

# Gabor Systems about any Symmetric Points in Two Dimensions

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## Abstract

Gabor system plays an important role in signal processing, image processing and other applications because of their redundancy properties. In this paper, symmetric or antisymmetric Gabor frames in two dimensions about any symmetric points are constructed from Gabor frames given. In special, symmetric or antisymmetric Parseval Gabor frames with better properties are obtained. Some thoughts of existing results in wavelet analysis are borrowed. At last, some examples are provided to prove the theory.

**Keywords:** Wavelet system, Gabor system, Frame

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## 1. Introduction

Today we are living in a data world. On the one hand, people have to develop the way to process all kind of types of data. On other hand, they are faces with analyzing the accuracy of such methods and providing a deeper understanding of the underlying structures. There is a pressing need for those tasks deriving from various fields such as signal processing, image processing, digital communications, medical imaging, and so on.

In the late 18th century, the Fourier Transform is the first tool to analyze the data. When the Fast Fourier Transform (FFT) was developed, it achieved the greatest achievements. Today, FFT is still one of the most fundamental algorithms and can be found in various applications. However, the Fourier Transform itself has a serious disadvantage: a local perturbation of  $f$  leads to a change of all Fourier coefficients simultaneously, since it merely analyzes the global structure of a signal. However, in many signal processing we have to detect the location of the signal, and this indicates a defect in an engineering process.

This deficiency led to the birth of the new fields of applied harmonic analysis, which is nowadays already one of the major research areas in applied mathematics. It exploits not only methods from harmonic analysis, but also borrows from areas such as approximation theory, numerical mathematics and operator theory.

Frames were first introduced by Duffin and Schaeffer [1] in the field of nonharmonic Fourier series. Since the ground breaking work of Daubechies, Grossmann, and Meyer [2], the theory of frames has been widely studied. Traditionally, frames were used in signal processing, image processing and other applications [3-4]. Recently, frames are also used to mitigate the effect of losses in packet-based communication systems and hence to improve the robustness of data transmission [5].

An important example about frame is wavelet frame, which is obtained by shifting and dilating a finite family of functions. Wavelet theory has been studied extensively in both theory and applications since 1980's. The main advantage of wavelets is their time-frequency localization property.

The theory of wavelets plays undoubtedly an important role in all mathematical, engineering and other fields throughout the last decade. Thus, few other mathematical theoretical sciences have enjoyed this much attention and popularity.

Then, we briefly review the history of wavelet analysis. In 1985, Stephane Mallat [6] gave wavelets an additional jump-start through his work in digital signal processing. He discovered some relationships among quadrature mirror filters, pyramid algorithms and

orthonormal wavelet bases. Inspired by these results, Y. Meyer constructed the first non-trivial wavelets. Unlike the Haar wavelets, the Meyer wavelets are continuously differentiable, however they do not have compact support. Later, Ingrid Daubechies [7], [8] used Mallat's work to construct a set of wavelet orthonormal basis functions that are perhaps the most elegant, and have become the corner stone of wavelet applications today. Though people have given an algorithm for constructing the mother wavelet by multiresolution analysis (MRA), not every wavelet is generated from an MRA as J. Journe in 1992 demonstrated by his celebrated example.

Another important concrete realization of frames is Gabor frames. Gabor systems were first introduced by Gabor [2]. They are generated by modulations and translations of some functions. That is, we choose fixed functions  $g \in L^2(\mathbb{R}^2)$  and two parameters  $a, b > 0$ , and define the associated Gabor system  $G(g, a, b)$  by

$$G(g, a, b) = \{E_{lb}T_{ka}g : j, k \in \mathbb{Z}\}, \quad (1.1)$$

where  $T_{ka}f(x) = f(x - ka)$  and  $E_{lb}f(x) = e^{2\pi lbx}f(x)$ . Such systems, also called Weyl-Heisenberg systems, were introduced by Gabor with the aim of constructing efficient, time-frequency localized expansions of signals as (infinite) linear combinations of elements. A major development in the theory of Gabor systems is due to Daubechies, Grossmann and Meyer [2] who placed the problem of Gabor expansions in the framework of frames for a Hilbert space. We state the definition of frames and give their main properties in Section 2. Since the appearance of [2], Gabor systems are widely studied by many researchers by characterizing Gabor systems being frames and efficiently computing canonical duals.

