Indonesian Journal of Electrical Engineering and Computer Science
Vol. 28, No. 3, December 2022, pp. 1486~1492
ISSN: 2502-4752, DOI: 10.11591/ijeecs.v28.i3.pp1486-1492

Some remarks concerning on a fuzzy real normed space

Mayada N. Mohammedali
Mathematics and Computer Applications, Applied Sciences Department, University of Technology, Baghdad, Iraq

Article Info

Article history:

Received Jul 26, 2022
Revised Aug 23, 2022
Accepted Sep 12, 2022

Keywords:

Fuzzy quasinormed space
Fuzzy real norm
Fuzzy real norrned space
Quasinorm
Sublinear functional

ABSTRACT

In mathematics, computer science, statistics, and other fields, the notion of fuzzy metric and fuzzy normed is crucial. Finding an acceptable fuzzy metric or norm for making some measurements can solve a lot of difficulties. In the sense of Mohammedali, fuzzy real normed spaces (FRNS) are the issue of this paper. Real normed and fuzzy real normed are advanced together with a ground-breaking analysis of their interactions. The structure of FRNS in terms of families of sublinear functional is established. After investigating some properties of "sublinear functional" that corresponding to an FRN, the concept of the family of star sublinear funtional based on the popular t-conorm (a parameter some families) is introduced with proved that a descending and separating family of star sublinear functional, denoted by Q^*produces a FRNS. In addition, the concept of generating space of quasinorm and quasi-norm family (GSQNF) has been presented. Furthermore, the decomposition theorem of a FRN $\tilde{N}_f$ into a family of quasinorm is formulated. The correlation between FRNS and the quasinorm family's generating space is explored and demonstrated.

Corresponding Author:

Mayada N. Mohammedali
Mathematics and Computer Applications, Applied Sciences Department, University of Technology
Baghdad, Iraq
Email: mayada.n.mohammedali@uotechnology.edu.iq

1. INTRODUCTION

Imprecise data is used in a variety of real-world problems in engineering, medical sciences, social sciences, economics, agriculture, and other fields, and their solution requires the application of mathematical principles based on uncertainty and imprecision. Probability, fuzzy set theory, intuitionistic fuzzy set, hazy set, theory of interval mathematics, and other topics are used to deal with such uncertainty. Zadeh [1] proposed the concept of a fuzzy set. As a result, fuzzy set theory has applications in various fields of mathematics as well as other sciences [2]-[12]. The concept of fuzzy metric spaces was presented by Kramosil and Michalek [13]. Fuzzy normed spaces are an important type of fuzzy metric space that provides a suitable framework for inexact measurements of ordinary lengths in linear space. In recent years, the research of fuzzy normed spaces and probabilistic normed spaces has gotten a lot of interest (see for instance [14]-[21]). On the other hand, Bag and Samanta [22] looked at a general t-norm in the notion of fuzzy normed linear space presented in [23], and showed that if the t-norm isn't "min," the decomposition theorem of fuzzy norm may not hold. On the other hand, the idea of a fuzzy norm on a real linear space based on the concept of continuous triangular conorm (t-conorm) was suggested in the paper [24]. On the basis of this idea, and given the importance of the decomposition theorem in the development of fuzzy functional analysis and its applications, it is worthwhile to construct a new type of fuzzy real norm decomposition theorem based on a t-conorm. This is one of the work's objectives. How far the outcomes of fuzzy real normed space (FRNS) can be established using the FRN in its general form, that is, by ignoring the restricted "max" t-conorm in the triangle inequality. Addressing this issue is to other objective of this work.

This work is structured as follows: section 2 comprises some preliminary results. In section 3, a formulation of the structure of FRNS in terms of families of sublinear functional based on the particular choice of (max) t-conorm is presented, as well as some fundamental results. In section 4, the concept of a quasinorm family and generating space of quasi-norm family (GSQNF) are given. Moreover, it is investigated how FRNS and GSQNF interact.

2. SOME PRELIMINARY RESULTS

In FRNS, which were offered on Mohammedali viewpoint, there are some fundamental hypotheses and and significant findings. We will provide the background information and supplemental findings that are required. An definition of a fuzzy real normed space is now provided.

Definition 2.1. [24]: Let $\varDelta_c$ be a triangular t- conorm, V be a linear space over the field $\boldsymbol{R}$. The fuzzy norm $\widetilde{N}_f$ on V is a fuzzy subset mapping $\widetilde{N}_f: V \times [0,\infty)$ into I if the following requirements apply $\forall\, \mathrm{v}_1, \mathrm{v}_2 \in V$ and $\forall\, \mathrm{t} \in [0,\infty)$;

(RN1) $\widetilde{N}_f(\mathrm{v}_1,\mathrm{t}) > 0$, for all $\mathrm{t} > 0$

(RN2) $\widetilde{N}_f(\mathrm{v}_1,\mathrm{t}) = 1$ if and only if $\mathrm{v}_1 = 0$, for all $\mathrm{t} > 0$

(RN3) $\forall\, \boldsymbol{r} \neq \boldsymbol{0} \in \boldsymbol{R}$, $\widetilde{N}_f(r\, \mathrm{v}_1,\mathrm{t}) = \widetilde{N}_f\left(\mathrm{v}_1, \frac{\mathrm{t}}{|r|}\right)$

(RN4) $\widetilde{N}_f(\mathrm{v}_1 + \mathrm{v}_2,\mathrm{t}) \leq \widetilde{N}_f(\mathrm{v}_1,\mathrm{t})\, \varDelta_c\, \widetilde{N}_f(\mathrm{v}_2,\mathrm{t})$

(RN5) $\widetilde{N}_f(\mathrm{v}_1,.): (0,\infty] \to [0,1]$ is continuous with respect to t

Consequently, $(V, \widetilde{N}_f, \varDelta_c)$ is known as FRNS.

