

## An investigation of even ordered magic squares (4, 6, and 8): characteristic polynomials, eigenvalues, and encryption

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### ABSTRACT

In this paper, we discuss and mathematically compute the eigenvalues and the characteristic polynomials of special even square matrices of orders 4x4 and 8x8. Also, we introduce two 8th order compound magic squares. The computed values are verified using Maple software. First, for the 4th order square matrix, the characteristic polynomial was derived to be:  $\lambda(\lambda - 2s)(\lambda^2 + 4\theta)$  with the eigenvalues:  $0, 2s$ , and two other conjugates. In further analysis, we performed numerical classification of the squares for the matrices of order 4. Second, for the 8th order magic square, the characteristic polynomial was obtained in the form:  $\lambda^3(\lambda - 4s)(\lambda^4 + \Omega\lambda^2 + \theta)$  where  $\Omega, \theta$  are constants; the eigenvalues are  $0, 4s, \pm\sqrt{\lambda_1}, \pm\sqrt{\lambda_2}$ ; where  $\lambda_1, \lambda_2$  are the roots of the quadratic equation:  $\lambda^2 + \Omega\lambda + \theta = 0$ . Third, for the franklin square, we obtained the eigenvalues  $0, 4s$ , and the roots of the equation:  $\lambda^2 + a\lambda + b$ . Finally, we suggested a hybrid image encryption technique based on Franklin magic square matrices and improved substitution technique. The proposed a grayscale image encryption/decryption algorithm uses circular rotation of bits and Franklin magic squares' properties in conjunction with substitution techniques to obtain a very secure algorithm against attacks.

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## 1. INTRODUCTION

In this investigation, we derive the characteristics polynomial and compute the eigenvalues related to special matrices called magic squares. Also, a semi magic square is a matrix that has the same sum of the matrix entries of every row and column, called the magic constant. The semi magic square has the sum of the main diagonals equal to the magic constant. On the other hand, the  $n^{\text{th}}$  order natural magic square is a square matrix with integer entries from one to  $n^2$ . The magic constant is obtained as (1).

$$\frac{1}{2}n(n^2 + 1) \tag{1}$$

The combinations which appear in the columns, rows, and both diagonals of this square are the only distinct three elements' mixture of the numbers from 1 to 9 that sum up to the value of 15. A self complementary magic matrix of the order  $2k+1$  that has the  $(2k+1)s$  as a magic constant is still considered a magic square satisfying the conditions (2).

$$a_{(k+1)(k+1)} = s \ \& \ a_{ij} + a_{(2k+2-i)(2k+2-j)} = 2s, 1 \leq i \leq k, 1 \leq j \leq 2k + 1 \tag{2}$$

Also, a  $2k$  self complementary magic matrix with  $2ks$  as a magic constant is a magic square having the condition (3).

$$a_{ij} + a_{(2k+1-i)(2k+1-j)} = 2s, 1 \leq i \leq k, 1 \leq j \leq 2k \quad (3)$$

In other words, a natural self-complementary magic square is a magic square of order  $n$  such that the sum of both elements of each couple of dual (opposite entries) is usually equal to  $n^2 + 1$ .

An off-diagonal is a combination of two parallel diagonal lines to the same main diagonal. The two parallel diagonal lines must occur on opposite sides of the primary diagonal and they can only be combined if the combination has the same number of entries as the primary diagonal. Two examples of an off-diagonal line are 10, 3, 8, 13 and 7, 14, 9, 4 are shown in Figure 1. The Figure shows 3 off-diagonals corresponding to the main diagonal. The square in Figure 1 is called semi pandiagonal since the off-diagonal lines 10, 3, 8, 13 and 7, 14, 9, 4 sum to the magic constant. A pandiagonal square is a magic square where the off-diagonals sum to the magic constant. It must be noted that even by swapping all rows and columns of a pandiagonal square, the matrix remains pandiagonal.

The results of different order squares are verified using linear algebra while Maple software is used to perform lengthy calculations; some complicated calculations are difficult to manipulate manually and must be performed using a computing device. Additionally, we introduce special cases of 4 by 4 squares which have a simplified form of the eigenvalues. Also, we give some numerical examples that substantiate our results.