The fundamental problems of Gabor theory are: how should we choose functions such that Gabor systems possess the spanning properties. When do the Gabor systems span a dense subspace of  $L^2(\mathbb{R}^2)$ ? When do Gabor systems constitute frames or linearly independent families for  $L^2(\mathbb{R}^2)$ ? The investigation about these problems is nowadays referred to as Gabor analysis [9]. At last, we think the monograph of Grochenig [10] provides an elaborate depict about time frequency analysis and Gabor systems.

Shearlet systems [11] are systems generated by one single generator with parabolic scaling, shearing, and translation operators applied to it, in the same way wavelet systems are dyadic scalings and translations of a single function, but including a directionality characteristic owing to the additional shearing operation (and the anisotropic scaling). In fact, from above shearlet's definition, it is obvious that shearlet system is a kind of special composite dilation wavelet system.

The main advantages of shearlet theory lie in that the shearing filters can have smaller support sizes than the directional filters used in the contourlet transform and can be implemented much more efficiently. An additional appealing point to make in favor of the shearlets approach is that they provide a unified treatment of such continuum models as well as digital models, allowing, for instance, a precise resolution of wavefront sets, optimally sparse representations of cartoon-like images, and associated to fast decomposition algorithms. Shearlet systems can be designed to efficiently encode anisotropic features. In order to achieve optimal sparsity, shearlets are scaled according to a parabolic scaling law. They parameterize directions by slope encoded in a shear matrix. Readers can refer to papers [12-13] for more knowledge about shearlet theory.

Shearlet system plays an important role in image compression, denoising, edge analysis and detection [14-15]. With the improvements of shearlet theory, people will obtain more huge success.

It is well known that the symmetry of wavelet plays an important role in image processing. In [16], authors gave a simple way to construct symmetric or antisymmetric wavelet frames from any wavelet frames given. Motivated by the way of [16], we will discuss the case of Gabor system. We obtain some similar results in Gabor analysis.

In this paper, symmetric or antisymmetric Gabor frames in two dimensions about any symmetric points are constructed from Gabor frames given. In special, symmetric or antisymmetric Parseval Gabor frames are obtained. This way makes the amount of Gabor

frames largely increase. We borrow some thoughts of existing result in wavelet analysis. At last, some examples are provided to prove the theory.

## 2. Preliminaries

In this section, some notations and some results which will be used later are introduced.

Throughout this paper, the following notations will be used.  $R^2$  and  $Z^2$  denote the set of n-dimensional real numbers and the set of integers, respectively.  $L^2(R^2)$  is the space of all square-integrable functions, and  $\langle \cdot, \cdot \rangle$  and  $\|\cdot\|$  denote the inner product and norm in  $L^2(R^2)$ , respectively, and  $l(Z^2)$  denotes the space of all square summable sequences.

We use the Fourier transform in the form  $\hat{f}(\omega) = \int_{R^2} f(x)e^{-2\pi i x \cdot \omega} dx$ , where  $\cdot$  denotes the standard inner product in  $R^2$ .

Let us recall the definition of frame.

**Definition 1.** Let  $H$  be a separable Hilbert space. A sequence  $\{f_i\}_{i \in N}$  of elements of  $H$  is a frame for  $H$  if there exist constants  $0 < C \leq D < \infty$  such that for all  $f \in H$ , we have

$$C\|f\|^2 \leq \sum_{i \in N} \|\langle f, f_i \rangle\|^2 \leq D\|f\|^2. \quad (2.1)$$

The numbers  $C, D$  are called lower and upper frame bounds, respectively (the largest  $C$  and the smallest  $D$  for which (2.1) holds are the optimal frame bounds). Those sequences which satisfy only the upper inequality in (2.1) are called Bessel sequences. A frame is tight if  $C = D$ . If  $C = D = 1$ , it is called a Parseval frame.