We suppose that the fuzzy real norm also meets the following criterion in order to get some significant results.

(RN6) $\widetilde{N}_f(\mathrm{v}_1,\mathrm{t}) > 0, \forall\, \mathrm{t} > 0 \Rightarrow \mathrm{v}_1 = 0$

Lemma 2.2. [24]: Let $\left(V,\, \widetilde{N}_f, \varDelta_c\right)$ be a FRNS. Then $\widetilde{N}_f(\mathrm{v}_1 - \mathrm{v}_2,\mathrm{t}) = \widetilde{N}_f(\mathrm{v}_2 - \mathrm{v}_1,\mathrm{t})\ \forall\, \mathrm{v}_1,\mathrm{v}_2 \in V$ and $\mathrm{t} > 0$.

Example 2.3. [24]: Let $(R^2, ||.||)$ be a normed space, where $V = R^2$ is a linear space which is obtained if the set of ordered pairs of real numbers $\mathrm{v}_1 = (\rho_1,\rho_2) \in R^2$ is taken with a function $\|\mathrm{v}_1\| = \|\mathrm{v}_1\| = (|\rho_1|^2 + |\rho_2|^2)^{\frac{1}{2}}$. Define $\widetilde{N}_f(\mathrm{v}_1,\mathrm{t}) = |\|\mathrm{v}_1\| - \mathrm{t}|/\mathrm{t} + \|\mathrm{v}_1\|$ for $\mathrm{t} < \|\mathrm{v}_1\|$ and $\widetilde{N}_f(\mathrm{v}_1,\mathrm{t}) = 0\ \forall\, \mathrm{t} \geq \|\mathrm{v}_1\|$. Also $a\, \varDelta_c b = a + b - ab$ for all $a, b \in I$. Then $(V, \widetilde{N}_f,\, \varDelta_c)$ is FRNS.

Definition 2.4. [24]: Let $(V, \widetilde{N}_f, \varDelta_c)$ be a FRN. Open is the name for the set A of V if for any point $\mathrm{v}_1 \in V$ $\exists$ an $\alpha \in (0,1)$ center v_1 and radius, $B(\mathrm{v}_1, \alpha, \mathrm{t}) = \{\mathrm{v}_2 \in V, \widetilde{N}_f(\mathrm{v}_1, \mathrm{v}_2, \mathrm{t}) > 1 - \alpha\}$ is contained entirely in A.

Definition 2.5. [24]: A sequence $\{\mathrm{v}_n\}$ in a FRNS $(V, \widetilde{N}_f, \varDelta_c)$ is said to be $\check{l}$-convergent if there exists a vector v in V such that $\widetilde{N}_f(\mathrm{v}_\mathrm{n} - \mathrm{v},\mathrm{t}) \to 1$ as $\mathrm{n} \to \infty$, for all $\mathrm{t} > 0$.

Definition 2.6. [24]: A sequence $\{\mathrm{v}_n\}$ in a FRNS $(V, \widetilde{N}_f, \varDelta_c)$ is said $\check{l}$-Cauchy sequence if lim $\widetilde{N}_f(\mathrm{v}_\mathrm{n}\mathrm{v}_{\mathrm{n+s}}) = 1$ as $\mathrm{n} \to \infty, \mathrm{s} > 0$ and $\forall\, \mathrm{t} > 0$.

Definition 2.7. [24]: A FRNS $(V, \widetilde{N}_f, \varDelta_c)$ is said to be $\check{l}$-complete if every $\check{l}$-Cauchy sequence in V is $\check{l}$-convergent.

3. SUBLINEAR FUNCTIONAL STRUCTURES IN A FRNS

This section focuses on formulating the structure of FRNS in terms of families of sublinear functional based on the particular choice of (max) t-conorm with investigating some properties of sublinear functional corresponding to a FRN. The concept of a family of star sublinear functional based on the popular t-conorm (a parameter some families) is established. Furthermore, a descending and separating family of star sublinear functions, referred to as Q^* has been proven to produce FRN.

A sublinear functional is given in [25] as: Let V be a linear space. The function q_α from V into $\boldsymbol{R}$ is called sublinear functional on V if the following conditions are satisfied.

i) $q_\alpha\,(\rho v_1) = \rho q(v_1)\ \forall\ v_1 \in\ V$ and $\rho \in (0,\infty)$

ii) $q_\alpha\,(v_1) + q(v_2) \geq q(v_1 + v_2)\ \forall\ v_1, v_2 \in V$

The following theorem gives a new characterization of sublinear functional family on V.

Theorem 3.1: Let $(V, \widetilde{N}_f, \Delta_c)$ be a FRNS where $\Delta_c = (max)$ and let $0 < \alpha < 1$. The function $q_\alpha\,(\cdot): V \to [0,\infty)$ is given by;

$$q_\alpha\,(v_1) = \ inf\ \{t > 0: \widetilde{N}_f(v_1, t) < \alpha\} \tag{1}$$

Then $Q = \{q_\alpha : 0 < \alpha < 1\}$ is a decreasing family of sublinear functional on V.

