

# High SNR Gain by Stochastic Resonance in a Tristable System

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## Abstract

We report that the signal-to-noise ratio (SNR) can be improved by the stochastic resonance (SR) in a tristable system. The system is driven by Gaussian white noise and a sinusoidal signal, and studied by using the second-order Runge-Kutta method. We find that the SNR gain exhibits the stochastic resonance behavior, and greatly exceeds unity on some occasions. This result is the latest development of the tristable stochastic resonance, and has potential applications in the signal detection, processing and communications.

**Keywords:** tristable stochastic resonance, SNR gain, numerical simulation, nonlinearity

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## 1. Introduction

The single-mode nonlinear optical system is a pretty complex nonlinear system. Y. M. Kang et al observed stochastic resonance with the spectral power amplification  $\eta$  in the single-mode nonlinear optical system which is monostable, bistable and tristable, respectively [1]. Based on Reference [1], we further study the SNR gain of the tristable system, and find that the SNR gain exhibits characteristic signature of SR and greatly exceeds unity.

## 2. Tristable SR in a Single-mode Nonlinear Optical System [1]

The phenomenological dynamic equation of a single-mode optical system with six-order potential function is [1]:

$$\frac{dx}{dt} = -\frac{x^5}{5} + \frac{(1+C)x^3}{3} - Cx + y + \Gamma(t) \quad (1)$$

Where  $y$  is dimensionless incident field;  $x$  is dimensionless emergent field;  $C$  is system parameter;  $\Gamma(t)$  is Gaussian white noise, and satisfies  $\langle \Gamma(t) \rangle = 0$ ,  $\langle \Gamma(t+\tau)\Gamma(t) \rangle = 2D\delta(\tau)$ , where  $D$  is noise intensity. Equation (1) is a more complex nonlinear optical system. It's monostable, bistable and tristable, respectively, with tuning the parameter  $C$  and  $y$ . This paper focuses on the tristable system due to space limitations, the other two cases have been studied respectively in our other papers. According to Reference [1], when  $C = 0.1$ ,  $y = 0$ , the system is tristable, the corresponding potential  $U(x)$  is:

$$U(x) = \frac{x^6}{30} - \frac{1.1x^4}{12} + \frac{0.1x^2}{2} \quad (2)$$

Equation (2) shows in Figure 1.

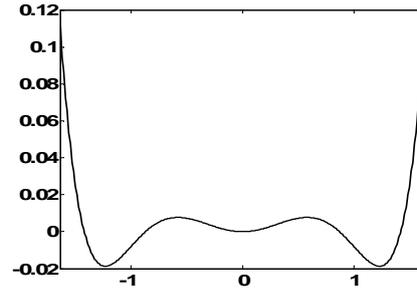


Figure 1. The Potential  $U(x)$  with  $C=0.1$ ,  $y=0$ .

Actually, Equation (1) can be regarded as an overdamped Langevin equation. A weak periodic signal  $A \cos(2\pi Ft)$  is added to Equation (1) [1].

$$\frac{dx}{dt} = -U'(x) + A \cos(2\pi Ft) + \Gamma(t) \quad (3)$$

After the transient processes in the system (3) have died out, the noise-averaged value of the system's coordinate  $x$  performs driven oscillations around equilibrium [2],

$$\langle x \rangle(t) = \langle x \rangle_{eq} + |\chi_x| A \cos(2\pi Ft + \delta_x) + o(A^2) \quad (4)$$

For small driving amplitudes, the higher-order terms  $o(A^2)$  can be neglected.  $\chi_x$  is called linear-response susceptibility of  $x$ . To characterize SR, we use the spectral power amplification [3, 4].

$$\eta = |\chi_x|^2 \quad (5)$$

Reference [1] adopted the variational method to numerically calculate the relaxation rate  $\lambda_{\min}$ , and obtained the numerical result of  $\eta$  using the linear response theory.  $\eta(D)$  shows SR with a single hump in Figure 2.

