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Global Exponential Stability Analysis of Dynamic Neural Networks with Distributed Delays

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Abstract

In this paper, the existence, uniqueness and globally exponential stability of the equilibrium point of a dynamic neural network with distributed delays were studied without assumption of boundedness and differentiability of activation functions. Sufficient criteria for existence, uniqueness and global exponential stability of the equilibrium point of such neural networks were obtained based on the knowledge of M-matrix, topology and Lyapunov stability theory. A test matrix was constructed by the weight matrix and the conditions satisfying activation functions of the neural networks. A neural network has a unique equilibrium point and is globally exponential stable if the test matrix is an M-matrix. Since the criterion is independent of the delays and simplifies the calculation, it is easy to test the conditions of the criterion in practice.

Keywords: *neural network, global exponential stability, lyapunov function, distributed delays*

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1. Introduction

There has recently been increasing interest in the potential applications of the dynamic cellular neural networks in the field of image processing, pattern recognition and associative memory because of its nonlinear transformation characteristics and the great ability of parallel computing. The existence, uniqueness and globally exponential stability of the equilibrium point of neural networks have a direct impact on the performance of its hardware circuit. But in the realization of circuit in neural network, the time delay factor is inevitable which may cause the system performance is often not stable. So, it has an important theoretical and practical significance in researching the existence, uniqueness and globally exponential stability of the equilibrium point of a dynamic neural network with distributed delays.

Some research results on stability have been derived for the dynamic neural network with distributed delays. Reference [1-3], [9] studied on the Global exponential stability of the static neural network. Reference [4, 5] studied on the stability of the neural network without time delays. Reference [6-8], [10-12] studied on the stability of the neural network with variable delays, but the unbounded delays were not involved. Reference [13-17] studied on the stability of the neural network with distributed delays and Reference [14] investigated the stability of the neural network under the assumption that the activation function is monotonically non-decreasing, and obtained the asymptotic stability criterion.

Although many results were derived for testing the stability solutions of the dynamic cellular neural networks with distributed or unbounded delays, to the best of our knowledge, the globally exponential stability of dynamic cellular neural networks with distributed delays are seldom considered. In this paper, we study the globally exponential stability of dynamic cellular neural networks with distributed delays. By constructing test matrix based on the weight matrix and the conditions satisfying activation functions of the neural networks, applying M-matrix theory and Lyapunov stability theory, we obtain the sufficient conditions for globally exponential stability of the equilibrium points of dynamic cellular neural networks with distributed delays.

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2. Preliminaries

The dynamic cellular neural networks with distributed delays can be described by the following nonlinear differential equations:

$$u_i^*(t) = -a_i d_i(u_i(t)) + \sum_{j=1}^{n} c_{ij} \int_{-\infty}^{t} k_{ij}(t-s) g_j(u_i(s)) d_s + J_i , \quad i = 1, 2, \ldots, n \tag{1}$$

Where i and j are the neuron number; u_i is the state of neuron and $d_i(u_i)$ is damping function; a_i is damping coefficient and $a_i > 0$; c_{ij} is connection weights; $g_i(u_i)$ is the activation function of the ith neuron; J_i is an input constant, and $k_{ij}(t-s)$ is the kernel function; The initial state of system (1) is $u_i(s) = \phi_i(s), s \le 0, \phi_i(s)$ is bounded and continuous in $(-\infty, 0]$.

For system (1), we assume the functions of g_j, d_j and k_{ij} meet the following conditions.

Assumption 1. $\forall j \in \{1, 2, \ldots, n\}$, the activation function $g_j : R \to R$ is the globally Lipschitz function, i.e. $|g_j(x_j) - g_j(y_j)| \le L_j |x_j - y_j|$ for all x_j, y_j, where L_j is a Lipschitz constant and $L_j > 0$.

Assumption 2. Suppose d_i are differentiable in R and $b_i = \inf\{d_i^*(u_i)\} > 0$, $\sup\{d_i^*(u_i)\} < +\infty$, $i = 1, 2, \ldots, n$.

Assumption 3. $k_{ij} : [0, \infty) \to [0, \infty)$ are piecewise continuous on $[0, \infty)$ and satisfy $\int_0^\infty e^{\beta s} k_{ij}(s) ds = p_{ij}(\beta)$, $i, j = 1, 2, \ldots, n$, where $p_{ij}(\beta)$ are continuous functions in $[0, \delta)$, $\delta > 0$, and $p_{ij}(0) = 1$, $i, j = 1, 2, \ldots, n$.