Proof:

i) For $\rho \in (0,\infty)$, we have $q_\alpha\,(\rho v_1) = inf\{t > 0: \widetilde{N}_f(\rho v_1, t) < \alpha\}$

$$= \ inf\left\{t > 0: \widetilde{N}_f\left(v_1, \frac{t}{|\rho|}\right) < \alpha\right\}$$

$$= inf\{|\rho|\ \frac{t}{|\rho|} > 0: \widetilde{N}_f\left(v_1, \frac{t}{|\rho|}\right) < \alpha\}$$

$$= |\rho| inf\{\ \frac{t}{|\rho|} > 0: \widetilde{N}_f(v_1, t) < \alpha\} = |\rho| q_\alpha(v_1)$$

ii) $q_\alpha(v_1) + q_\alpha(v_2)\ = \ inf\{t > 0: \widetilde{N}_f(v_1, t) < \alpha\} + inf\{t > 0: \widetilde{N}_f(v_2, t) < \alpha\}, \forall\ 0 < \alpha < 1$

$$= inf\{2t > 0: \widetilde{N}_f(v_1, t) < \alpha, \widetilde{N}_f(v_2, t) < \alpha\}$$

$$= inf\ \{t > 0: \widetilde{N}_f(v_1 + v_2) < \alpha\ \Delta_c \alpha\}$$

$$\geq inf\{t > 0: \widetilde{N}_f(v_1 + v_2, t) < \alpha\} = q_\alpha(v_1 + v_2)$$

it remains to be proven that Q is a decreasing family. For $0 < \alpha_1 < \alpha_2 < 1$, we note $\{t > 0: \widetilde{N}_f(v_1, t) < \alpha_2\} \subseteq \{t > 0: \widetilde{N}_f(v_1, t) < \alpha_1\}$. Thus $inf\{t > 0: \widetilde{N}_f(v_1, t) < \alpha_1\} \geq inf\{t > 0: \widetilde{N}_f(v_1, t) < \alpha_2\}$, namely $q_{\alpha 1}(v_1) \geq q_{\alpha 2}(v_1)\ \forall\ v_1 \in V$.

The sublinear functional corresponding to a FRN $(\widetilde{N}_f)$ behaves as stated in the following proposition.

Proposition 3.2: Let $(V, \widetilde{N}_f, \Delta_c)$ be an FRNS and $q_\alpha(v_1) = inf\{t > 0: \widetilde{N}_f(v_1, t) < \alpha\}, 0 < \alpha < 1$. Then $\widetilde{N}_f(v_1, l) < \alpha$ and only if $q_{\alpha 1}(v_1) < l$ for $v_1 \in V, l > 0$ and $0 < \alpha < 1$.

Proof:

As $\widetilde{N}_f(v_1, l) < \alpha$, we obtain that $l \in \{t > 0: \widetilde{N}_f(v_1, t) < \alpha\}$. Thus $q_\alpha(v_1) \leq l$. We presumes that $q_\alpha(v_1) = l$. Since $\widetilde{N}_f(v_1, .)$ is continuous in l then $\widetilde{N}_f(v_1, t) = \widetilde{N}_f(v_1, l)$. Thus $\exists\ t_\circ < l$ with $\widetilde{N}_f(v_1, t_\circ) < \alpha$. (Contrary, $\widetilde{N}_f(v_1, t) \geq \alpha$, $\forall\ t \leq l$. Therefore $\widetilde{N}_f(v_1, t) \geq \alpha$ so $\widetilde{N}_f(v_1, l) \geq \alpha$, which is a contradiction). But $t_\circ < l$ with $\widetilde{N}_f(v_1, t_\circ) < \alpha$ are in contradiction with the fact that $l = inf\{t > 0: \widetilde{N}_f(v_1, t) < \alpha\}$. Thus $q_\alpha(v_1) \neq l$. Hence $q_\alpha(v_1) < l$.

Conversely, we must show that $l \in \{t > 0: \widetilde{N}_f(v_1, t) < \alpha\}$. We assume that $l \notin \{t > 0: \widetilde{N}_f(v_1, t) < \alpha\}$, then $\exists\ t_\circ \in \{t > 0: \widetilde{N}_f(v_1, t) < \alpha\}$ with $t_\circ < l$. (Contrary, $l \leq t$, $\forall\ t \in \{t > 0: \widetilde{N}_f(v_1, t) < \alpha\}$. So $\leq inf\{t > 0: \widetilde{N}_f(v_1, t) < \alpha\}$, this means $l \leq q_\alpha(v_1)$ which is contradiction). As $t_\circ \in \{t > 0: \widetilde{N}_f(v_1, t) < \alpha\}$, $t_\circ < l$ and $\widetilde{N}_f(v_1, .)$ is decreasing, we obtain that $\widetilde{N}_f(v_1, l) \leq \widetilde{N}_f(v_1, t_\circ) < \alpha$. Therefore, $\widetilde{N}_f(v_1, l) < \alpha$, which leads to a contradiction.

The following theorem characterizes an open set.