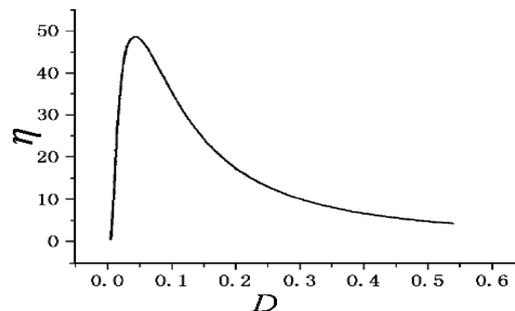


Figure 2.  $\eta$  vs.  $D$  for the Stochastic Dynamics (3) ( $F=0.016\text{Hz}$ )

There are two potential barriers in the potential function (Equation (2)), the corresponding Equation (3) is a tristable system. The basic mechanism leading to SR is similar to that of the conventional bistable system possessing a barrier, namely, it is also that the

transition between wells results in SR since noise provides sufficient energy to surmount the barriers.

### 3. SNR Gain

There are a variety of definitions about the measure of stochastic resonance, for example, SNR gain, SNR, spectral power amplification and linear-response susceptibility. Among these definitions SNR gain is the most important, which is defined as follows:

$$SNR_{gain} = \frac{SNR_{out}}{SNR_{in}} \quad (6)$$

Where  $SNR_{gain}$  is SNR gain,  $SNR_{out}$  is output SNR,  $SNR_{in}$  is input SNR.

So far, there's no uniform definition about the SNR of stochastic resonance systems [5-8]. Two definitions are mainly adopted, one is widely used in the field of SR: [3, 5, 9].

The stochastic resonance system is driven by the sinusoidal signal and noise, the output SNR.

$$SNR = \frac{S(F_0)}{N(F_0)} \quad (7)$$

Where  $F_0$  is the frequency of input sinusoidal signal,  $S(F_0)$  is the power of  $F_0$  frequency component in the output port, and  $N(F_0)$  is the background noise spectrum at input frequency  $F_0$  in the output port.

The other is widely used in the field of the signal detection and communication [5-8]:

$$SNR = \frac{S(F_0)}{P - S(F_0)} \quad (8)$$

Where  $S(F_0)$  is the power of  $F_0$  frequency component;  $P$  is total power, including the signal power and noise power,  $P$  minus  $S(F_0)$  equals to the noise power.

The distinction between the two definitions is that the interpretations of the noise power are different. The former is the local noise power, and the latter is the whole noise power. Equation (8) is considered to be better and more comprehensive description of the comparison between the signal and noise power, and is more widely accepted especially in practical detection and communication [5-8]. Obviously the input SNR in SNR gain (6) should adopt the latter definition because the input signal and noise are not processed by stochastic resonance systems. As a result, we use the latter henceforth.

The inputs of stochastic resonance system are  $A \cos(2\pi F_0 t)$  and  $\Gamma(t)$ . The discrete sequence  $z(l)$  with the length  $L$  is obtained when the noisy signal is sampled at sampling frequency  $F_s$ . The output of SR system solved by the Runge-Kutta numerical method is a discrete sequence  $x(l)$  with the length  $L$ . Its frequency spectrum  $X(k)$  is calculated using *FFT* :

$$X(k) = \sum_{l=0}^{L-1} x(l) e^{-j2\pi k l / L} \quad (9)$$

We assume that the signal  $X(k_0)$  is the magnitude of the output at the input frequency  $F_0$ , and  $k_0 = \frac{F_0}{F_s} L$ . According to Equation (8), the output SNR can be obtained.

$$SNR_{out} = \frac{2|X(k_0)|^2}{\sum_{k=0}^{L-1} |X(k)|^2 - 2|X(k_0)|^2} \quad (10)$$

Similarly, the input SNR can be calculated.

$$SNR_{in} = \frac{2|Z(k_0)|^2}{\sum_{k=0}^{L-1} |Z(k)|^2 - 2|Z(k_0)|^2} \tag{11}$$

Where  $Z(k_0)$  is the magnitude of the input at the frequency  $F_0$ .

