

On image restoration problems using new conjugate gradient methods

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ABSTRACT

The nonlinear conjugate gradient algorithm is one of the effective algorithms for optimization since it has low storage and simple structure properties. The coefficient conjugate is the basis of conjugate gradient algorithms with the desirable conjugate property. In this manuscript, we have derived a new second order information for the Hessian from objective function, which can give a new search direction. Based on new search direction, we have proposed the update formula interesting and nonlinear conjugate gradient method. Under wolfe line search and mild assumptions on objective function, the method possess sufficient descent property and are always globally convergent. Numerical results show that the method is effective and competitive to recover the original image from an image corrupted by impulse noise.

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1. INTRODUCTION

Images are often corrupted by impulse noise. Salt-and-pepper and random-valued impulse noise are removed using two-phase techniques. The first phase employs decision-based median filters to identify pixels that are prone to noise corruption (noise candidates). These noise candidates are restored in the second phase using a detail-preserving regularization approach that preserves edges and noise-free pixels. Let X be the genuine image and $A = \{1,2,3,\dots,M\} \times \{1,2,3,\dots,N\}$ be the index set of X . The collection of indices of the noise pixels observed in the first phase is denoted by $N \subset A$. The noise pixels are then recovered in the second phase by (1):

$$F_{\alpha}(u) = \sum_{(i,j) \in N} \left[|u_{i,j} - y_{i,j}| + \frac{\beta}{2} (2 \times S_{i,j}^1 + S_{i,j}^2) \right] \quad (1)$$

where β is the regularization parameter, $S_{i,j}^1 = 2 \sum_{(m,n) \in P_{i,j} \cap N^c} \phi_{\alpha}(u_{i,j} - y_{m,n})$, $S_{i,j}^2 = \sum_{(m,n) \in P_{i,j} \cap N} \phi_{\alpha}(u_{i,j} - y_{m,n})$ and edge-preserving potential function is $\phi_{\alpha} = \sqrt{\alpha + x^2}$, $\alpha > 0$. Let $P_{i,j}$ be the set of four closest neighbors of the pixel at location $(i,j) \in A$, $y_{i,j}$ the image's observed pixel value at position (i,j) , and $u_{i,j} = [u_{i,j}]_{(i,j) \in N}$ a lexicographically ordered column vector of length c . The number of elements in N is represented by the letter c . In actuality, the smooth data-fitting term is eliminated from the minimization, and only noisy pixels are restored. The smooth functional that results is as shown in (2).

$$F_{\alpha}(u) = \sum_{(i,j) \in N} [2 \times S_{i,j}^1 + S_{i,j}^2] \quad (2)$$

For more details see Xue *et al.* [1] and Yu *et al.* [2]. Our aim is to minimize a function of n variables by using iteration methods:

$$\text{Min}f(u) , u \in R^N \tag{3}$$

where f is smooth and its gradient g_{k+1} is available. Many practical problems can be translated into the problems, see Dai and Wen [3] and Hajmohammadi [4].

One of the most widely remarkable and powerful used minimization functions is conjugate gradient methods. Begins the minimization process with an initial estimate x_0 and an initial search direction $d_0 = -g_0$. As we noted that the iterative formula given by (4):

$$u_{k+1} = u_k + \alpha_k d_k \tag{4}$$

where α_k is a step length generated by a suitable line search and d_k is the search direction, see [5]. Obviously, the approximate adequate step size associated to quadratic function f_k is (5).

$$\alpha_k = \frac{-g_k^T d_k}{d_k^T Q d_k} \tag{5}$$

For general nonlinear functions, it is necessary to use an iterative procedure [6]. We use the typical Wolfe criteria to determine the step length throughout our operation, as shown in (6) and (7):

$$f(u_k + \alpha_k d_k) \leq f(u_k) + \delta \alpha_k g_k^T d_k \tag{6}$$

$$d_k^T g(u_k + \alpha_k d_k) \geq \sigma d_k^T g_k \tag{7}$$

where $0 < \delta < \sigma < 1$, Moghrabi [7]. In conjugate gradient methods, the search directions can be defined recursively:

$$d_{k+1} = -g_{k+1} + \beta_k s_k \tag{8}$$

where the coefficient β_k is so chosen that d_k and d_{k+1} must fulfill the conjugacy property. In practical computations promising, the CG methods include suggested by Fletcher and Reeves (FR) [8] and Dai and Yuan (DY) [9] is believed of the efficient methods, respectively given by:

$$\beta_k^{FR} = \frac{g_{k+1}^T g_{k+1}}{g_k^T g_k}, \beta_k^{DY} = \frac{g_{k+1}^T g_{k+1}}{d_k^T s_k} \tag{9}$$

or via other formulaes (e.g. see [4], [10]-[12]). Some of conjugate gradient (CG) approaches are numerically successful, while the others are conceptually successful.