Theorem 3.3: Let $(V, \widetilde{N}_f, \Delta_c)$ be a FRNS with $\Delta_c = (max)$. If E is open set in $(V, \widetilde{N}_f, \Delta_c)$ then:

$$E = \{v_1 \in V, q_\alpha(v_1) < t\} \tag{2}$$

Proof:
Let $B = \{v_1 \in V, q_\alpha(v_1) < t\}$. Suppose that $v_1 \in E$ then $\exists\ 0 < l < 1$, $t \in R^+$and $t > 0$ with $B(v_1, l, t) \subset E$ then $B(v_1, l, t) = \{v_2 \in V: \widetilde{N}_f(v_1 - v_2, t) < 1 - l\} \subset E$. Now, if $v_2 = 0$ in V (since V linear space) then $B(v_1, l, t) = \{0 \in V: \widetilde{N}_f(v_1, t) < 1 - l\} \subset E$. Let $1 - l \geq \alpha$ then $0 < \alpha < 1$ this implies that $\widetilde{N}_f(v_1, t) < \alpha$ thus $q_\alpha(v_1) < t$, i.e $v_1 \in B$. If $v_2 \neq 0$ then $\widetilde{N}_f(v_1, t) = \widetilde{N}_f(v_1 - v_2 + v_2, t) \leq \widetilde{N}_f(v_1 - v_2, t)\ \Delta_c\ \widetilde{N}_f(v_2, t) < (1 - l\ )\Delta_c \widetilde{N}_f(v_2, t)$. $\widetilde{N}_f(v_2, t) \neq 1$ because $v_2 \neq 0$ and $\widetilde{N}_f(v_2, t) > 0$ then $0 < \widetilde{N}_f(v_2, t) < 1$. Assume that $l^* = \widetilde{N}_f(v_2, t)$ then $\widetilde{N}_f(v_1, t) < 1 - l\ \Delta_c\ l^* = \alpha$, $0 < \alpha < 1$, it means that $E \subset B$.

Conversely, suppose that $v_1 \in B$ then $q_\alpha(v_1) < t$ this implies that $\widetilde{N}_f(v_1, t) < \alpha$, $0 < \alpha < 1$. Let $1 - \alpha = l$, $0 < l < 1$ then $1 - l \leq \alpha$ it suggests that $\widetilde{N}_f(v_1, t) < 1 - l = \widetilde{N}_f(v_1 - 0, t) < 1 - l$, it signifies that $B(v_1, l, t) \subset E$, i.e., $v_1 \in E$. Therefore, $E = \{v_1 \in V: q_\alpha(v_1) < t\}$.

Definition 3.4: Let V be a linear space, Δ_c be a continuous t-conorm. $q^*_\alpha: V \to (0, \infty)$ is a function for each $0 < \alpha < 1$. $Q^* = \{q^*_\alpha, 0 < \alpha < 1\}$ is called a family of star sublinear functional if it satisfies the following conditions, $\forall\ v_1, v_2 \in V$, $0 < \alpha, B < 1$ and $\rho \geq 0$:

$$(Q^*_1)\ q^*_\alpha(\rho v_1) = \rho q^*_\alpha(v_1)$$

$$(Q^*_2)\ q^*_\alpha(\rho v_1) + q^*_\beta(v_2) \geq q^*_{\alpha\ \Delta_c\ \beta}(v_1 + v_2)$$

if Q^* satisfies the condition $(Q^*_3)\ q^*_\alpha(v_1) = 0\ \forall\ 0 < \alpha < 1$ implies $v_1 = 0$, then Q^* is said to be separating.

Remark 3.5: from $({Q_1}^*)$, we conclude that $q_\alpha(0) = 0\ \forall\ 0 < \alpha < 1$.

Definition 3.6: $q^*_\alpha(.)$ is said to be continuous with respect to $0 < \alpha < 1$ if for any sequence (α_n) in (0,1) with $\alpha_n \to \alpha$ implies $q^*_{\alpha_n}(v_1) \to q^*_\alpha(v_1)\ \forall\ v_1 \in V$.

The following theorem constructs a FRNS of a star sublinear functional family when the family Q^* satisfies the criteria for descending and separating.

Theorem 3.7: Let V be a real linear space and $Q^* = \{q^*_\alpha(.) : 0 < \alpha < 1\}$ be a sufficient, descending and separating family of star sublinear functional satisfies that $q^*_\alpha(.)$ is continuing in respect to $0 < \alpha < 1$. Define a function ${\widetilde{N}_f}^*: V \times [0, \infty) \to I$ as;

$$ {\widetilde{N}_f}^*: (v_1, t) = \begin{cases} \{inf\ \{\alpha \in (0,1) : q_\alpha(v_1) < t, & if\ t > 0 \\ 0, & if\ \{\alpha \in (0,1) : q_\alpha(v_1) < t\} = \emptyset \end{cases} \tag{3}$$

Then ${\widetilde{N}_f}^*$ is a FRN on V.

Proof:
$(\boldsymbol{RN_1})$ Is obvious.
$(\boldsymbol{RN_2})$ If $v_1 = 0$, then $q^*_\alpha(v_1) = 0 < t\ \forall\ t > 0$ from Remark 3.5. Hence ${\widetilde{N}_f}^*(v_1, t) = 1$. Conversely, if ${\widetilde{N}_f}^*(v_1, t) = 1\ \forall\ t > 0$, then $q^*_\alpha(v_1) < t\ \forall\ 0 < \alpha < 1$ so $q^*_\alpha(v_1) = 0\ \forall\ 0 < \alpha < 1$. Since Q^* is separating, then $v_1 = 0$.
$(\boldsymbol{RN_3})$ Let $r \neq 0\ \in R$, then ${\widetilde{N}_f}^*(r v_1, t) = inf\{\alpha \in (0,1): q^*_\alpha(r v_1) < t\}$

$$= inf\left\{\alpha \in (0,1): q^*_\alpha(v_1) < \frac{t}{|r|}\right\} = {\widetilde{N}_f}^*(v_1, \frac{t}{|r|}).$$