In the case of large  $L$ , SNR equation ((10) (11)) was proved by Reference [5]. SNR gain can be calculated by Equation (6).

**4. SR behavior of SNR Gain**

Reference [1] demonstrated SR behavior of the spectral power amplification  $\eta$ , but an important parameters such as the SNR gain was not considered. This section focuses on the SNR gain of the tristable system. So far no research results have been reported on the analytical solution and approximate solution about the tristable system. Therefore, the tristable system (12) is studied further using the second-order Runge-Kutta method in this paper.

$$\frac{dx}{dt} = -\frac{x^5}{5} + \frac{1.1x^3}{3} - 0.1x + A\cos(2\pi Ft) + \Gamma(t) \tag{12}$$

In order to compare with the result of Reference [1], we will demonstrate SR behaviors of  $SNR_{gain}$  and  $\eta$  simultaneously. According to  $|\chi_x|A\cos(2\pi Ft + \delta_x)$  in Equation (4),  $\chi_x$  can be approximated to the ratio between the output-to-input amplitude at the frequency  $F$ ,

consequently  $\eta \approx \frac{|X(k_0)|^2}{|Z(k_0)|^2}$  can be calculated

In essence, the parameters ( $SNR$ ,  $SNR_{gain}$  and  $\eta$ ) are random variables in this paper, therefore, their final values are obtained by averaging over many numerical solutions [5, 6].

Periodic input signal  $A\cos(2\pi Ft)$ ,  $A=0.05$ ,  $F$  equals 0.008Hz, 0.016Hz and 0.133Hz, respectively.  $\Gamma(t)$  is the same as in Equation (1). Number of sampling points  $L=16384$ , sampling frequency  $F_s=128F$ , the final values are obtained by averaging over 100 numerical solutions. The results are shown in Figure 3.

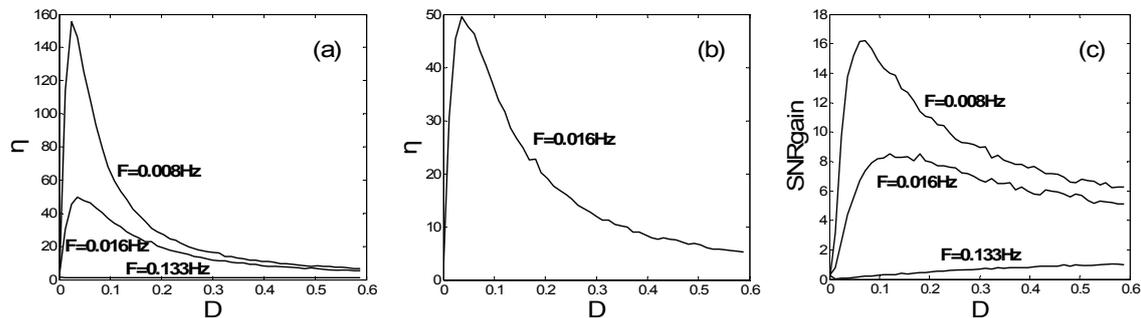


Figure 3. The Results are Obtained by the Second-order Runge-Kutta Method. (a) SR behavior of  $\eta$ ; (b) SR behavior of  $\eta$  ( $F=0.016\text{Hz}$ ); (c) SR behavior of  $SNR_{gain}$

In order to compare with the result obtained in Reference [1], we extract the curve ( $F=0.016\text{Hz}$ ) from Figure 3(a) to put purposely in Figure 3(b). Comparing Figure 3(b) with Figure

2, we can easily find that the two curves are much the same, the result obtained by the Runge-Kutta method is in good agreement with that obtained by the variational method.

We can also observe the SR behavior of  $SNR_{gain}$  in Figure 3(c). Most importantly, it's noted that the  $SNR_{gain}$  greatly exceeds unity on some occasions. When  $F$  equals 0.008Hz and 0.016Hz respectively, the  $SNR_{gain}$  values greatly exceed unity and their maxima are reached at around 16 and 8, respectively; when  $F$  equals 0.133Hz, the  $SNR_{gain}$  begins to exceed unity only when  $D \approx 0.5$ .