1	7	10	16
14	12	5	3
8	2	15	9
11	13	4	6

Figure 1. Natural semi pandiagonal square

## 2. LITERATURE REVIEW

There are many papers in literature related to magic squares and their applications. Nordgren [1] proved using Mattingly well-known theorem that magic squares called quasi-regular (QR) have signed eigenvalues pairs are similar to that of regular magic squares. The researchers concluded that the odd powers of the QR magic matrices are magic squares that can be obtained from the QR condition. They generalized that all 4<sup>th</sup> order pandiagonal magic squares are QR because they are all most-perfect (MP). Also, the authors argued that the QR conditions holds for all 5<sup>th</sup> order pandiagonal magic squares but the same cannot be said of higher order matrices. Similarly, Stephens [2] studied 4<sup>th</sup> and 5<sup>th</sup> order magic squares and elaborated on the previously obtained eigenvalues. An  $n$  by  $n$  matrix  $A = (A_{i,j})$  is called a pentadiagonal matrix if  $A_{i,j} = 0$  when  $|i - j| > 2$  condition is true. Mattingly [3] argued that even order regular magic squares are singular because they always have a zero eigenvalue and since each positive eigenvalue has the same algebraic multiplicity as the negative one. Chu *et al.* [4] analyzed magic matrices for which the Moore-Penrose inverse is also magic. They introduced singular magic matrices in which the numbers in the rows and columns and in the two main diagonals all add up to the same sum. They then introduced the concept of a “philatelic magic square” as a square arrangement of images of postage stamps for which the nominal values make a magic square. Ibrahim and Salman [5] presented definitions and some special properties of magic squares with proofs of two theorems. Neeradha and Mallayya [6] presented the famous strongly magic square, which has the strong property that the sum of the entries of the sub-squares taken without any gaps between the rows or columns is also a magic. The authors detailed the properties of 4×4 strongly magic squares dot products and different eigen values and eigen vectors properties. Zhaolin *et al.* [7] suggested a new Sylvester-Kac matrix, i.e., Fibonacci-Sylvester-Kac matrix where they discussed the eigenvalues, eigenvectors and characteristic polynomial of the matrix in two categories based on whether the Fibonacci-Sylvester-Kac matrix order is odd or even. Additionally, the authors gave the explicit formulas for its determinant and inverse. Abdulkhaev and Shirokov [8] discussed the characteristic polynomials in (Clifford) geometric algebras  $G_{p,q}$  of vector space of dimension  $n=p+q$ . They presented explicit formulas for all characteristic polynomial coefficients in the case  $n=5$ . The formulas involved the operations of geometric product, summation, and conjugation.

Tridiagonal and pentadiagonal matrices appear in several areas of mathematics and engineering; specially the ones that involve linear systems of differential equations. Since every 3 by 3 matrix is a pentadiagonal one, it is clear that not every pentadiagonal matrix is comparable to a symmetric matrix.

Alvarez *et al.* [9] considered two classes of pentadiagonal matrices and obtained recursive formulas for the characteristic polynomials and explicit formulas for the eigenvalues of these classes of pentadiagonal matrices. They showed that if  $A$  is a pentadiagonal matrix and if it satisfies some certain conditions on the sign of the product of the entries, then  $A$  is similar to a symmetric pentadiagonal matrix. Thus, all the eigenvalues of  $A$  are real and  $A$  is diagonalizable. Elouafi [10] explained that the characteristic polynomial for such matrices is the product of two polynomials presented in terms of Chebyshev polynomials.

Al-ashhab [11] considered some properties of the magic 4 by 4 squares such as their determinant; while Al-ashhab [12] introduced new special types such as the four corner magic squares. The researcher explained ways of counting them using parallel programming. He also discussed the characteristic polynomial of Franklin squares. Rosser and Walker [13] described the general structure of pandiagonal magic 4 by 4 squares. The authors proved for the first time a well-known structure. Ahmed [14] described many features of Franklin squares, which is used in this paper in a cryptographic application of magic squares. Fahimi *et al.* [15] classified twelve groups of 4 by 4 squares where he discussed these groups with the minimum values of electrostatic potential. The potential and its related properties are calculated on the grids numerically using MATLAB 2018a. Equipotential points and certain constants are found among the electrostatic potential sums along horizontal and vertical lines on the square lattice.

Rungratgasame *et al.* [16] introduced new special magic squares and called them reflective magic squares, corner magic squares, and skew-regular magic squares by combining the concepts of magic squares and linear algebra. They found the dimensions of the vector spaces of these magic squares under the standard addition and scalar multiplication of matrices using the rank-nullity theorem.

