{x_i^{(1)}(t_1), x_i^{(1)}(t_2), \dots, x_i^{(1)}(t_j), \dots, x_i^{(1)}(t_m)\}$ . If  $x_i^{(1)}(t_1) = x_i^{(0)}(t_1)$  and  $x_i^{(1)}(t_j) = x_i^{(1)}(t_{j-1}) + x_i^{(0)}(t_j) \cdot \Delta t_j, j = 2, \dots, m, i = 1, 2, \dots, n, \Delta t_j = t_j - t_{j-1}$ , we call  $\mathbf{X}_i^{(1)}$  the first-order accumulated generation (1-AG0) of the non-equidistant distance sequence  $\mathbf{X}_i^{(00)}$ .

Suppose the original data matrix of multiple variables is

$$\mathbf{X}^{(0)} = \{\mathbf{X}_1^{(0)}, \mathbf{X}_2^{(0)}, \dots, \mathbf{X}_n^{(0)}\}^T = \begin{bmatrix} x_1^{(0)}(t_1) & x_1^{(0)}(t_2) & \cdots & x_1^{(0)}(t_m) \\ x_2^{(0)}(t_1) & x_2^{(0)}(t_2) & \cdots & x_2^{(0)}(t_m) \\ \cdots & \cdots & \cdots & \cdots \\ x_n^{(0)}(t_1) & x_n^{(0)}(t_2) & \cdots & x_n^{(0)}(t_m) \end{bmatrix} \quad (1)$$

where the measured values of variables of  $\mathbf{X}^{(0)}(t_j)(j=1,2,\dots,m)$  at  $t_j$  are displayed as  $\mathbf{X}^{(0)}(t_j)=[x_1^{(0)}(t_j), x_2^{(0)}(t_j), \dots, x_n^{(0)}(t_j)]$ , and  $[x_i^{(0)}(t_1), x_i^{(0)}(t_2), \dots, x_i^{(0)}(t_j), \dots, x_i^{(0)}(t_m)](i=1,2,\dots,n, j=1,2,\dots,m)$  is non-equidistant which means  $t_j - t_{j-1}$  is not a constant.

For building the model, first conduct accumulation once on the original data and a new matrix is derived as

$$\mathbf{X}^{(1)} = \{\mathbf{X}_1^{(1)}, \mathbf{X}_2^{(1)}, \dots, \mathbf{X}_n^{(1)}\}^T = \begin{bmatrix} x_1^{(1)}(t_1) & x_1^{(1)}(t_2) & \dots & x_1^{(1)}(t_m) \\ x_2^{(1)}(t_1) & x_2^{(1)}(t_2) & \dots & x_2^{(1)}(t_m) \\ \dots & \dots & \dots & \dots \\ x_n^{(1)}(t_1) & x_n^{(1)}(t_2) & \dots & x_n^{(1)}(t_m) \end{bmatrix} \quad (2)$$

where  $x_i^{(1)}(t_j)(i=1,2,\dots,n)$  is equivalent to

$$x_i^{(1)}(t_j) = \begin{cases} \sum_{j=1}^k x_i^{(0)}(t_j)(t_j - t_{j-1}) & (k=2,\dots,m) \\ x_i^{(0)}(t_1) & (k=1) \end{cases} \quad (3)$$

Generate multivariable non-equidistant MGRM(1,n) model as  $n$ -element first-order differential equations based on reciprocal accumulation:

$$\begin{cases} \frac{dx_1^{(1)}}{dt} = a_{11}x_1^{(1)} + a_{12}x_2^{(1)} + \dots + a_{1n}x_n^{(1)} + b_1 \\ \frac{dx_2^{(1)}}{dt} = a_{21}x_1^{(1)} + a_{22}x_2^{(1)} + \dots + a_{2n}x_n^{(1)} + b_2 \\ \dots \\ \frac{dx_n^{(1)}}{dt} = a_{n1}x_1^{(1)} + a_{n2}x_2^{(1)} + \dots + a_{nn}x_n^{(1)} + b_n \end{cases} \quad (4)$$

$$\text{Take } \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{bmatrix}, \text{ then equation (4) could be wrote as}$$

$$\frac{d\mathbf{X}^{(1)}(t)}{dt} = \mathbf{A}\mathbf{X}^{(1)}(t) + \mathbf{B} \quad (5)$$