Later, if the point x_{k+1} is close enough to a local minimizer u^* in pure Newton's technique, the search direction of good to follow, that is, $d_{k+1} = -Q_{k+1}^{-1}g_{k+1}$, is determined. Nazareth [13], inspired by this, rewrites in (5).

$$-Q_{k+1}^{-1}g_{k+1} = -g_{k+1} + \beta_k s_k \tag{10}$$

On this guideline, this work is improved to investigate ideas from Newton updates, in order to build definite new method for conjugate gradient methods. They proved to be really effective in practice and showed mature convergence properties. This idea was presented as a technical method in some manuscripts, for example, to maximize in the advantages of the original respective conjugate gradient methods [7], [14]-[16]. To introduce a new method, we will find the Hessian approximation of the minimum of a function $f(u)$, which give a new search direction and choose the coefficient conjugate satisfies above relation.

2. OUR NEW COEFFICIENT CONJUGATE

A very important coefficient conjugate in minimization is that for conjugate gradient method. In order to deriving a new coefficient conjugate, we exposed a second order Taylor series as:

$$f(x) = f(x_{k+1}) + g_{k+1}^T(x - x_{k+1}) + \frac{1}{2}(x - x_{k+1})^T Q(x_{k+1})(x - x_{k+1}) \tag{11}$$

can see derivative as (12).

$$g_{k+1} = g_k + Q(x_{k+1})s_k \quad (12)$$

A good second order curvature information, we obtained (13).

$$s_k^T Q(x_{k+1})s_k = (f_{k+1} - f_k) - 3/2s_k^T g_k \quad (13)$$

The yielded matrix $Q(x_{k+1})$ can be as (14).

$$Q(x_{k+1}) = \frac{(f_{k+1}-f_k)-3/2s_k^T g_k}{s_k^T s_k} \quad (14)$$

Putting $Q(x_{k+1})$ in (10) we get (15).

$$\beta_k = \left(1 - \frac{s_k^T s_k}{(f_{k+1}-f_k)-3/2s_k^T g_k}\right) \frac{g_{k+1}^T y_k}{s_k^T y_k} \quad (15)$$

In addition by (14) is gained (16).

$$\beta_k = \left(1 - \frac{s_k^T s_k}{(f_{k+1}-f_k)+3/2s_k^T y_k}\right) \frac{g_{k+1}^T y_k}{s_k^T y_k} \quad (16)$$

To ensure the sufficient descent condition, we will do some algebra manipulations on above formula, we get:

$$\beta_k = \frac{1}{s_k^T y_k} \left(y_k - \varpi \frac{\|y_k\|^2}{s_k^T y_k} s_k \right)^T g_{k+1} \quad (17)$$

where:

$$\varpi = \frac{(s_k^T y_k)}{\|y_k\|^2} \left[\frac{s_k^T y_k}{s_k^T s_k} * \frac{s_k^T s_k}{(f_{k+1}-f_k)-3/2s_k^T g_k} \right] \quad (18a)$$

$$\varpi = \frac{(s_k^T y_k)}{\|y_k\|^2} \left[\frac{s_k^T y_k}{s_k^T s_k} * \frac{s_k^T s_k}{(f_{k+1}-f_k)+3/2s_k^T y_k} \right] \quad (18b)$$

this formulas will assist us to proving that the direction is satisfied the sufficient descent property. We denote by binarized neural networks (BNN) and block truncated-Newton (BTN) the methods defined by (17). We introduced the BNN and BTN methods by the following algorithm, which would be used for its convergence analysis. New algorithm (BNN and BTN Algorithms):

- Stage 1. Give initial point u_1 . Set $k = 1$ and $d_1 = -g_1$. If $\|g_1\| \leq 10^{-6}$, then stop.
- Stage 2. Evaluate α_k satisfying the Wolfe conditions (6) and (7).
- Stage 3. Let $u_{k+1} = u_k + \alpha_k d_k$. If $\|g_{k+1}\| \leq 10^{-6}$, then stop.
- Stage 4. Evaluate β_k by the formulae (17) then generate d_{k+1} by (8).
- Stage 5. Set $k = k + 1$ and continue with step 2.

Theorem 1.