Many algorithms and schemes have been proposed to protect data of various formats including images. Ozturk and Sogukpinaar [17] classified image encryption techniques into three types: position permutation, value transformation and visual transformation. For example, Caesar's magic square works by removing spaces between words of sentences. The letters are arranged in a square starting from the top left downward, then moving to the second column from the top and so on. It is preferable to have the letters of the sentence as squares of an integer, such as 1, 4, 9, 16, 25, 36, and so on, so that the square root of the letters can be obtained. Al-qdah [18] proposed and implemented a secure two operation image encryption technique by dividing a selected image into some blocks before applying bit rotation using one random key followed by chaotic map scrambling using another random key. Gupta *et al.* [19] introduced a hybrid image encryption technique for protection against statistical attacks using Arnold map and S-box of advanced encryption standard (AES). The Arnold cat map spatially shuffles the pixels of the image and then the image is encrypted using substitution-box (S-box). Irom and Ningthoujam [20] considered the applicability of a magic squares/weak magic squares matrix of any order in evaluating numerals for encryption and decryption. The authors indicated that singly even  $n$  weak magic squares can produce possibly different ciphertext from plaintext than that of the actual magic squares. Also, they added dummy letters to the English letters to eliminate the duplication of vowels in encrypting a message.

Alattar and Rahma [21] proposed a hybrid method to encrypt images using franklin magic square matrices and improved substitution techniques. They developed two image or text encryption algorithms based on 5 by 5 Magic Squares using GF(P) and GF( $2^8$ ): the first used a message length = 10 and the second used a message length = 14. A number of rounds were added and a mask is used in the even round for an addition operation while the odd round is used for multiplication operation so that the text resulting from the first round is used as input text for the next Round. Al-Hashemy and Mehdi [22] presented a hybrid image encryption algorithm based on chaotic systems and image squares. The encryption keys are generated chaotically equal to the size of an image, which is then subdivided into equal sub images; each sub-image is multiplied by a magic matrix. The resulting matrices are XORed produce one encrypted image.

Alattara and Rahma [23] developed an encryption algorithm using magic squares (5 by 5) with multi message lengths for added security. The algorithm used two rounds and selected a message interchangeably. The key was fixed in one position while the remaining positions are filled with the message; some sums were calculated to obtain an encrypted text. The authors concluded that their proposed algorithm gave better encryption speeds and added levels of security. Umar [24] presented an added security layer to the Rivest-Shamir-Adleman (RSA) algorithm approach using a magic square of order 32 to generate a non-duplicate random numbers to represent the numeral aspect of the message instead of the ASCII values and also the author used the magic squares values as a key in the encryption process. Mohammed and Hasan [25] suggested an encryption technique that hides a message in a submatrix of order 4 by 4, which is selected from a 16 by 16 magic square matrix. The author used two stages of hiding the ciphertext using a magic square of size 3x3 and Latin square of size 3x3 for more permutations and to obtain an inverse matrix for the decryption operation. The algorithm hides the ciphertext into a 16 by 16 matrix with 16 sub matrices so that duplicates are eliminated. The elements of the matrices were polynomial numbers of a finite field of degree Galois fields GF ( $2^8$ ).

### 3. METHODS AND DISCUSSIONS

In this section, we particularly discuss the characteristics of magic squares of sizes: 4 by 4, 8 by 8, nested 6 by 6, and Franklin squares. We first perform some mathematical proofs and calculations before we utilize Franklin magic squares to develop a cryptographic technique in subsection 3.6.

#### 3.1. Magic square (4 by 4)

It is accepted that the magic constant is always an eigen value since one of the corresponding eigen vectors has all ones. Hence, any 4 by 4 semi magic square has its characteristic polynomial in the form (4).

$$(\lambda - \text{magic constant})(\lambda^3 + a \lambda^2 + b \lambda + c) \tag{4}$$

The determinant of the pandiagonal magic square 4 by 4 is zero. Generally, the pandiagonal magic square of size 4 by 4 has the structure shown in Figure 2.