In the following, we let:

$$A = diag[a_1 \;\; a_2 \ldots a_n],\; B = diag[b_1 \;\; b_2 \ldots b_n],\; L = diag[L_1 \;\; L_2 \ldots L_n], C = (b_{ij})_{n \times n},$$
$$g(u) = [g_1(u_1) \;\; g_2(u_2) \ldots g_n(u_n)]^T,\; J = [J_1 \;\; J_2 \ldots J_n]^T,\; d(u) = [d_1(u_1) \;\; d_2(u_2) \ldots d_n(u_n)]^T.$$

In order to obtain our results, we give the following definitions and lemmas.

Definition 1. The equilibrium point u^* of system (1) is said to be globally exponentially stable, if there exist constant $\alpha > 0$ and $M > 0$ such that $\| u_i(t) - u^* \| \le M \| \phi - u^* \| e^{-\alpha t}$ for all $t \ge 0$, where $\phi = diag[\phi_1 \;\; \phi_2 \ldots \phi_n]^T$, $\| \phi - u^* \| = \max_{1 \le i \le n} \{ \sup_{s \in (-\infty, 0]} | \phi_i(s) - u_i^* | \}$.

Lemma 1. Let $A = (a_{ij})_{n \times n}$ be a matrix with non-positive off-diagonal elements. Then the following statements are equivalent:

(1) *A* is an M-matrix;

(2) The real parts of all eigenvalues of *A* are positive;

(3) There exists a vector $\xi > 0$, such that $\xi^T A > 0$;

(4) A is an nonsingular and all the elements of the A^T are nonnegative;

(5) There exists a positive definite diagonal matrix Λ, such that $\Lambda A + A^T \Lambda$ is a positive definite matrix;

Lemma 2. If $H(u) \in C^0$ is a continuous function (C^0 is a continuous function space) and satisfies the following conditions:

(1) $H(u)$ is an injective function on R^n;

(2) $\lim_{\|u\| \to \infty} \|H(u)\| \to \infty$, then the function $H(u)$ is a homeomorphism mapping on R^n.

3. Existence and Uniqueness of the Equilibrium Point

In Reference [14], the activation function must satisfy $0 \le (g_j(x_j) - g_j(y_j))(x_j - y_j) \le K_j(x_j - y_j)^2$. It is not difficult to see that the constraint conditions on the activation function are loose in the **Assumption 1**. Under the condition satisfying the requirements of **Assumption 1**, the nonlinear mapping associated with the system (1) can be described by the following equations:

$$H(u) = -Ad(u) + Cg(u) + J \tag{2}$$

Where $H(u) = [H_1(u_1)\ H_2(u_2) \ldots H_n(u_n)]^T$. The solution of $H(u) = 0$ is the equilibrium point of the system. If $H(u)$ is an homeomorphism mapping on R^n, then there exists only one point u^* which satisfying the equation $H(u^*) = 0$, i.e. the system (1) has the unique equilibrium point u^*.

Theorem 1. If system (1) satisfies the requirements of **Assumption 1~3** and $AB - |C|L$ is an M-matrix, then the system (1) has the unique equilibrium point.

Proof. To prove the system (1) has the unique equilibrium point u^*, only need to prove $H(u)$ is a homeomorphism mapping on R^n.

Step 1. Prove $H(u)$ is an injective function on R^n.

Here we use reduction to absurdity. Suppose there exists $x, y \in R^n$, $x \ne y$ and $H(x) = H(y)$. From **Assumption 1** and **Assumption 2**, we can get:

$$\begin{aligned} 0 &= |H(x) - H(y)| = \left|-A[d(x) - d(y)] + C[g(x) - g(y)]\right| \ge \left|A[d(x) - d(y)]\right| - \left|C[g(x) - g(y)]\right| \\ &\ge AB|x - y| - |C|L|x - y| = (AB - |C|L)|x - y| \end{aligned} \tag{3}$$

Since $AB - |C|L$ is an M-matrix, from **Lemma 1**, we can get that $AB - |C|L$ is a nonsingular matrix and all of its inverse elements are nonnegative. From formula (3) we can deduce that $|x - y| \le 0$, so $x - y = 0$, i.e. $x = y$. It is contradiction with the assumption which is $x \ne y$, so $H(u)$ is an injective function on R^n.

Step 2. Prove $\lim_{\|u\| \to \infty} \|H(u)\| \to \infty$.

Define $\bar{H}(u) = H(u) - H(0)$, then to prove $H(u)$ is a homeomorphism on R^n, only need to prove $\bar{H}(u)$ is a homeomorphism on R^n.

Due to $AB - |C|L$ is an M-matrix, from the **Lemma 1**, we get that there exists a positive definite diagonal matrix T, such that $T(AB - |C|L) + [AB - |C|L]^T T$ is a positive definite matrix, so there exists a sufficiently small positive number ε, satisfy:

$\frac{1}{2}\left\{T(-AB + |C|L) + [-AB + |C|L]^T T\right\} \le -\varepsilon E$, where E is an identity matrix.