Take  $\mathbf{x}_i^{(1)}(t_1)$  which is the first component of  $\mathbf{x}_i^{(1)}(t_j)(j=1,2,\dots,m)$  as the original condition of the grey differential equations, and the continuous time response of equation (5) is

$$\mathbf{X}^{(1)}(t) = e^{At}\mathbf{X}^{(1)}(t_1) + \mathbf{A}^{-1}(e^{At} - \mathbf{I})\mathbf{B} \quad (6)$$

where  $e^{At} = \mathbf{I} + \sum_{k=1}^{\infty} \frac{\mathbf{A}^k}{k!} t^k$  and  $\mathbf{I}$  is the unit matrix.

To distinguish **A** and **B**, we dissociate equation (4) and obtain:

$$x_i^{(0)}(t_j)\Delta t_j = \sum_{l=1}^n a_{il} \int_{t_{j-1}}^{t_j} x_i^{(1)}(t_j)dt + b_i \Delta t_j$$

$$(i = 1, 2, \dots, n; j = 2, 3, \dots, m)$$

To distinguish **A** and **B**, we integrate equation (4) on  $[t_{j-1}, t_j]$  and obtain

$$x_i^{(0)}(t_j) = \sum_{l=1}^n a_{il} \frac{\int_{t_{j-1}}^{t_j} x_i^{(1)}(t_j)dt}{\Delta t_j} + b_i$$

$$(i = 1, 2, \dots, n; j = 2, 3, \dots, m)$$
(7)

Take  $\mathbf{a}_i = (a_{i1}, a_{i2}, \dots, a_{in}, b_i)^T$  ( $i = 1, 2, \dots, n$ ) and  $\hat{\mathbf{a}}_i$  which is the value of  $a_i$  could be obtained through least square method as

$$\hat{\mathbf{a}}_i = [\hat{a}_{i1}, \hat{a}_{i2}, \dots, \hat{a}_{in}, \hat{b}_i]^T = (\mathbf{L}^T \mathbf{L})^{-1} \mathbf{L}^T \mathbf{Y}_i, i = 1, 2, \dots, n$$
(8)

Suppose  $z_i^{(1)}(t_j) = \frac{\int_{t_{j-1}}^{t_j} x_i^{(1)}(t_j)dt}{\Delta t_j}$ . When background values are generated using the

average value which means  $z_i^{(1)}(t_j) = 0.5(x_i^{(1)}(t_{j-1}) + x_i^{(1)}(t_j))$ , **L** is expressed as

$$\mathbf{L} = \begin{bmatrix} \frac{1}{2}(x_1^{(0)}(t_1) + x_1^{(0)}(t_2)) & \frac{1}{2}(x_2^{(0)}(t_1) + x_2^{(0)}(t_3)) & \dots & \frac{1}{2}(x_n^{(0)}(t_1) + x_n^{(0)}(t_2)) & 1 \\ \frac{1}{2}(x_1^{(0)}(t_2) + x_1^{(0)}(t_3)) & \frac{1}{2}(x_2^{(0)}(t_2) + x_2^{(0)}(t_3)) & \dots & \frac{1}{2}(x_n^{(0)}(t_2) + x_n^{(0)}(t_3)) & 1 \\ \dots & \dots & \dots & \dots & 1 \\ \frac{1}{2}(x_1^{(0)}(t_{m-1}) + x_1^{(0)}(t_m)) & \frac{1}{2}(x_2^{(0)}(t_{m-1}) + x_2^{(0)}(t_m)) & \dots & \frac{1}{2}(x_n^{(0)}(t_{m-1}) + x_n^{(0)}(t_m)) & 1 \end{bmatrix}$$
(9)

$$\mathbf{Y}_i = [x_i^{(0)}(t_2), x_i^{(0)}(t_3), \dots, x_i^{(0)}(t_m)]^T$$
(10)

And the identify values of **A** and **B** could be obtained as

$$\hat{\mathbf{A}} = \begin{bmatrix} \hat{a}_{11} & \hat{a}_{12} & \dots & \hat{a}_{1n} \\ \hat{a}_{21} & \hat{a}_{22} & \dots & \hat{a}_{2n} \\ \dots & \dots & \dots & \dots \\ \hat{a}_{n1} & \hat{a}_{n2} & \dots & \hat{a}_{nn} \end{bmatrix}, \mathbf{B} = \begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \dots \\ \hat{b}_n \end{bmatrix}$$
(11)