A	B	C	$2s - B - C - A$
E	$2s - B - A - E$	$A + E - C$	$B + C - E$
$s - C$	$A + B + C - s$	$s - A$	$s - B$
$C + s - A - E$	$s - C - B + E$	$s - E$	$A + B + E - s$

Figure 2. Symbolic pandiagonal square

The characteristic polynomial of the magic square was calculated using Maple software to be (5):

$$\lambda(\lambda - 2s)(\lambda^2 + 4\theta) \tag{5}$$

where

$$\theta = C^2 - A^2 - s^2 - AE - 2BE - CE - AB + BC + s(2E + 2A + B - C) \tag{6}$$

we can see that the characteristic polynomial has two missing coefficients ( $a$  and  $c$ ) and can be rewritten as (7):

$$(\lambda - 2s)(\lambda^3 + 4\theta\lambda) \tag{7}$$

Mathematically, we can obtain the characteristic polynomial in (1) using the trace eigen value theorem which postulates that the sum of the eigen values is the trace (in this case  $2s$ ). We can also conclude that both eigen values must be conjugates since the magic square has  $2s$  as eigen values, and it has been proven that its determinant is zero ( $0$  is the second eigen value). Figure 3 shows another example of a 4 by 4 magic square. The figure presents a semi magic square of order 4 that has the characteristic polynomial with no missing terms (8).

$$(\lambda - 60)(\lambda^3 - 120 \lambda^2 - 3453 \lambda + 980) \tag{8}$$

20	2	3	35
42	106	-46	-42
1	-46	46	59
-3	-2	57	8

Figure 3. A 4th order semi magic square matrix

Therefore, we can generally conclude that semi magic squares don't have any missing coefficients in their characteristic polynomial. On the other hand, the standard magic square in Figure 4 has  $2s$  as a magic constant. We observe that the matrix has seven independent variables in addition to the  $s$  variable. The characteristic polynomial, which was calculated by Maple, has one missing coefficient ( $a=0$ ) given in the form (9).

$$(\lambda - 2s)(\lambda^3 + b \lambda + c) \tag{9}$$

h	p	j	2s-j-h-p
v	q	2s-q-f	f-v
6s-2h-j-2q-f-v-p	2s-q	f+q-2s	2h+j+2q-4s+v+p
f+h+j+2q-4s+p	-p	2s-j	4s-h-2q-f

Figure 4. Magic square matrix with constant 2s

The mathematical justification for the above form can be performed using the trace coefficient theorem which states that the trace (in this instance 2s) is the negative of the second coefficients; then the characteristic polynomial must have the form (9).

$$(\lambda - 2s)(\lambda^3 + b\lambda + c) = \lambda^4 - 2s\lambda^3 + b\lambda^2 + (c - 2bs)\lambda - 2cs \tag{10}$$

In general, semi pandiagonal magic squares with magic constant 2s has six independent variables in addition to the s. Therefore, the square matrix can be written in the form given in Figure 5.

2s-t-m-z	f	d	z-d+t+m-f
z	t+m-f	2s-2t-m+f-z+d-g	g-d+t
m	g	z-d+t	2s-g-t-m-z+d
t	2s-t-m-g	g+t+m-f-d	d-t+f

Figure 5. Semi pandiagonal magic square

Furthermore, we calculated the semi pandiagonal magic square using Maple software; it has the characteristic polynomial in the form:

$$(\lambda - 2s)(\lambda^3 + b\lambda + c) \tag{15}$$

Figure 6 shows an example of a semi pandiagonal magic square with two independent variables. The characteristic polynomial was calculated as (16).

$$(\lambda - 2s)(\lambda^3 - (14f + 10s - 2fs + f^2 - 15)\lambda - 2M) \tag{16}$$

Where:

$$M = 4s^3 + (2f - 34)s^2 + (79 - f^2 - 14f)s + 2f^2 + 26f - 36 \tag{17}$$

2s-3	f	4	-f-1
2	1-f	f+2s-4	1
1	5	-2	2s-4
0	2s-6	2-f	f+4

Figure 6. Special semi pandiagonal magic square

in some particular cases, a semi pandiagonal magic square with less free elements can have a simple form of the eigenvalues; for example, Figure 7 shows a special case with five free elements. It can be observed that the eigen values are independent of d and are actually in the following form (18).

$$0, 2s, \pm 2\sqrt{(s-t-z)(m+t-s)} \tag{18}$$

2s-m-t-z	s-d	d	m-s+t+z
z	d+m-s+t	2s-d-m-t	s-z
m	d+s-t-z	t-d+z	s-m
t	s-d-m+z	d+m-z	s-t

Figure 7. Special semi pandiagonal magic square type 1

Figure 8 illustrates one more special case where the last two eigen values are independent of  $s$  and  $j$ .