Let  $T_f$  denote the synthesis operator of  $f = \{f_i\}_{i \in N}$ , i.e.,  $T_f(c) = \sum_i c_i f_i$  for each sequence of scalars  $c = (c_i)_{i \in N}$ . Then the frame operator  $Sh = T_f T_f^*(h)$  associated with  $\{f_i\}_{i \in N}$  is a bounded, invertible, and positive operator mapping of  $H$  on itself. This provides the reconstruction formula

$$h = \sum_{i=1}^{\infty} \langle h, \tilde{f}_i \rangle f_i = \sum_{i=1}^{\infty} \langle h, f_i \rangle \tilde{f}_i, \forall h \in H. \quad (2.2)$$

where  $\tilde{f}_i = S^{-1} f_i$ . The family  $\{\tilde{f}_i\}_{i \in N}$  is also a frame for  $H$  and is called the canonical dual frame of  $\{f_i\}_{i \in N}$ . If  $\{g_i\}_{i \in N}$  is any sequence in  $H$  which satisfies

$$h = \sum_{i=1}^{\infty} \langle h, g_i \rangle f_i = \sum_{i=1}^{\infty} \langle h, f_i \rangle g_i, \forall h \in H, \quad (2.3)$$

it is called an alternate dual frame of  $\{f_i\}_{i \in N}$ .

Then, we will give the definitions of composite dilation multiwavelet frame and the frame composite dilation multiwavelet.

**Definition 2.** We say that the Gabor system defined by (1.1) is a Gabor frame if it is a frame for  $L^2(R^2)$ .

### 3. Main Results

In this section, we consider to construct symmetric or antisymmetric Gabor frames with any symmetric points from any Gabor system frames given.

For some functions  $g \in L^2(\mathbb{R}^2)$  and fixed point  $y \in \mathbb{R}^2$ , define new symmetric or antisymmetric functions with symmetric point  $y$  as the following:

$$\begin{aligned} g_1(x) &= \frac{g(y+x) + g(y-x)}{2}, \\ g_2(x) &= \frac{g(y+x) - g(y-x)}{2}. \end{aligned} \quad (3.1)$$

Thus, we have

**Theorem 3.1** Suppose that Gabor system

$$\{E_l T_k g^m : k \in \mathbb{Z}^2, l \in \mathbb{Z}\}$$

defined by (1.1) is a frame for  $L^2(\mathbb{R}^2)$  with frame bounds  $C_1, C_2$ , then Gabor system

$$\{E_l T_k g_1 \cup E_l T_k g_2 : k \in \mathbb{Z}^2, l \in \mathbb{Z}\} \quad (3.2)$$

is a symmetric or antisymmetric frame for  $L^2(\mathbb{R}^2)$  about any symmetric points  $y$  with frame bounds  $C_1, C_2$ , where the functions  $g_1(x), g_2(x)$  are defined by (3.1).

*Proof.* Because Gabor system

$$\{E_l T_k g : k \in \mathbb{Z}^2, l \in \mathbb{Z}\}$$

is a frame with frame bounds  $C_1, C_2$ , for all  $f(x) \in L^2(\mathbb{R}^2)$ , we have

$$C_1 \|f\|^2 \leq \sum_{l \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^2} |\langle f, E_l T_k g \rangle|^2 \leq C_2 \|f\|^2. \quad (3.3)$$

In order to prove that Gabor system defined by (3.2) is a frame, we firstly calculate the series

$$\sum_{l \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^2} |\langle f, E_l T_k g_1 \rangle|^2 + \sum_{l \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^2} |\langle f, E_l T_k g_2 \rangle|^2 \quad (3.4)$$

According to definition of  $g_1$  and the property of inner product, we can obtain

$$|\langle f(\cdot), E_l T_k g_1 \rangle|^2 = \left| \langle f(\cdot), E_l T_k \frac{g(y+\cdot)}{2} \rangle + \langle f(\cdot), E_l T_k \frac{g(y-\cdot)}{2} \rangle \right|^2 \quad (3.5)$$

For any complex numbers  $z_1, z_2$ , it is well known that the following equality holds

$$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + z_1 \bar{z}_2 + \bar{z}_1 z_2. \quad (3.6)$$