The calculated value of model MGRM(1,n) based on reciprocal accumulation is

$$\mathbf{X}^{(1)}(t) = e^{At} \mathbf{X}^{(1)}(t_1) + \mathbf{A}^{-1}(e^{At} - \mathbf{I})\mathbf{B}$$

And the discrete solution is

$$\hat{\mathbf{X}}_i^{(1)}(t_j) = e^{\hat{\mathbf{A}}(t_j-t_1)} \mathbf{X}_i^{(1)}(t_1) + \hat{\mathbf{A}}^{-1}(e^{\hat{\mathbf{A}}(t_j-t_1)} - \mathbf{I})\hat{\mathbf{B}} \quad (12)$$

$$(j = 1, 2, \dots, m)$$

The equation is taken as the first column of the data and the original value of the solution. And the fitting value of the original data restored from it is

$$\hat{X}_i^{(0)}(t_1) = X_i^{(0)}(t_1)$$

$$\hat{\mathbf{X}}_i^{(0)}(t_j) = (\hat{\mathbf{X}}_i^{(1)}(t_j) - \hat{\mathbf{X}}_i^{(1)}(t_{j-1})) / (t_j - t_{j-1}) \quad (13)$$

$$(j = 2, 3, \dots, m)$$

Then we obtain  $\hat{x}_i^{(00)}(t_j) (j = 1, 2, \dots, m)$  which is the model value of the original sequence using Definition 1.

Define the absolute error of the  $i$ 'th variable as  $\hat{x}_i^{(00)}(t_j) - x_i^{(00)}(t_j)$ , then, the relative error (%) of the  $i$ 'th variable as  $e_i(t_j) = \frac{\hat{x}_i^{(00)}(t_j) - x_i^{(00)}(t_j)}{x_i^{(00)}(t_j)} * 100$ , the average value of

the relative error of the  $i$ 'th variable as  $\frac{1}{m} \sum_{j=1}^m |e_i(t_j)|$  and the mean error of all data as

$$f = \frac{1}{nm} \sum_{i=1}^n \left( \sum_{j=1}^m |e_i(t_j)| \right).$$

We could see that the non-equidistant model MGRM(1,n) degraded into the non-equidistant model GRM(1,1) when  $n = 1$ . And the non-equidistant model MGRM(1,n), a combination of  $n$  non-equidistant model GRM(1,n), is not only available for modeling but also suitable for predicting or data fitting and processing when  $B = 0$ . Models like MGRM(1,2), MGRM(1,3) and MGRM(1,4) were generated according to different specific  $n$  values.

### 3. Construct Background Values of MGRM (1,N) Model Based on the Theory of Vector Valued Continued Fractions

The background values of non-equidistant multivariable models MGRM (1,n) are generated using the average value, which results in unsatisfactory model precisions. Therefore, we adopted the trapezoid formula and extrapolation method using rational interpolation and numerical integration based on the theory of vector valued continued fractions to increase the model precision.

**Definition 1** <sup>[14]</sup> Suppose  $\{a_n\}$  and  $\{b_n\}$  are sequences of real numbers. We call fractions in the form of  $b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \frac{a_4}{b_4 + \dots}}}}$  are continued fractions which are written

as  $b_0 + \overline{K}(a_n / b_n)$ . And fractions like  $b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \frac{a_4}{b_4 + \dots + \frac{a_n}{b_n}}}}}$  are  $n$ -order advanced

fractions of the continued fraction.

**Definition 2** <sup>[14]</sup> Suppose  $v = (v_1, v_2, \dots, v_n)$  is a  $d$ -dimension vector and  $|v| = \sqrt{\sum_{j=1}^d v_j^2}$

is the norm of the vector. The generalized inverse of the vector is  $v^{-1} = \frac{v}{|v|^2}$ .

**Definition 3** <sup>[14]</sup> Let  $\varphi[x_i] = v_i, i = 0, 1, \dots$ ,  $\varphi[x_p, x_q] = \frac{x_q - x_p}{\varphi[x_q] - \varphi[x_p]}$ , and

$\varphi[x_i, \dots, x_j, x_k, x_l] = \frac{x_l - x_k}{\varphi[x_i, \dots, x_j, x_l] - \varphi[x_i, \dots, x_j, x_k]}$ .  $\varphi[x_i, \dots, x_j, x_k, x_l]$  determined by the

formulas above is the  $l$ -order inverse difference of vector set  $V^m$  at point  $x_0, x_1, \dots, x_l$ .