$g-b-j+s$	$b$	$s-b$	$b-g+j$
$j$	$s-g$	$g$	$s-j$
$2b+j$	$g+s$	$-g$	$s-j-2b$
$s-b-g-j$	$-b$	$b+s$	$b+g+j$

Figure 8. Special semi pandiagonal magic square type 2

Specifically, the eigen values are:  $0, 2s, \pm 2\sqrt{b^2 - g^2}$ , Figure 9 shows an additional special case where all the eigen values can be real numbers.

$2s$	$f$	$d$	$-d-f$
$0$	$-f$	$d+f-g+2s$	$g-d$
$0$	$g$	$-d$	$d-g+2s$
$0$	$2s-g$	$g-f-d$	$d+f$

Figure 9. Special semi pandiagonal magic square type 3

The eigen values are given in the form (19).

$$2s, 2s, -s \pm \sqrt{f^2 + 2fg - 3g^2 + 4dg + (6g - 2f - 4d)s - 3s^2} \tag{19}$$

The discriminant of the quadratic quantity under the root sign with respect to  $s$  is (20).

$$16(df + d^2 + f^2) > 0, \forall d, f \neq 0 \tag{20}$$

Hence, if  $f$  and  $d$  are both nonzero then there is always a bounded interval of  $s$  (between the roots of the quadratic quantity) that has four real eigen values regardless of the  $g$  value.

### 3.2. Magic square 8 by 8

We construct an 8 by 8 pandiagonal magic square using four separate pandiagonal magic squares of sizes 4 by 4 with each having the same magic constant. The square matrix structure is displayed in Figure 10. We require for the four submatrices to become pandiagonal that some conditions on  $D, F, H, L, N, O$  and  $P$  and the corresponding small letters hold. As a result, the generated 8 by 8 square becomes of magic type; but in order to obtain a pandiagonal 8 by 8 square as shown in Figure 11, we must impose more conditions on the independent variables:  $A+i=a+I, B+j=b+J, C+k=c+K, E+m=e+M$ . This correspondence will be passed onto the dependent variables automatically, e. g.  $D+l=d+L$ .

A	B	C	D	I	J	K	L
E	F	G	H	M	N	O	P
$s-C$	$s-D$	$s-A$	$s-B$	$s-K$	$s-L$	$s-I$	$s-J$
$s-G$	$s-H$	$s-E$	$s-F$	$s-O$	$s-P$	$s-M$	$s-N$
a	b	c	d	i	j	k	l
e	f	g	h	m	n	o	p
$s-c$	$s-d$	$s-a$	$s-b$	$s-k$	$s-l$	$s-i$	$s-j$
$s-g$	$s-h$	$s-e$	$s-f$	$s-o$	$s-p$	$s-m$	$s-n$

Figure 10. Special pandiagonal magic square of size 8 by 8

The matrix rank is five and its characteristic polynomial is (21).

$$\lambda^3(\lambda - 16)(\lambda^4 - 384\lambda^2 + 55328) \tag{21}$$

-4	-4	-3	19	0	0	1	7
1	15	0	-8	2	6	1	-1
7	-15	8	8	3	-3	4	4
4	12	3	-11	3	5	2	-2
1	2	3	2	5	6	7	-10
8	-3	6	-3	9	-12	7	4
1	2	3	2	-3	14	-1	-2
-2	7	-4	7	-3	0	-5	16

Figure 11. Special numerical pandiagonal magic square of size 8 by 8

Thus, we conclude that its rank is generally five and expects that the characteristic polynomial to have the form (22):

$$\lambda^3(\lambda - 4s)(\lambda^4 + \Omega\lambda^2 + \theta) \tag{22}$$

where  $\Omega, \theta$  are constants. Also, when we perform the substitution  $x = \lambda^2$ , we obtain the roots of this equation (the eigenvalues):  $0, 4s, \pm\sqrt{\lambda_1}, \pm\sqrt{\lambda_2}$ , where  $\lambda_1, \lambda_2$  are the roots of the quadratic in (23).