$(\boldsymbol{RN_4})$ Let v_1, $v_2 \in V$ and $t > 0$. Then ${\widetilde{N}_f}^*(v_1 + v_2, t) = inf\{\alpha \in (0,1): q^*_\alpha(v_1 + v_2) < t\}$. Now, ${\widetilde{N}_f}^*(v_1, t) = inf\{\alpha \in (0,1): q^*_\alpha(v_1) < t\} = \overline{\alpha}$ (say) and ${\widetilde{N}_f}^*(v_2, t) = inf\{\beta \in (0,1): q^*_\beta(v_2) < t\} = \overline{\beta}$ (say) where $q^*_{\overline{\alpha}}(v_1) = t$ and $q^*_{\overline{\beta}}(v_2) = t$.(Since $q^*_\alpha(.)$ is continuing in respect to $0 < \alpha < 1$). Now,

$$ {\widetilde{N}_f}^*(v_1, t)\Delta_c {\widetilde{N}_f}^*(v_2, t) = \overline{\alpha}\ \Delta_c\ \overline{\beta} \tag{4}$$

and

$$ q^*_{\overline{\alpha}\ \Delta_c\ \overline{\beta}}(v_1 + v_2) \leq q^*_{\overline{\alpha}}(v_1) + q^*_{\overline{\beta}}(v_2) = 2t \tag{5}$$

this implies that $\widetilde{N_f}^*(\mathrm{v}_1 + \mathrm{v}_2) \le \overline{\alpha}\,\Delta_c\,\overline{\beta} = \widetilde{N_f}^*(\mathrm{v}_1, t)\Delta_c\,\widetilde{N_f}^*(\mathrm{v}_2, t)$.
(RN_5) Let (t_n) be a sequence in [0,∞) with $t_n \to t$. Now, $\forall\, \mathrm{v}_1 \in V$

$$\begin{aligned} lim_{n\to\infty}\widetilde{N_f}^*(\mathrm{v}_1, t_n) &= lim_{n\to\infty}[inf\ \{\alpha \in (0,1): q_\alpha^*(\mathrm{v}_1) < t_n\}] \\ &= inf\{\alpha \in (0,1): q_\alpha^*(\mathrm{v}_1) < lim_{n\to\infty}t_n\} \\ &= inf\ \{\alpha \in (0,1): q_\alpha^*(\mathrm{v}_1) < t\} = \widetilde{N_f}^*(\mathrm{v}_1, t). \end{aligned}$$

Hence $\widetilde{N_f}^*(\mathrm{v}_1, t_n) \to \widetilde{N_f}^*(\mathrm{v}_1, t)$. It means that $\widetilde{N_f}^*: [0,\infty) \to t$ is continuous with respect to t.

4. CREATING SPACE OF QUASINORM FAMILY

In this section, the concept of generating space of quasinorm and quasinorm family are presented. Moreover, the decomposition theorem for a FRN ($\widetilde{N_f}$) with general t-conorm into a family of quasinorm is established. In addition The concepts of various $\check{l}$-convergent sequences types, $\check{l}$-Cauchy sequences, an $\check{l}$-completeness of a fuzzy real normed space, etc. are presented, and the relationships between these conceptions are analyzed.

In the definition that follows, the idea of GSQNF is introduced.

Definition 4.1: Let V be a linear space over the filed R. Let Q={$q_\alpha(\cdot), 0 < \alpha < 1$} be a family of mappings from V into $[0, \infty)$. Q is called a quasinorm family and (V, Q) is called GSQNF if the following conditions are satisfied $\forall\, \mathrm{v}_1, \mathrm{v}_2 \in V, \rho \in \boldsymbol{R}$.

(QRN_1) $q_\alpha(\mathrm{v}_1) = 0$, $\forall\, 0 < \alpha < 1$ if and only if $\mathrm{v}_1 = 0$

(QRN_2) $q_\alpha(\rho\mathrm{v}_1) = |\rho|\, q_\alpha(\mathrm{v}_1)$, $\forall\, 0 < \alpha < 1$

(QRN_3) For any $0 < \alpha < 1$, $\exists\, \beta \in (0, \alpha]$ with $q_\alpha(\mathrm{v}_1 + \mathrm{v}_2) \le q_\beta(\mathrm{v}_1) + q_\beta(\mathrm{v}_2)$

(QRN_4) If $0 < \alpha \le \beta < 1$ then $q_\alpha(\mathrm{v}_1) \ge q_\beta(\mathrm{v}_1)$

The following theorem illustrates how a FRN $\widetilde{N_f}$ based on (t-conorm) can be decomposed into a family of quasinorms.

Theorem 4.2: Let $(V, \widetilde{N_f}, \Delta_c)$ be a FRNS with Δ_c is upper semi-continuous. Define:

$$q_\alpha(\mathrm{v}_1) = inf\{t > 0, \widetilde{N_f}(\mathrm{v}_1, t) < \alpha\} \text{ for } 0 < \alpha < 1 \tag{6}$$

and Q={$q_\alpha(\cdot), 0 < \alpha < 1$} then (V, Q) is a GSQNF.

Proof:
(QRN_1) Let $q_\alpha(\mathrm{v}_1) = 0\ \forall\, 0 < \alpha < 1$ then $inf\{t > 0: \widetilde{N_f}(\mathrm{v}_1, t) < \alpha\} = 0$, $\forall\, 0 < \alpha < 1$. Suppose that $\in_\circ > 0$ be given then $inf\{t > 0: \widetilde{N_f}(\mathrm{v}_1, t) < \alpha\} < \in_\circ, \forall\, 0 < \alpha < 1$ implies that $\widetilde{N_f}(\mathrm{v}_1, \in_\circ) < \alpha$, $\forall\, 0 < \alpha < 1$ so $\widetilde{N_f}(\mathrm{v}_1, \in_\circ) = 1$. Since $\in_\circ > 0$ is arbitrary then, $\widetilde{N_f}(\mathrm{v}_1, t) = 1\ \forall\, t > 0$ hence $\mathrm{v}_1 = 0$.