From (3.5) and (3.6), we have

$$\begin{aligned}
\langle f(\cdot), E_l T_k g_1(\cdot) \rangle|^2 &= \frac{1}{4} |\langle f(\cdot), E_l T_k g(y+\cdot) \rangle|^2 + \frac{1}{4} |\langle f(\cdot), E_l T_k g(y-\cdot) \rangle|^2 \\
&+ \frac{1}{4} \langle f(\cdot), E_l T_k g(y+\cdot) \rangle \overline{\langle f(\cdot), E_l T_k g(y-\cdot) \rangle} \\
&+ \frac{1}{4} \overline{\langle f(\cdot), E_l T_k g(y+\cdot) \rangle} \langle f(\cdot), E_l T_k g(y-\cdot) \rangle.
\end{aligned} \tag{3.7}$$

In the similar way, we can prove

$$\begin{aligned}
\langle f(\cdot), E_l T_k g_2(\cdot) \rangle|^2 &= \frac{1}{4} |\langle f(\cdot), E_l T_k g(y+\cdot) \rangle|^2 + \frac{1}{4} |\langle f(\cdot), E_l T_k g(y-\cdot) \rangle|^2 \\
&- \frac{1}{4} \langle f(\cdot), E_l T_k g(y+\cdot) \rangle \overline{\langle f(\cdot), E_l T_k g(y-\cdot) \rangle} \\
&- \frac{1}{4} \overline{\langle f(\cdot), E_l T_k g(y+\cdot) \rangle} \langle f(\cdot), E_l T_k g(y-\cdot) \rangle.
\end{aligned} \tag{3.8}$$

Comparing with (3.7) and (3.8), we get

$$\begin{aligned}
&\sum_{l \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^2} \langle f, E_l T_k g_1 \rangle|^2 + \sum_{l \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^2} \langle f, E_l T_k g_2 \rangle|^2 \\
&= \frac{1}{2} \sum_{l \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^2} |\langle f(\cdot), E_l T_k g(y+\cdot) \rangle|^2 \\
&+ \frac{1}{2} \sum_{l \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^2} |\langle f(\cdot), E_l T_k g(y-\cdot) \rangle|^2
\end{aligned} \tag{3.9}$$

By simple calculation, we have

$$\sum_{l \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^2} |\langle f(\cdot), E_l T_k g(y-\cdot) \rangle|^2 = \sum_{l \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^2} |\langle f(y-\cdot), E_l T_k g(\cdot) \rangle|^2. \tag{3.10}$$

According to (3.3), we obtain

$$C_1 \|f(-y-\cdot)\|^2 \leq \sum_{l \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^2} |\langle f(-y-\cdot), E_l T_k g \rangle|^2 \leq C_2 \|f(-y-\cdot)\|^2, \tag{3.11}$$

From (3.10), (3.11) and the equality  $\|f(-x_m-\cdot)\|^2 = \|f(\cdot)\|^2$ , we deduce

$$C_1 \|f\|^2 \leq \sum_{l \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^2} |\langle f, E_l T_k g(y-\cdot) \rangle|^2 \leq C_2 \|f\|^2, \quad \forall f(x) \in L^2(\mathbb{R}^2). \tag{3.12}$$

At last, comparing with (3.3), (3.9) and (3.12), we get

$$C_1 \|f\|^2 \leq \sum_{l \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^2} |\langle f, E_l T_k g_1 \rangle|^2 + \sum_{l \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^2} |\langle f, E_l T_k g_2 \rangle|^2 \leq C_2 \|f\|^2. \tag{3.13}$$

Therefore, we have completed the proof of Theorem 3.1.

**Remark**

(1) In particular, if that Gabor system is an orthonormal base, then, by simpler deduction, we obtain the construction of symmetric or antisymmetric Parseval Gabor frames about origin from any Gabor frames given.

**Corollary 3.1.** Suppose that Gabor system

$$\{E_l T_k g : k \in \mathbb{Z}, l \in \mathbb{Z}\}$$

is an orthonormal base for  $L^2(\mathbb{R})$ , then, Gabor system

$$\{E_l T_k g_1 \cup E_l T_k g_2 : k \in \mathbb{Z}, l \in \mathbb{Z}\}$$

is a symmetric or antisymmetric Parseval frame for  $L^2(\mathbb{R})$ , where the functions  $g_1(x), g_2(x)$  are defined by (3.1).

(2) By adjusting the proof of theorem 3.1, we easily obtain the generalization of theorem 3.1.