$$x^2 + \Omega x + \theta = 0 \tag{23}$$

The proof of this eigen values conjecture is achieved by computing the determinants using a computing device since the amount of computation of the determinants is extremely huge. Now, we again consider the characteristic polynomial and a special type of matrices that have the form (26).

$$\lambda^5(\lambda - 4s)(\lambda^2 + \zeta) \tag{24}$$

Where  $\zeta$  is a constant. We arrange two 4 by 4 pandaigonal magic squares A and B (magic sum 2s) that satisfy the conditions in the following matrix (25).

$$D = \left( \begin{array}{c|c} A & A \\ \hline B & B \end{array} \right) \tag{25}$$

Accordingly, we conclude that the nullspace of this 8 by 8 matrix contains the following four independent vectors  $(x_i)$  such that (26).

$$x_i = -1, x_{i+4} = 1, x_j = 0 \text{ for } j \neq i, i + 4, i = 1, 2, 3, 4 \tag{26}$$

Thus, the characteristic polynomial must be in the form (27).

$$\lambda^4(\lambda - 4s)(\lambda^3 + b\lambda + c) \tag{27}$$

But the square of the roots of this cubic polynomial must be eigen values for  $D^2$ . The structure of  $D^2$  is as (28):

$$\left( \begin{array}{c|c} G & G \\ \hline H & H \end{array} \right) \tag{28}$$

where G and H are matrices, which has the following structure (29).

$$N = \begin{pmatrix} \rho & \varphi & \sigma & \phi \\ \chi & \tau & \vartheta & \xi \\ \sigma & \phi & \rho & \varphi \\ \vartheta & \xi & \chi & \tau \end{pmatrix} \tag{29}$$

In addition, we observe that G and H have the same nullspace. If we take the vector  $(v_1, v_2, v_3, v_4)^t$  in this common nullspace, then a basis of the nullspace of  $D^2$  can be chosen as (30):

$$(v_1, v_2, v_3, v_4, 0, 0, 0, 0)^t, (x_i), x_i = -1, x_{i+4} = 1, x_j = 0 \text{ for } j \neq i, i + 4, i = 1, 2, 3, 4 \tag{30}$$

We notice that N multiplied by the vector  $(1, 1, 1, 1)^t$  yields  $(8s^2, 8s^2, 8s^2, 8s^2)^t$ . On the other hand, N multiplied by the vector  $(1, -1, 1, -1)^t$  yields  $(t, -t, t, -t)^t$  for a real number t. Therefore,  $D^2$  is a magic square with a magic sum  $16s^2$ ; and this magic sum is one of the  $D^2$  eigen values. The eigen vectors are

multiples of the vector one. Also, due to these properties we can deuce that the value  $t_1 + t_2$  is an eigenvalue of  $D^2$  where:

$$t_1 = G(1, -1, 1, -1)^t, t_2 = H(1, -1, 1, -1)^t, \tag{31}$$

an eigen vector can be selected as  $(t_1, -t_1, t_1, -t_1, t_2, -t_2, t_2, -t_2)^t$ . Actually, we can check that there exists a second linear independent eigen vector for this eigen value, and

$$X(X - 16s^2)(X - (t_1 + t_2)) \tag{32}$$

is the minimum polynomial of  $D^2$ . Therefore, the only eigen values of  $D^2$  are 0,  $16s^2$  and  $t_1 + t_2$ . The other eigen values of  $D$  (except 0 and  $4s$ ) must sum up to zero and their squares is one of the three eigen values of  $D^2$ . Thus, the eigen values must be 0,  $4s$  and  $\mp\sqrt{t_1 + t_2}$  and the characteristic polynomial is given by (33).

$$\lambda^5(\lambda - 4s)(\lambda^2 - (t_1 + t_2)) \tag{33}$$

### 3.3. Franklin squares

Franklin squares are semi magic squares of order 8. They have some similarities to pandiagonal magic squares of size 8 by 8 and resemblance to the special type of 8 by 8 square, which is a combination of four squares. As explained by Franklin, each row and column of the square has the common sum  $2s$  and half of each row or column sums up to half of  $s$ . In addition, each of the "bent rows" (as Franklin called them) have the sum  $2s$ . Franklin described the 8 by 8 magic square and its special properties. Figure 12 shows an example of a Franklin magic square with magic sum 260.