Let $s_k, y_k, g_{k+1} \in R^n, \beta_k \in R$ and $\beta_k^{BNN} = \frac{1}{s_k^T y_k} \left(y_k - \varpi \frac{\|y_k\|^2}{s_k^T y_k} s_k \right)^T g_{k+1}$. If $s_k^T y_k \neq 0$, then

$$d_{k+1}^T g_{k+1} \leq - \left[1 - \frac{1}{4\varpi} \right] \|g_{k+1}\|^2$$

Proof :

Since $d_0 = -g_0$, we have $g_0^T d_0 = -\|g_0\|^2$, which satisfy (17). Multiplying (8) by g_{k+1} , we have:

$$d_{k+1}^T g_{k+1} = -\|g_{k+1}\|^2 + \left(\frac{g_{k+1}^T y_k}{s_k^T y_k} - \varpi \frac{\|y_k\|^2}{(s_k^T y_k)^2} g_{k+1}^T s_k \right) s_k^T g_{k+1} \quad (19)$$

Yielding:

$$d_{k+1}^T g_{k+1} = \frac{(g_{k+1}^T y_k)(s_k^T g_{k+1})(s_k^T y_k) - \|g_{k+1}\|^2 (s_k^T y_k)^2 - \varpi \|y_k\|^2 (g_{k+1}^T s_k)^2}{(s_k^T y_k)^2} \tag{20}$$

now, by applying:

$$w = \frac{1}{\sqrt{2\varpi}} (s_k^T y_k) g_{k+1} \text{ and } v = \sqrt{2\varpi} (g_{k+1}^T s_k) y_k \tag{21}$$

into the inequality:

$$w^T v \leq \frac{1}{2} (\|w\|^2 + \|v\|^2) \tag{22}$$

we can obtain:

$$(g_{k+1}^T y_k)(s_k^T g_{k+1})(s_k^T y_k) \leq \frac{1}{2} \left[\frac{1}{2\varpi} (s_k^T y_k)^2 \|g_{k+1}\|^2 + 2\varpi (s_k^T g_{k+1})^2 \|y_k\|^2 \right] \tag{23}$$

therefore by (23) and (20), we have that:

$$d_{k+1}^T g_{k+1} \leq \frac{\left[\frac{1}{4\varpi} - 1 \right] (s_k^T y_k)^2 \|g_{k+1}\|^2 + [\varpi - \varpi] (s_k^T g_{k+1})^2 \|y_k\|^2}{(s_k^T y_k)^2} \tag{24}$$

therefore, we get:

$$d_{k+1}^T g_{k+1} \leq - \left[1 - \frac{1}{4\varpi} \right] \|g_{k+1}\|^2 \tag{25}$$

which completes our proof. In BTN method similar to the proof done in BNN.

3. CONVERGENCE ANALYSIS

To understand the global convergence theorem of the BNN technique, we must first understand the assumptions listed below. 1. The $\Omega = \{x \in R^n / f(x) \leq f(x_1)\}$ is a bounded level set. 2. In some neighborhood Λ of Ω , the gradient of function g is Lipschitz continuous, namely, there exists a constant $L > 0$ such that:

$$\|g(o) - g(\tau)\| \leq L \|o - \tau\|, \forall \tau, o \in \Lambda \tag{26}$$

We demonstrate the Dai *et al.* [17] theorem it is very important for deducing global convergence.

Lemma 1.

Let x_k be generated by (4), d_k satisfy descent property and α_k be satisfy (6)-(7). If:

$$\sum_{k \geq 0} \frac{1}{\|d_{k+1}\|^2} = \infty, \tag{27}$$

then:

$$\lim_{k \rightarrow \infty} \inf \|g_{k+1}\| \tag{28}$$

We introduced our main theorem of this paper.

Theorem 2.

Consider the method formed by (8), (17) and $f(x)$ satisfies Assumption 1 and 2. If step size α_k satisfies wolfe conditions (6) and (7), then we obtain:

$$\lim_{k \rightarrow \infty} \inf \|g_k\| = 0 \tag{29}$$

Proof:

We may deduce the following from (8) and the definition of β_k given by (17)

$$\|d_{k+1}\| = \|-g_{k+1} + \beta_k^{BNN} d_k\| \leq \|g_{k+1}\| +$$

$$\begin{aligned}
 & \frac{\|y_k\| \|g_{k+1}\| + \varpi \frac{\|g_{k+1}\| \|y_k\|^2 \|s_k\|}{\|s_k\| \|y_k\|}}{\alpha_k \|d_k\| \|y_k\|} \|d_k\| \\
 & \leq \|g_{k+1}\| + \frac{\|y_k\| \|g_{k+1}\| + \varpi \|g_{k+1}\| \|y_k\|}{\alpha_k \|d_k\| \|y_k\|} \|d_k\| \\
 & \leq \left[1 + \frac{1}{\alpha_k} + \frac{\varpi}{\alpha_k}\right] \|g_{k+1}\| \leq \left[\frac{\alpha_k + 1 + \varpi}{\alpha_k}\right] \|g_{k+1}\|
 \end{aligned} \tag{30}$$