From **Assumption 1** and **Assumption 2**, we can get:

$$\begin{aligned} &[Tu]^T \bar{H}(u) = [Tu]^T (H(u) - H(0)) = [Tu]^T [-d(u) - d(0) + C(g(u) - g(0)] \\ &\le |u|^T [T(-AB + |C|L)]|u| \le |u|^T \frac{1}{2}\left\{T(-AB + |C|L) + [-AB + |C|L]^T T\right\} \le -\varepsilon\|u\|^2 \end{aligned} \tag{4}$$

From formula (4), we get that $\varepsilon\|u\|^2 \le \|T\|\|u\|\left\|\bar{H}(u)\right\|$, as a result $\left\|\bar{H}(u)\right\| \ge \frac{\varepsilon\|u\|}{\|T\|}$, therefore when $\|u\| \to +\infty$, $\left\|\bar{H}(u)\right\| \to +\infty$, so we can deduce that $\|H(u)\| \to +\infty$.

According to the above proof and **Lemma 2**, we can obtain the conclusion that $H(u)$ is a homeomorphism on R^n for arbitrary input u, the proof is completed.

4. Global Exponential Stability

Theorem 2. If system (1) satisfies the requirements of **Assumption 1~3** and $AB-|C|L$ is an M-matrix, then the system (1) has the unique and globally exponential stability of the equilibrium point.

Proof. Due to $AB-|C|L$ is an M-matrix, from **Theorem 2**, system (1) has a unique equilibrium point u^*. Let $x(t)=u(t)-u^*$, then system (1) can be expressed as following:

$$x_i^*(t)=-a_i\bar{d}i(x_i(t))+\sum_{j=1}^{n}c_{ij}\int_{-\infty}^{t}k_{ij}(t-s)g_j(x_i(s))d_s \tag{5}$$

Where $i=1,2,\cdots n$, $G_j(x_j)=g_j(x_j+u_j)-g_j(u_j)$ are still satisfy the **Assumption 1** and $\bar{d}_i(x_i)=d_i(x_i+u_i^*)-d_i(u_i^*)$ are still satisfy the **Assumption 2**.

According to the initial conditions of system (1), we can reduce the initial conditions of Equation (5) expressed as following:

$\psi(s)=\phi(s)-u^*$, where $-\infty<s\le 0$.

Since system (1) has a unique equilibrium point u^*, so Equation (5) has a unique solution $x=0$

Due to $AB-|C|L$ is an M-matrix, from the **Lemma 1**, we that there exist positive constant numbers ξ_i, $i=1,2,\ldots,n$, satisfy:

$$-a_ib_i\xi_i+\sum_{j=1}^{n}\xi_j\,|\,c_{ij}\,|\,L_j<0, i=1,2,...,n\,.$$

Because $p_{ij}(\beta)$ are continuous functions on $[0,\delta)$ and $p_{ij}(0)=1$, so there exists a constant $\alpha>0$ such that:

$$\xi_i(-a_ib_i+\alpha)+\sum_{j=1}^{n}\xi_j\,|\,c_{ij}\,|\,p_{ij}(\alpha)L_j<0 \tag{6}$$

Define the Lyapunov functions as following:

$$v_i(t)=e^{\alpha t}\,|\,x_i(t)\,|, v=\left[v_1,v_2,\ldots,v_n\right]^T.$$

According to the **Assumption 1** and **Assumption 2**, we can get the upper right of $v_i(t)$.

$$\begin{aligned}D^+(v_i(t))&=e^{\alpha t}\operatorname{sgn}x_i\left\{-a_i\bar{d}_i(x_i(t))+\sum_{j=1}^{n}c_{ij}\int_{-\infty}^{t}k_{ij}(t-s)g_j(x_i(s))d_s\right\}+\alpha e^{\alpha t}\left|x_i(t)\right|\\&\le e^{\alpha t}\left\{(-a_ib_i+\alpha)\left|x_i(t)\right|+\sum_{j=1}^{n}L_j\left|c_{ij}\right|\left|\int_{-\infty}^{t}k_{ij}(t-s)\right|x_i(s)\left|d_s\right.\right\}\\&=(-a_ib_i+\alpha)v_i(t)+\sum_{j=1}^{n}L_j\left|c_{ij}\right|\left|\int_{-\infty}^{t}k_{ij}(t-s)e^{\alpha(t-s)}\right|v_j(s)\left|d_s\right., i=1,2,\ldots,n.\end{aligned} \tag{7}$$

Define $\gamma=\{z(l):z_i=\xi_il,\ l>0,\ i=1,2,\ldots,n\}$, $O(z)=\{w:0\le w\le z,\ z\in\gamma\}$.