**Theorem 1** <sup>[14]</sup> Suppose  $R_n(x) = \varphi[x_0] + \frac{x - x_0}{\varphi[x_0, x_1] + \frac{x - x_1}{\varphi[x_0, x_1, x_2] + \dots + \frac{x - x_{n-1}}{\varphi[x_0, x_1, x_2, \dots, x_n]}}$

where  $\varphi[x_0, x_1, \dots, x_k] \neq 0, \infty, k = 0, 1, \dots, n$  is the  $k$ -order inverse difference of vector set  $V^m$  at point  $x_0, x_1, \dots, x_k$ , and  $\varphi[x_i, \dots, x_j, x_k, x_l]$  is the  $l$ -order inverse difference of vector set  $V^m$  at point  $x_0, x_1, \dots, x_l$ . And we could obtain:

$R_n(x_i) = v_i = (x_1^{(1)}(i), x_2^{(1)}(i), \dots, x_n^{(1)}(i)), i = 0, 1, \dots, n$  Suppose interval  $[a, b]$  is equivalently divided into  $m_1$  parts, and the step length is  $h = (b - a) / m_1$ . Then the generalized trapezoid integration formula would be:

$$T_m(f) = h[\frac{1}{2}f(a) + f(a+h) + \dots + f(a+(m_1-1)h) + \frac{1}{2}f(b)]$$

$$\text{When } m_1 = 4, z^{(1)}(k+1) = \frac{1}{4}[\frac{1}{2}x^{(1)}(k) + \dots + x^{(1)}(k+\frac{3}{4}) + \frac{1}{2}x^{(1)}(k+1)]$$

$$\text{When } m_1 = 8, z^{(1)}(k+1) = \frac{1}{8}[\frac{1}{2}x^{(1)}(k) + \dots + \frac{1}{2}x^{(1)}(k+1)]$$

$$\text{Combine the formulas } z^{(1)}(k+1) = \frac{4}{3}ze^{(1)}(k+1) - \frac{1}{3}zf^{(1)}(k+1), k = 1, 2, \dots, m-1$$

And take  $z_i^{(1)}(t_{j+1}) = \frac{4}{3}ze_i^{(1)}(t_{j+1}) - \frac{1}{3}zf_i^{(1)}(t_{j+1})$  ( $j = 1, 2, \dots, m-1$ ) as the background values of the grey derivative vector, and take it as  $a_i = (a_{i1}, a_{i2}, \dots, a_{im}, b_i)^T, i = 1, 2, \dots, n$ .  $\hat{a}_i$ , the estimated value of  $a_i$ , is obtained through least square method as:

$$\hat{a}_i = (\hat{a}_{i1}, \hat{a}_{i2}, \dots, \hat{a}_{im}, \hat{b}_i)^T = (\mathbf{L}^T \mathbf{L})^{-1} \mathbf{L}^T \mathbf{Y}_i, i = 1, 2, \dots, n$$

where

$$\mathbf{L} = \begin{bmatrix} \frac{4}{3}ze_1^{(1)}(t_2) - \frac{1}{3}zf_1^{(1)}(t_2) & \frac{4}{3}ze_2^{(1)}(t_2) - \frac{1}{3}zf_2^{(1)}(t_2) & \dots & \frac{4}{3}ze_n^{(1)}(t_2) - \frac{1}{3}zf_n^{(1)}(t_2) & 1 \\ \frac{4}{3}ze_1^{(1)}(t_3) - \frac{1}{3}zf_1^{(1)}(t_3) & \frac{4}{3}ze_2^{(1)}(t_3) - \frac{1}{3}zf_2^{(1)}(t_3) & \dots & \frac{4}{3}ze_n^{(1)}(t_3) - \frac{1}{3}zf_n^{(1)}(t_3) & 1 \\ \dots & \dots & \dots & \dots & \dots \\ \frac{4}{3}ze_1^{(1)}(t_m) - \frac{1}{3}zf_1^{(1)}(t_m) & \frac{4}{3}ze_2^{(1)}(t_m) - \frac{1}{3}zf_2^{(1)}(t_m) & \dots & \frac{4}{3}ze_n^{(1)}(t_m) - \frac{1}{3}zf_n^{(1)}(t_m) & 1 \end{bmatrix} \tag{14}$$

The non-equidistant model built through equation (14) instead of equation (9) according to the first section is the non-equidistant multivariable grey model MGRM(1,n) based on reciprocal accumulated generating operation.