52	61	4	13	20	29	36	45
14	3	62	51	46	35	30	19
53	60	5	12	21	28	37	44
11	6	59	54	43	38	27	22
55	58	7	10	23	26	39	42
9	8	57	56	41	40	25	24
50	63	2	15	18	31	34	47
16	1	64	49	48	33	32	17

Figure 12. Numerical Franklin magic square

The general form of a Franklin square with magic constant  $2m$  is (34):

$$D = \begin{pmatrix} A & C \\ B & D \end{pmatrix} \tag{34}$$

where  $A, B, C$  and  $D$  are semi magic 4 by 4 squares. Neeradha and Mallayya [6], it was proven that the characteristic polynomial is in the form (35):

$$\lambda^5(\lambda - 4s)(\lambda^2 + a\lambda + b) \tag{35}$$

where the constant  $a$  depends only on the value of two entries in the square. Actually, there are only eight independent variables in addition to the magic constant.

### 3.4. Compound magic squares 8x8

In this section we present an 8<sup>th</sup> order square shown in Figure 13. The matrix consists of four nonregular semi magic squares where each one has the number 130 as a magic constant value. We are going to show that the characteristic polynomial is simpler due to the form and the way its constructed.

1	62	44	23	16	51	26	37
63	4	22	41	50	13	40	27
24	43	61	2	25	38	15	52
42	21	3	64	39	28	49	14
17	46	60	7	32	35	10	53
47	20	6	57	34	29	56	11
8	59	45	18	9	54	31	36
58	5	19	48	55	12	33	30

Figure 13. Compound magic square that can be divided into four 4th order semi magic squares

The matrix in Figure 13 has a nontrivial nullspace and one of its bases is given in (36).

$$(1,0,1,0, -1, 0, -1, 0), (-1, -1,0,0,1,1,0,0), ( 1, 0, 0, -1, -1, 0, 0, 1 ). \tag{36}$$

Also, the characteristic polynomial is given as (37).

$$(X - 260)X^4(X^3 + 8X^2 - 5120X + 81920) \tag{37}$$

Therefore, the nonzero eigenvalues are 260, 17.53, 56.77 and -82.3. But, there are various examples of squares that consist of four self complementary semi pandiagonal magic squares with the nullity of exactly three. In some cases they are diagonalizable.

**3.5. Nested magic square 6 by 6**

The nested magic squares of order n by n can be generally written as shown in Figure 14. Where we set some conditions on F, G, H, N, U and V to ensure that the middle 4 by 4 square is the exact pandiagonal matrix that was given in figure 2. We can mathematically prove that the characteristic polynomial has the form (38).

I	J	K	L	M	N
O	A	B	C	D	s-O
P	E	F	G	H	s-P
R	s-C	s-D	s-A	s-B	s-R
T	s-G	s-H	s-E	s-F	s-T
U	s-J	s-K	s-L	s-M	V

Figure 14. Nested magic square of size 6 by 6

$$\lambda(\lambda - 3s)(\lambda^4 + H\lambda^3 + L\lambda^2 + I\lambda + K) \tag{38}$$

In general, the coefficients H, L, I and K are nonzero. On the other hand, in some cases such as the magic matrix shown in Figure 15, we obtain for the coefficients in the characteristic polynomial (39).

$$H = 0 \text{ and } I = 4 + 8g - 6a - 4j \tag{39}$$

Consequently, the odd powers disappear by imposing the condition as (40).

$$j = 1 + 2g - 1.5a \tag{40}$$

1	2	3	4	0	2
1	g	h	j	8-h-j-g	3
2	a	8-g-h-a	a+g-j	h-a+j	2
4	4-j	g+h+j-4	4-g	4-h	0
2	j-g-a+4	a-h-j+4	4-a	a+g+h-4	2
2	2	1	0	4	3