Conversely, Let $\mathrm{v}_1 = 0$ and $0 < \alpha < 1$ then by (RN_2), $\widetilde{N_f}(\mathrm{v}_1, t) = 1$, $\forall\, t > 0$ this suggests that $\widetilde{N_f}(\mathrm{v}_1, t) < \alpha$, $\forall\, t > 0$ means $q_\alpha(\mathrm{v}_1) = 0$. Since $0 < \alpha < 1$ is arbitrary hence $q_\alpha(\mathrm{v}_1) = 0$, $\forall\, 0 < \alpha < 1$.
(QRN_2) Let $\mathrm{v}_1 \in V$, $\rho \in R$ and $0 < \alpha < 1$ then:

$$\begin{aligned} q_\alpha(\rho\mathrm{v}_1) &= inf\left\{t > 0, \widetilde{N_f}(\rho\mathrm{v}_1, t) < \alpha\right\} \\ &= inf\left\{t > 0, \widetilde{N_f}\left(\mathrm{v}_1, \tfrac{t}{|\rho|}\right)\alpha\right\} \\ &= inf\left\{|\rho|.\tfrac{t}{|\rho|} > 0,\ \widetilde{N_f}\left(\mathrm{v}_1, \tfrac{t}{|\rho|}\right) < \alpha\right\} \\ &= |\rho|\ inf\left\{|\rho|.\tfrac{t}{|\rho|}, \widetilde{N_f}\left(\mathrm{v}_1, \tfrac{t}{|\rho|}\right) < \alpha\right\} = |\rho| q_\alpha(\mathrm{v}_1) \end{aligned} \tag{7}$$

(QRN_3) since Δ_c is upper semi-continuous, for any $0 < \alpha < 1$, $\exists\, \beta \in (0, \alpha]$ with $\beta\, \Delta_c \beta \le \alpha$.
Now, $q_\beta(\mathrm{v}_1) + q_\beta(\mathrm{v}_2) = inf\{t > 0: \widetilde{N_f}(\mathrm{v}_1, t) < \beta\} + inf\{t > 0: \widetilde{N_f}(\mathrm{v}_2, t) < \beta\}$

$\geq inf\{2t > 0: \widetilde{N}_f(\mathrm{v}_1, t) < \beta, \widetilde{N}_f(\mathrm{v}_2, t) < \beta\}$

$\geq inf\{t > 0: \widetilde{N}_f(\mathrm{v}_1 + \mathrm{v}_2, t) < \beta\, \Delta_c\, \beta \ \leq \alpha\}$ (8)

(QRN_4) suppose that $\mathrm{v}_1 \in V, 0 < \alpha, \beta < 1$ then for $\alpha < \beta$ we note $inf\{t > 0: \widetilde{N}_f(\mathrm{v}_1, t) < \alpha\} \geq inf\{t > 0: \widetilde{N}_f(\mathrm{v}_1, t) < \beta\}$, it means $q_\alpha(\mathrm{v}_1) \geq q_\beta(\mathrm{v}_1)$.
Therefore, $Q = \{q_\alpha(.): 0 < \alpha < 1\}$ is quasinorm family on V and (V, Q) is a GSQNF.

Definition 4.3: Let (V, Q) be GSQNF and v_n be a sequence in V. Then (v_n) is said to be $\check{l}$-convergent if $\exists\, \mathrm{v} \in V$ with $\lim_{n\to\infty} q_\alpha(\mathrm{v}_n - \mathrm{v}) = 0\ \forall\ 0 \leq \alpha < 1$.

Definition 4.4: Let (V, Q) be a gsqnf and v_n be a sequence in V. Then (v_n) is said to be $\check{l}$-Cauchy sequence, if $\lim_{n\to\infty} q_\alpha |\mathrm{v}_{n+s} - \mathrm{v}_n| = 0\ \forall\ 0 \leq \alpha < 1, s > 0$.

Definition 4.5: A GSQNF (V, Q) is said to be $\check{l}$-complete if every $\check{l}$-Cauchy sequence in V converge to some point in V.

This section's final propositions are devoted to demonstrating how FRNS and GSQNF are related.
Proposition 4.6: Let $(V, \widetilde{N}_f, \Delta_c)$ be a FRNS satisfying (R$\boldsymbol{N_6}$). If (x_n) be sequence in V, then (v_n) is $\check{l}$-convergent to v with respect to Q if and only if (v_n) is $\check{l}$-converges to v with respect to $\widetilde{N}_f$.

Proof:
Let (v_n) be a sequence in V, with (v_n) is $\check{l}$-converges to V with respect to Q. Consider $\lim_{n\to\infty} q_\alpha(\mathrm{v}_n - \mathrm{v}) = 0\ \forall\ 0 < \alpha < 1$. Then corresponding to any $t > 0$,$\exists\, n_\circ(\alpha, t) > 0$ with $q_\alpha(\mathrm{v}_n - \mathrm{v}) < t, \forall\, n \geq n_\circ(\alpha, t) \to \widetilde{N}_f(\mathrm{v}_n - \mathrm{v}, t) < \alpha, \forall\, n \geq n_\circ(\alpha, t) \to \lim_{n\to\infty} \widetilde{N}_f(\mathrm{v}_n - \mathrm{v}, t) < \alpha, \forall\ 0 < \alpha < 1 \to \lim_{n\to\infty} \widetilde{N}_f(\mathrm{v}_n - \mathrm{v}, t) = 1, \forall\, t > 0$.