This relationship demonstrates that:

$$\sum_{k \geq 1} \frac{1}{\|d_k\|^2} \geq \left(\frac{\alpha_k}{\alpha_k + 1 + \varpi}\right) \frac{1}{\Gamma} \sum_{k \geq 1} 1 = \infty \tag{31}$$

we obtain $\lim_{k \rightarrow \infty} \inf \|g_k\| = 0$ from Lemma 1, which is identical to $\lim_{k \rightarrow \infty} \|g_k\| = 0$ for a uniformly convex function. In BTN way, one can find its proof similar to the proof done in BNN.

4. NUMERICAL RESULT

The goal of the results of applying the proposed BNN and FR methods to test images (Lena, House, Cameraman and Elaine), will be reported in the Table 1. is to analyze the practical performance of Algorithms. The method was implemented in MATLAB on a PC-type computer with the stopping criterion:

$$\frac{|f(u_k) - f(u_{k-1})|}{|f(u_k)|} \leq 10^{-4} \text{ and } \|f(u_k)\| \leq 10^{-4}(1 + |f(u_k)|) \tag{32}$$

more details can be found in [18]-[24]. The examined pictures are presented in Table 1, and the comprehensive numerical results of our testing are reported in Table 1, where NI, NF, peak signal to noise ratio (PSNR) denotes the number of iterations, function evaluations, and PSNR, which is defined as:

$$PSNR = 10 \log_{10} \frac{255^2}{\frac{1}{MN} \sum_{i,j} (u_{i,j}^r - u_{i,j}^*)^2} \tag{33}$$

where pixel values for the restored and original images are denoted by $u_{i,j}^r$ and $u_{i,j}^*$, respectively.

Table 1. Performance of FR, BNN and BTN algorithms

Image	Noise level r (%)	FR-Method			BNN-Method			BTN-Method		
		NI	NF	PSNR (dB)	NI	NF	PSNR (dB)	NI	NF	PSNR (dB)
Le	50	82	153	30.5529	2	8	37.7683	2	8	37.8792
	70	81	155	27.4824	2	8	36.6362	2	8	36.5749
	90	108	211	22.8583	9	29	29.5153	2	8	35.6453
ho	50	52	53	30.6845	10	34	38.0645	3	14	40.588
	70	63	116	31.2564	12	33	34.9569	4	14	40.2693
	90	111	214	25.287	12	33	31.6781	4	14	38.94
El	50	35	36	33.9129	10	35	34.6177	3	17	36.4966
	70	38	39	31.864	12	41	33.32	3	17	35.667
	90	65	114	28.2019	14	47	32.1012	4	21	33.935
c512	50	59	87	35.5359	10	31	39.1136	3	12	42.176
	70	78	142	30.6259	12	39	35.7245	3	14	39.7155
	90	121	236	24.3962	12	42	33.333	4	19	37.3434

Figures 1-4 show the obtained results by denoised images. Figure 1(a), Figure 2(a), Figure 3(a), and Figure 4(a) are the images corrupted with 70% salt-and-pepper noise; Figure 1(b), Figure 2(b), Figure 3(b) and Figure 4(b) are results of FR method; Figure 1(c), Figure 2(c) and Figure 3(c), and Figure 4(c) are results of the BNN method; Figure 1(d), Figure 2(d), Figure 3(d) and Figure 4(d) are results of the BTN method. Take everything together, the numerical experiments show that the proposed method performs well for removing impulse noise images. The field of research work in optimization is wider as in [25]-[30].