**Corollary 3.2.** Suppose that Gabor system

$$\{E_{la} T_{kb} g : k \in \mathbb{Z}, l \in \mathbb{Z}\}$$

defined by (1.1) is a frame for  $L^2(\mathbb{R})$  with frame bounds  $C_1, C_2$ , then Gabor system

$$\{E_{la} T_{kb} g_1 \cup E_{la} T_{kb} g_2 : k \in \mathbb{Z}, l \in \mathbb{Z}\} \quad (3.14)$$

is a symmetric or antisymmetric frame for  $L^2(\mathbb{R})$  about origin with frame bounds  $C_1, C_2$ , where the functions  $g_1(x), g_2(x)$  are defined by (3.1).

**4. Some examples**

In paper [12], authors constructed several examples of symmetric or antisymmetric wavelet frames from any wavelet frames given.

In the following, we mainly devote to constructing symmetric or antisymmetric Gabor frames about any symmetric points. We only provide several examples to prove our theory.

**Example 1.** Let  $g \in L^2(\mathbb{R})$  be a real-valued bounded function with  $\text{supp } g \subset [0, L]$  for which

$$\sum_{n \in \mathbb{Z}} g(x-n) = 1.$$

Let  $b \in (0, \frac{1}{2L-1}]$ . Consider any scalar sequence  $\{a_n : n = -N+1, -N+2, \dots, N-2, N-1\}$  for which

$$a_0 = b, a_n + a_{-n} = 2b, b = 1, 2, \dots, N-1,$$

and define  $h \in L^2(\mathbb{R})$  by

$$h(x) = \sum_{n=-N+1}^{N-1} a_n g(x+n).$$

Then, According to theorem 3.1 of [13], the functions  $g \in L^2(R)$  and  $h \in L^2(R)$  generate dual frames  $\{E_{la}T_k g : k \in Z, l \in Z\}$  and  $\{E_{la}T_k h : k \in Z, l \in Z\}$  for  $L^2(R)$ .

For a fixed point  $y \in R$ , define new symmetric or antisymmetric functions with the symmetry as the following:

$$g_1(x) = \frac{g(y+x) + g(y-x)}{2}, g_2(x) = \frac{g(y+x) - g(y-x)}{2};$$

$$h_1(x) = \frac{h(y+x) + h(y-x)}{2}, g_2(x) = \frac{h(y+x) - h(y-x)}{2}.$$

Then, according to Theorem 3.1, both Gabor system

$$\{E_{la}T_k g_1 \cup E_{la}T_k g_2 : k \in Z, l \in Z\}$$

and Gabor system

$$\{E_{la}T_k h_1 \cup E_{la}T_k h_2 : k \in Z, l \in Z\}$$

are symmetric Gabor frames about symmetric point  $y$  for  $L^2(R)$ . Furthermore, we can prove that they generate a symmetric or antisymmetric dual frames for  $L^2(R)$ .

**Example 2.** If we partition  $[0, a)$  into disjoint measurable sets  $(A_n)_{n \in Z}$ , let  $B_n = A_n + \{n\}$  and define  $f(x, y) = d_n + nx$  for all  $y \in A_n$ .

Then, according to [13], the function  $g(x)$  by the following

$$g(x) = \sum_{n \in Z} c_n \chi_{B_n}$$

yields a Parseval Gabor frame.

Define new symmetric or antisymmetric functions with the symmetry as the following:

$$g_1(x) = \frac{g(y+x) + g(y-x)}{2}, g_2(x) = \frac{g(y+x) - g(y-x)}{2}.$$

Then, according to Theorem 3.1, Gabor system

$$\{E_{la}T_k g_1 \cup E_{la}T_k g_2 : k \in Z, l \in Z\}$$

is symmetric Gabor frames about symmetric point  $y$  for  $L^2(R)$ .

## 5. Conclusion

In this paper, symmetric or antisymmetric Gabor frames in two dimensions about any symmetric points are constructed from Gabor frames given. In special, symmetric or antisymmetric Parseval Gabor frames are obtained. This way makes the amount of Gabor

frames largely increase. We borrow some thoughts of existing result in wavelet analysis. At last, some examples are provided to prove the theory.

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