Conversely, Let (v_n) be a sequence in $V$with (v_n) is $\check{l}$-converges to v with respect to $\widetilde{N}_f$ means $\lim_{n\to\infty} \widetilde{N}_f(\mathrm{v}_n - \mathrm{v}, t) = 1\ \forall\, t > 0$. We choose $0 \leq \alpha < 1 \to \lim_{n\to\infty} \widetilde{N}_f(\mathrm{v}_n - \mathrm{v}, t) < \alpha, \forall\, t > 0 \to \exists\, n_\circ(\alpha, t) > 0$ with $\widetilde{N}_f(\mathrm{v}_n - \mathrm{v}, t) < \alpha\ \forall\, n \geq n_\circ(\alpha, t) \to q_\alpha(\mathrm{v}_n - \mathrm{v}) \leq t, \forall\, n \geq n_\circ(\alpha, t) \to \lim_{n\to\infty} q_\alpha(\mathrm{v}_n - \mathrm{v}) \leq t, \forall\, t > 0 \to \lim_{n\to\infty} q_\alpha(\mathrm{v}_n - \mathrm{v}) = 0$. Therefore, (v_n) is $\check{l}$-converges to v with respect to Q.

Proposition 4.7: Let $(V, \widetilde{N}_f, \Delta_c)$ be a FRNS satisfying (R$\boldsymbol{N_6}$) and (x_n) be a sequence in V,then (x_n) is a $\check{l}$-Cauchy sequence in $(V, \widetilde{N}_f, \Delta_c)$ if and only if (v_n) is a $\check{l}$-Cauchy sequence in (V, Q).

Proof:
Suppose that (x_n) be a $\check{l}$-Cauchy sequence in $(V, \widetilde{N}_f, \Delta_c)$, then $\lim\ \widetilde{N}_f(\mathrm{v_n} - \mathrm{v_{n+s}}) = 1$ as $\mathrm{n} \to \infty, \mathrm{s} > 0$ and $\forall\, \mathrm{t} > 0 \leftrightarrow \forall\ 0 \leq \alpha < 1\ \exists\, n_\circ(\alpha, t) > 0$ with $\widetilde{N}_f(\mathrm{v_n} - \mathrm{v_{n+s}}) < \alpha\ \forall\, n \geq n_\circ(\alpha, t), \forall\, s > 0 \leftrightarrow q_\alpha|\mathrm{v}_{n+s} - \mathrm{v}_n| = 0\ \forall\, s > 0 \leftrightarrow (\mathrm{v}_n)$ is a $\check{l}$-Cauchy sequence in (V, Q).

Proposition 4.8: If $(V, \widetilde{N}_f, \Delta_c)$ be a FRNS satisfying an $\check{l}$-complete property then (V, Q) is a $\check{l}$-complete GSQNF.

Proof: The evidence is clear.

5. CONCLUSIONS

In this paper, an investigation was conducted between real normed space and fuzzy real normed space (FRNS). After presenting the notion of the family of star sublinear functional and through pioneering analysis of their reciprocal interactions, we've been able to advance both disciplines and increase their attractiveness. Additionally, the concept of generating space of quasinorm and quasi-norm family (GSQNF) has been presented, and with the formation of the theory of decomposition theorem for an FRN (N ̃_f) into a family of quasinorm. Several findings regarding fuzzy functional analysis can be easily deduced from the corresponding formulations in functional analysis.

REFERENCES

[1] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, Jun. 1965, doi: 10.1016/S0019-9958(65)90241-X.

[2] S. Kumar, V. Arya, S. Kumar, and A. Dahiya, "A New picture fuzzy entropy and its application based on combined picture fuzzy methodology with partial weight information," *International Journal of Fuzzy Systems*, Jul. 2022, doi: 10.1007/s40815-022-01332-w.

[3] R. I. Sabri, M. N. Mohammedali, and J. Abbas, "An application of non-additive measures and corresponding integrals in tourism management," *Baghdad Science Journal*, vol. 17, no. 1, p. 0172, Mar. 2020, doi: 10.21123/bsj.2020.17.1.0172.

[4] Z. M. Yusoff *et al.*, "Real time robustness test evaluation performance for intelligent fuzzy controller in extraction process," *Indonesian Journal of Electrical Engineering and Computer Science (IJEECS)*, vol. 15, no. 3, pp. 1290–1296, 2019, doi: 10.11591/ijeecs.v15.i3.pp1290-1296.

[5] S. Kabir and Y. Papadopoulos, "A review of applications of fuzzy sets to safety and reliability engineering," *International Journal of Approximate Reasoning*, vol. 100, pp. 29–55, 2018, doi: 10.1016/j.ijar.2018.05.005.

[6] K. Ouarid, M. Essabre, A. El Assoudi, and E. H. El Yaagoubi, "State and fault estimation based on fuzzy observer for a class of Takagi-Sugeno singular models," *Indonesian Journal of Electrical Engineering and Computer Science (IJEECS)*, vol. 25, no. 1, pp. 172–182, Jan. 2022, doi: 10.11591/ijeecs.v25.i1.pp172-182.

[7] A. Ghaliba and A. Oglah, "Design and implementation of a fuzzy logic controller for inverted pendulum system based on evolutionary optimization algorithms," *Engineering and Technology Journal*, vol. 38, no. 3A, pp. 361–374, Mar. 2020, doi: 10.30684/etj.v38i3a.400.