**4. Application**

In contact strength calculation, parameters like the principle curvature function  $F(\rho)$  and coefficients  $m_a$  and  $m_b$  of the long radius  $a$  and the short radius  $b$  of the point contact ellipse are usually looked up in the parameter table. The data below was extracted from Table 1 [19].

Table 1. Values of  $F(\rho)$ ,  $m_a$  and  $m_b$

$F(\rho)$	0.9995	0.9990	0.9980	0.9970	0.9960	0.9930	0.9920	0.9910
$m_a$	23.95	18.53	14.25	12.26	11.02	8.92	8.47	8.10
$m_b$	0.163	0.185	0.212	0.228	0.241	0.268	0.275	0.281
$F(\rho)$	0.9900	0.9890	0.9880	0.9870	0.9860	0.9850	0.9840	0.9810
$m_a$	7.76	7.49	7.25	7.02	6.84	6.64	6.47	6.06
$m_b$	0.287	0.292	0.297	0.301	0.305	0.310	0.314	0.325
$F(\rho)$	0.9800	0.9790	0.9780	0.9770				
$m_a$	5.95	5.83	5.72	5.63				
$m_b$	0.328	0.332	0.335	0.338				

Take  $m_b$  which is the parameter of the ellipse short radius  $b$  as  $t_j$ , the principle curvature function  $F(\rho)$  as  $x_1$ , and  $m_a$  which is the parameter of the ellipse short radius  $a$  as  $x_2$ . We built a non-equidistant MGRM(1,2) model based on reciprocal accumulated generating operation according to the method proposed in the paper. And the parameters of the model are shown as

$$\mathbf{A} = \begin{vmatrix} -0.1249 & 2.4327 \\ 0.4032 & 3.4483 \end{vmatrix}, \quad \mathbf{B} = \begin{vmatrix} 1.0237 \\ -0.5031 \end{vmatrix}$$

The fitted value of the principle curvature function  $F(\rho)$  is:

$$\hat{F}(\rho) = [0.9995, 0.99978, 0.99938, 0.99832, 0.99717, .99601, 0.99487, 0.99371, 0.99258, 0.99152, 0.99046, 0.98944, 0.98845, 0.98752, 0.98666, 0.98565, 0.9846, 0.98376, 0.98289, 0.98186, 0.98094, 0.97999, 0.97901, 0.97815, 0.97697]$$

The absolute error of the principle curvature function is:

$$q = [0, 0.00078371, 0.001381, 0.0013223, 0.0011731, 0.0010093, 0.00087083, 0.00071486, 0.00058468, 0.00051998, 0.00046475, 0.00043722, 0.00045081, 0.00052141, 0.00066052, 0.00065248, 0.00060466, 0.00076081, 0.00089019, 0.00086448, 0.0009399, 0.00098793, 0.001011, 0.0011521]$$

The relative error of the principle curvature function is:

$$e = [0, 0.07845, 0.13838, 0.13263, 0.11778, 0.10144, 0.087609, 0.07199, 0.05894, 0.052471, 0.046945, 0.044208, 0.045628, 0.052828, 0.06699, 0.066242, 0.061449, 0.077397, 0.090651, 0.088122, 0.095909, 0.10091, 0.10338, 0.11792]$$

And the average value of the relative error is 0.079094% which shows a high model precision.

## 5. Conclusion

The trapezoid formula and extrapolation method using rational interpolation and numerical integration was proposed based on the theory of vector valued continued fractions. And a non-equidistant multivariable grey model MGRM (1,n) was built through applying reciprocal accumulated generating operation. The model is suitable for building both equidistant and non-equidistant models, and it broadens the application range of the grey model and effectively increases both the fitting and the prediction precisions of the model. The applicability and the reliability of the model built were proven by real cases. Therefore, this model is of important practical and theoretical model, and it is worthy of popularization.

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