The aforementioned result is the latest development of the tristable stochastic resonance.

Similar to the conventional bistable SR, it can be observed in Figure 3 that the SR behavior occurs in the lower-frequency range.  $\eta$  and  $SNR_{gain}$  are inversely proportional to the input frequency  $F$ . The humps of  $\eta$  and  $SNR_{gain}$  decrease with the increase of  $F$ . Meanwhile, the hump of  $SNR_{gain}$  moves right.

We set  $A=0.39$ ,  $F=0.01\text{Hz}$ ,  $D=0.1$ ,  $L=16384$ ,  $F_s=128F$ . The tristable system processes the noisy signals when the input signal is sinusoidal and rectangular respectively. Figure 4 and Figure 5 below illustrate the results.

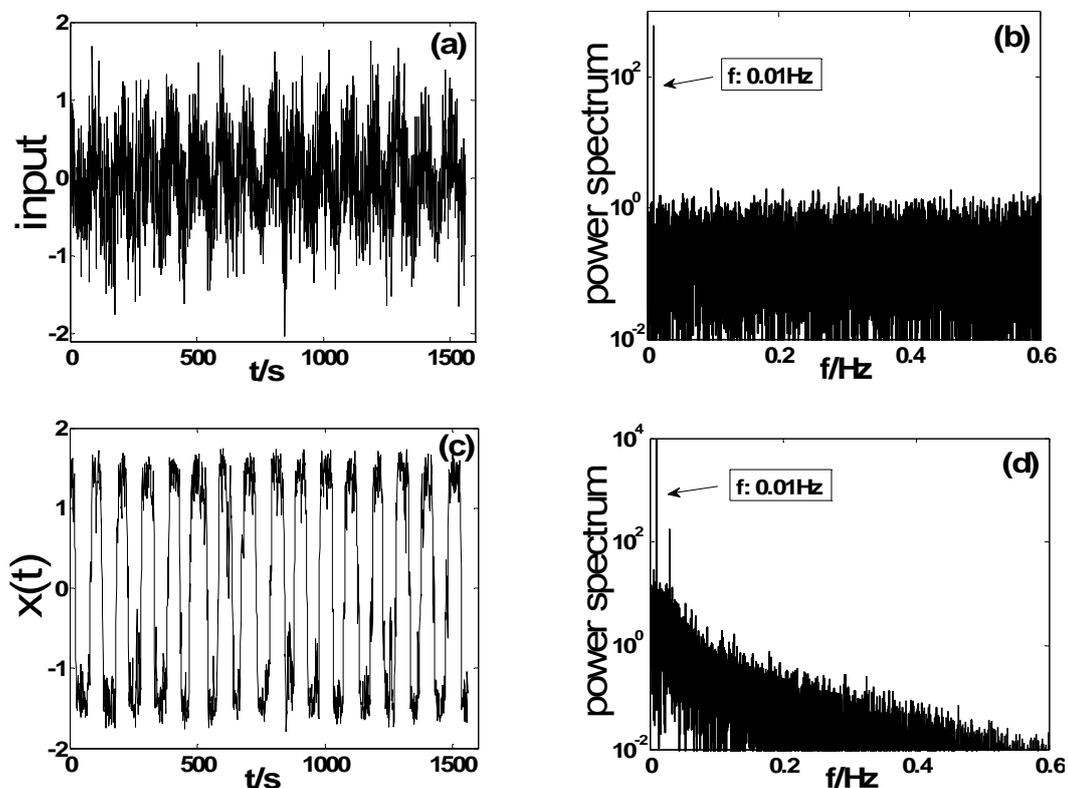


Figure 4. The Input Consists of the Sinusoidal Signal and Gaussian white noise ((a) The input; (b) The power spectrum of the input; (c) The output of the tristable system; (d) The power spectrum of the output)

The output of the tristable system (Fig. 4(c)) is significantly improved in comparison with the input (Fig. 4(a)). Noise, especially higher frequency noise is largely suppressed. It can be observed in Fig. 5(a)-(c) that the tristable system can more significantly improve the noisy sequence of rectangular pulses.