Lena Image

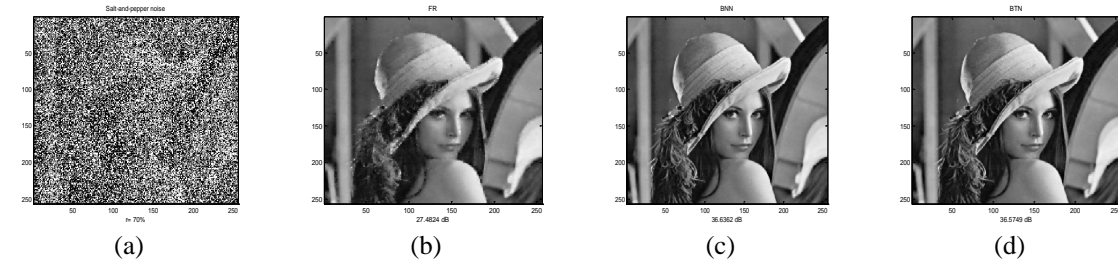


Figure 1. Comparing images results of algorithms: (a) denoised images with 70% salt-and-pepper noise, (b) recovered images through FR, (c) and (d) restored images using BNN and BTN of 256 * 256

House image

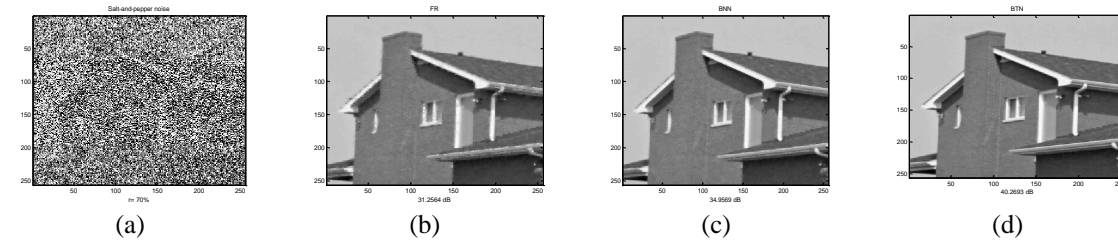


Figure 2. Comparing images results of algorithms: (a) denoised images with 70% salt-and-pepper noise, (b) recovered images through FR, (c) and (d) restored images using BNN and BTN of 256 * 256

Elaine image

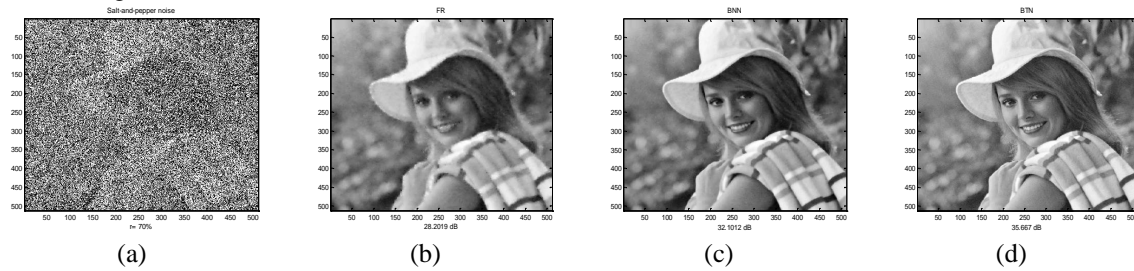


Figure 3. Comparing images results of algorithms: (a) denoised images with 70% salt-and-pepper noise, (b) recovered images through FR, (c) and (d) restored images using BNN and BTN of 256 * 256

Camerman picture image

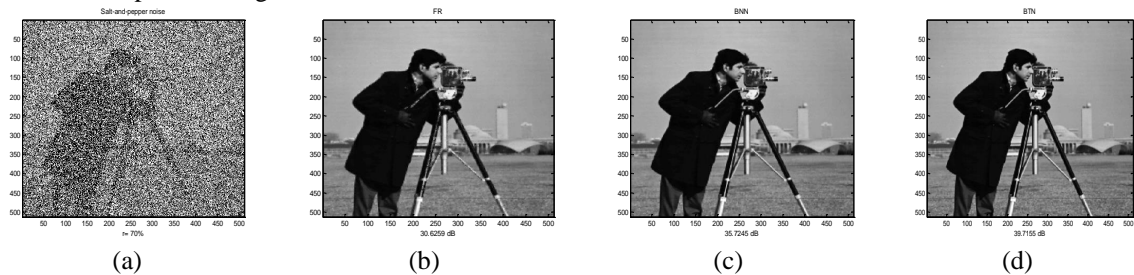


Figure 4. Comparing images results of algorithms: (a) denoised images with 70% salt-and-pepper noise, (b) recovered images through FR, (c) and (d) restored images using BNN and BTN of 256*256

5. CONCLUSION

Using the second order Taylor series to deriving a new coefficient conjugate for conjugate gradient method for image restoration problems. The algorithms exhibit global convergence and the necessary descent property holds. In numerical tests, the new method has made significant progress. It demonstrates that the novel conjugate gradient approach outperforms the conventional FR conjugate gradient method.




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