[8] M. Sudha, "Classifying the fault type in underground distribution system based on fuzzy logic algorithm," *Indonesian Journal of Electrical Engineering and Computer Science (IJEECS)*, vol. 8, no. 2, pp. 557–560, 2017, doi: 10.11591/ijeecs.v8.i2.pp557-560.

[9] H. Mohammed, "Determination of suitable areas for establishment of sports-city in Iraq's center using an integrated fuzzy logic algorithm and geomatic techniques," *Engineering and Technology Journal*, vol. 39, no. 8, pp. 1291–1300, Aug. 2021, doi: 10.30684/etj.v39i8.2094.

[10] I. E. Fattoh, F. A. Mousa, and S. Safwat, "Converting cumulative grade point average to an equivalent percentage value based on a fuzzy logic," *Indonesian Journal of Electrical Engineering and Computer Science (IJEECS)*, vol. 24, no. 3, pp. 1823–1831, Dec. 2021, doi: 10.11591/ijeecs.v24.i3.pp1823-1831.

[11] I. H. Hadi and A. K. Abdul-Hassan, "A proposed speaker recognition method based on long-term voice features and fuzzy logic," *Engineering and Technology Journal*, vol. 39, no. 1B, pp. 1–10, Mar. 2021, doi: 10.30684/etj.v39i1b.343.

[12] A. A. Sadat Asl, M. M. Ershadi, S. Sotudian, X. Li, and S. Dick, "Fuzzy expert systems for prediction of ICU admission in patients with COVID-19," *Intelligent Decision Technologies*, vol. 16, no. 1, pp. 159–168, Apr. 2022, doi: 10.3233/IDT-200220.

[13] I. Kramosil and J. Michalek, "Fuzzy metrics and statistical metric spaces.," *Kybernetika*, vol. 11, no. 5, pp. 336–344, 1975.

[14] B. Y. Hussein and F. K. Al-Basri, "On the completion of Quasi-fuzzy normed algebra over fuzzy field," *Journal of Interdisciplinary Mathematics*, vol. 23, no. 5, pp. 1033–1045, Jul. 2020, doi: 10.1080/09720502.2020.1776942.

[15] R. Saadati and S. M. Vaezpour, "Some results on fuzzy banach spaces," *Journal of Applied Mathematics and Computing*, vol. 17, no. 1–2, pp. 475–484, 2005, doi: 10.1007/BF02936069.

[16] A. A. Borubaev, "On some generalizations of metric, normed, and unitary spaces," *Topology and its Applications*, vol. 201, pp. 344–349, Mar. 2016, doi: 10.1016/j.topol.2015.12.045.

[17] D. Sain, "On the norm attainment set of a bounded linear operator and semi-inner-products in normed spaces," *Indian Journal of Pure and Applied Mathematics*, vol. 51, no. 1, pp. 179–186, Mar. 2020, doi: 10.1007/s13226-020-0393-9.

[18] N. Sookoo, "F-normed spaces and linear operators," *Journal of Interdisciplinary Mathematics*, vol. 24, no. 4, pp. 911–919, May 2021, doi: 10.1080/09720502.2020.1826628.

[19] Mami Sharma and Debajit Hazarika, "On linear operators in Felbin's fuzzy normed linear space," *Annals of Fuzzy Mathematics and Informatics*, vol. 13, no. 6, pp. 749–758, Jun. 2017, doi: 10.30948/afmi.2017.13.6.749.

[20] M. N. Mohammed Ali, R. I. Sabri, and F. A. Sadiq, "A new properties of fuzzy b-metric spaces," *Indonesian Journal of Electrical Engineering and Computer Science (IJEECS)*, vol. 6, no. 1, pp. 221–228, 2022, doi: 10.11591/ijeecs.v26.i1.pp221-228.

[21] S. Abed and Z. Mohamed Hasan, "Weak convergence of two iteration schemes in banach spaces," *Engineering and Technology Journal*, vol. 37, no. 2B, pp. 32–40, 2019, doi: 10.30684/etj.37.2b.1.

[22] T. Bag and S. K. Samanta, "Some fixed point theorems in fuzzy normed linear spaces," *Information Sciences*, vol. 177, no. 16, pp. 3271–3289, 2007, doi: 10.1016/j.ins.2007.01.027.

[23] M. N. Mohammedali, "Fuzzy real pre-hilbert space and some of their properties," *Baghdad Science Journal*, vol. 19, no. 2, pp. 313–320, Apr. 2022, doi: 10.21123/bsj.2022.19.2.0313.

[24] T. Bag and S. K. Samanta, "Finite dimensional fuzzy normed linear spaces," *Journal of Fuzzy Mathematics*, vol. 11, no. 3, pp. 687–706, 2003.

[25] A. T. Diab, S. I. Nada, and D. L. Fearnley, "The sandwich theorem for sublinear and superlinear functionals," *arXiv preprint*, Nov. 2016, doi: 10.48550/arXiv.1611.02670.

BIOGRAPHIES OF AUTHORS

Mayada N. Mohammedali Received the B.Sc. degree in applied mathematics from the University of Technology, Iraq (2006), the M.Sc. degree in fuzzy functional analysis from the University of Technology, Iraq (2008). She has become an assistant professor in March 2021. Her researches are in fields of fuzzy functional analysis, abstract Algebra, multi-criteria decision making, fuzzy sets, and system and numerical analysis. She has served as an invited reviewer. She has 19 published articles inside Iraq and some in international journals. She can be contacted at email: mayada.n.mohammedali@uotechnology.edu.iq.