Let $\xi_{\max} = \max_{1\le i\le n}\{\xi_i\}$, $\xi_{\min} = \min_{1\le i\le n}\{\xi_i\}$, $l_0 = (1+\sigma)\|\psi\|/\xi_{\min}$, where $\sigma > 0$ is a constant, then $v(s)\in O(z(l_0))$, $v(s)\notin\gamma$, i.e.

$$v_i(s) = e^{\alpha s}\left|\psi_i(s)\right| < \xi_i l_0 \quad , \tag{8}$$

Where $-\infty < s \le 0, i = 1,2,\ldots,n$. In the following we prove:

$$v_i(t) < \xi_i l_0,\ t \ge 0, i = 1,2,\ldots,n. \tag{9}$$

If (9) is not true, then from (8), there exist $t_1 > 0$ and some i such that:

$$V_i(t_1) = \xi_i l_0\ ,\ D^+(V_i(t_1)) \ge 0\ ,\ V_j(t) \le \xi_j l_0\ , \tag{10}$$

Where $j = 1,2,\cdots,n$, $t\in(-\infty, t_1]$.

According to (6), (7), we can get:

$$D^+\{v_i(t_1)) \le \left\{\xi_i(-a_i b_i + \alpha) + \sum_{j=1}^{n} L_j \mid c_{ij} \mid p_{ij}(\alpha)\xi_i\right\} l_0 < 0\ .$$

However in (10), $D^+(v_i(t_1)) \ge 0$, this is a contradiction. So $v_i(t) < \xi_i l_0$ for all $t > 0$. Furthermore, we can obtain $|x_i(t)| \le \xi_i l_0 e^{-\alpha t} \le (1+\sigma)\|\psi\|\frac{\xi_{\max}}{\xi_{\min}} e^{-\alpha t} = M\|\psi\| e^{-\alpha t}$, where $i = 1,2,\ldots,n$.

So $|u_i(t) - u_i^*| \le M\|\phi - u^*\| e^{-\alpha t}$, where $M = (1+\sigma)\frac{\xi_{\max}}{\xi_{\min}}$.

From the **Definition 1**, system (1) is globally exponential stable at the equilibrium point. The proof is completed.

5. An Illustrative Example

The dynamic cellular neural networks with distributed delays can be described by the following differential equations:

$$\left.\begin{aligned} u_1^*(t) &= -0.8d_1(u_1(t)) + 0.15\int_{-\infty}^{t} k_1(t-s)g_1(u_1(s))ds + 0.3\int_{-\infty}^{t} k_2(t-s)g_2(u_2(s))ds + J_1 \\ u_2^*(t) &= -d_2(u_2(t)) + 0.1\int_{-\infty}^{t} k_1(t-s)g_1(u_1(s))ds + 0.1\int_{-\infty}^{t} k_2(t-s)g_2(u_2(s))ds + J_2 \end{aligned}\right\} \tag{11}$$

Where $d_1(u_1(t)) = 0.5u_1; d_2(u_2(t)) = 0.3u_2 - 1; g_1(u_1(t)) = \dfrac{e^{u_1} - e^{-u_1}}{e^{u_1} + e^{-u_1}}; g_2(u_2(t)) = \dfrac{e^{u_2} - e^{-u_2}}{e^{u_2} + e^{-u_2}}; k_1(t) = e^{-t}; k_2(t) = \dfrac{2}{1+t^3}e^{-t}$.

It is easy to verify that $g_1(u_1)$ and $g_2(u_2)$ are satisfy the **Assumption 1** with and $L_1 = L_2 = 1$; $d_1(u_1)$ and $d_2(u_2)$ are satisfy the **Assumption 2** with and $b_1 = 0.5, b_2 = 0.3$; $k_1(t)$ and $k_2(t)$ satisfy the **Assumption 3**. Thus we get:

$$A = \begin{bmatrix} 0 & 8 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}, C = \begin{bmatrix} 0.15 & 0.30 \\ 0.10 & -0.10 \end{bmatrix}, L = \begin{bmatrix} 0.25 & -0.30 \\ -0.10 & 0.20 \end{bmatrix}.$$

So we obtain $AB - |C|L = \begin{bmatrix} 0.25 & -0.30 \\ -0.10 & 0.20 \end{bmatrix}$ is an M-matrix. From **Theorem 2**, we can determine that (11) has equilibrium points and (11) is globally exponential stable on these points.

6. Conclusion

Based on the knowledge of M-matrix, topology and Lyapunov stability theory, by constructing proper vector Lyapunov functions, the existence and uniqueness of the equilibrium point and its global exponential stability are investigated for a class of neural networks with distributed delays. Without assuming the boundedness and differentiability of the activation functions, several new sufficient criterions ascertaining the existence, uniqueness and global exponential stability of the equilibrium point of such neural networks are obtained. Since the criterions is independent of the delays and simplifies the calculation, it is easy to test the conditions of the criterion in practice. Furthermore, the research method of this paper can be applied to study the stability for other types of neural network.

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