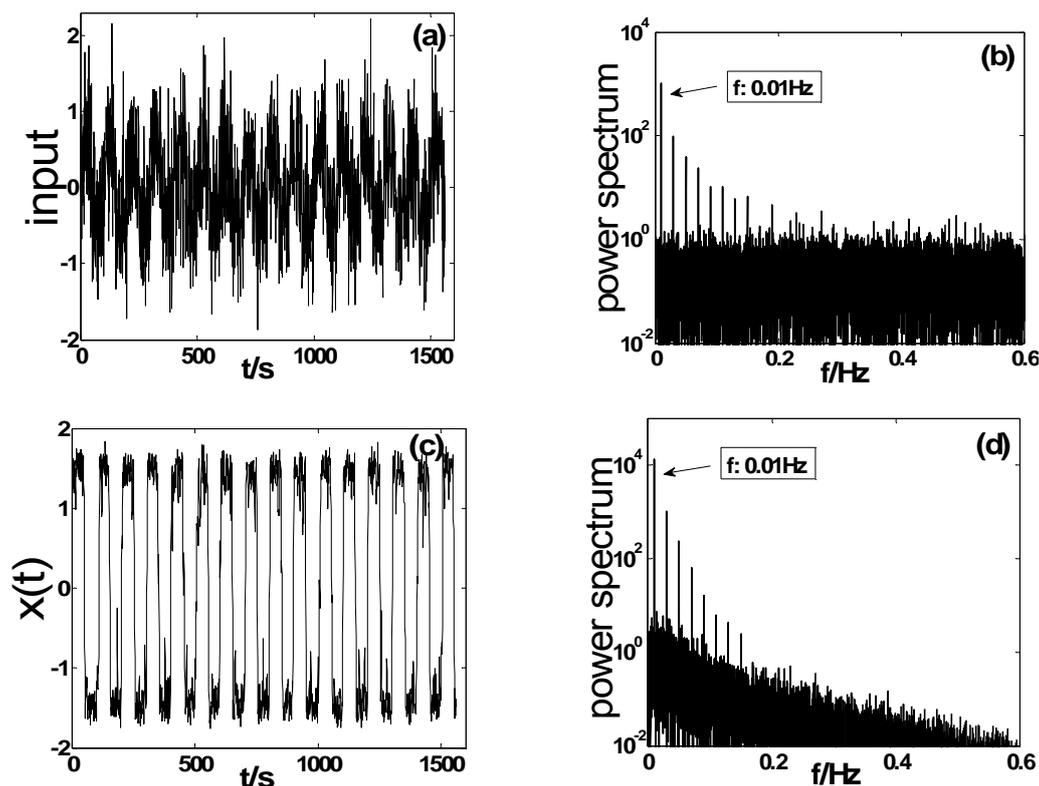


Figure 5. The Input Consists of a Sequence of Rectangular Pulses and Gaussian White Noise ((a) The input; (b) The power spectrum of the input; (c) The output of the tristable system; (d) The power spectrum of the output)

Similar to the conventional bistable system, the output power spectrum of the tristable system also shows the distribution characteristic that spectral energy is concentrated in the low frequency region [10]. Obviously, Figure 4(d) and Figure 5(d) exhibit this characteristic.

## 5. Conclusion

Using the second-order Runge-Kutta method, we study the SNR gain of a tristable SR system. Remarkably, we notice that the SNR gain exhibits the SR behavior. Most importantly, we find that the *SNR gain* greatly exceeds unity on some occasions. This result is the latest development of the tristable SR system. Two above-mentioned examples (Figure 4 and Figure 5) demonstrate that the system can effectively suppress noise. The frequency range of the input signal can be expanded by means of the twice sampling method [10, 11]. Consequently, the application range of the tristable SR system can be extended. This research result has potential applications in the signal detection, processing and communications.

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