Figure 15. Nested magic square of order 6 by 6

**3.6. Image encryption based on Franklin magic squares and modified substitution techniques**

Many researchers have proposed hundreds of numeric and alphabetic ciphers that transform letters into numbers, or symbols, or transform alphabets into other forms. Recently, many intelligent algorithms have been suggested with many permutations and lengthy keys of millions of kilobytes. Therefore, the literature is full of various algorithms with varying complexities and speed. Al-qdah [18] explained that cryptography is a secure technique that achieves confidentiality of messages. The author elaborated that the term had a specific meaning for the Greek: secret organizations is provided through cryptography at a high level, making sure that the exchanged information is sent in a secure way that only the authorized receiver can access it. Nowadays, cryptography is globally used by many organizations and individuals to protect different kinds of stored and network data. In this section, we propose a hybrid technique to encrypt/decrypt grayscale images using substitution techniques and the 8 by 8 Franklin magic square matrices of distinct values. Magic square encryption offers the possibility of checking the encrypted image for tamperment since the magic condition has to hold after applying the encryption algorithm. Thus, attacking the encrypted image might render the matrices not magic, which would indicate some attack has been inflicted on the encrypted

image. Therefore, this work designs an algorithm that checks for the franklin square condition on a partitioned image before applying rotational and substitution techniques. The proposed encryption and decryption procedures are explained below.

Proposed encryption procedure: i) generate four magic squares of sizes 8 by 8 and number them m1, m2, m3, m4; ii) store only the values of the 9 independent variables from each matrix (total=36) that are needed to regenerate the four matrices (m1, m2, m3, m4); iii) read a grayscale image of size 256 pixels (pixel numerical range is: 0 -255); iv) divide the image into 4 partitions (8 by 8 pixels each); v) generate a random number to be utilized as a key; vi) apply circular rotation of bits on each pixel in the 4 matrices based on the encryption key (right rotation when an even number is encountered in the key and left rotation when an odd number is found in the key); vii) store the image's four rotated partitions into 4 matrices (8 by 8 each) and number them p1, p2, p3, and p4. The partitioned matrices are shown in Figure 16; viii) substitute the values of every magic square matrix for its corresponding image partitioned rotated matrix (m1 for p1, m2 for p2, m3 for p3, and m4 for p4); then check each substituted matrix for franklin magic condition; ix) combine the substituted matrices into one 16×16 matrix to create one encrypted image.

Proposed decryption procedure: i) read the encrypted image and the encryption key; ii) divide the encrypted image into 4 partitions (8 by 8 pixels each); iii) store the 4 partitions into 4 matrices (8 by 8 pixels each) and number them e1, e2, e3, and e4. The partitions are shown in Figure 17; iv) check the partitions for the Franklin magic condition; v) read the 4 matrices (p1, p2, p3, p4); vi) perform opposite bits' rotations for each pixel using the encryption key (left rotation with even number and right rotation with odd number); vii) substitute the original image matrices' values for their corresponding encrypted image partitioned matrices' values (p1 for e1, p2 for e2, p3 for e3, and p4 for e4); viii) combine the substituted matrices into one combined image and compare it with the original image.

p1	p2
p3	p4

Figure 16. The partitions of an Image into four matrices

e1	e2
e3	e4

Figure 17. The four encrypted partitioned matrices of the image

#### 4. CONCLUSION

We have computed and analyzed the characteristic polynomial of magic square matrices of order n by n for the distinctive even cases of 4, 8 and 6. We specifically demonstrated that at least one linear term for any magic square can be factored; but also in some cases more linear terms might be factored in the squares: pandiagonal 4 by 4 and Franklin magic 8 by 8. Furthermore, we examined the structure of a non-linear remaining polynomial for several square matrices and discovered that some kinds of polynomials possess zero coefficients, where we proved this fact for some special types of 6<sup>th</sup> order squares. We demonstrated that such types of polynomials can be written as a quadratic equation with squared variable resulting in conjugate eigen values. In addition, we introduced examples of 4<sup>th</sup> order matrices: pandiagonal magic squares and special types of semi magic pandiagonal magic square. We considered several special types of squares of order 4 by 4 whose eigenvalues could be written in a simpler form compared to the general case of a magic square. Also, we introduced a subset of an 8<sup>th</sup> order pandiagonal magic square with a characteristics polynomial whose quadratic factors have even powers. In the case of nested 6 by 6 magic square, we concluded that the characteristic polynomial has a 4<sup>th</sup> order equation in the factorization. The odd powers are cancelled under some conditions; the eigen values are  $0, 3s, \mp\sqrt{\lambda_1}, \mp\sqrt{\lambda_2}$ ; where  $\lambda_1, \lambda_2$  are the zeros of a quadratic equation. Finally, we utilized Franklin square matrix properties in conjunction with improved substitution technique to design a secure cryptographic grayscale image algorithm. The algorithm examines a franklin square matrix and its characteristics otherwise a temperament is detected.

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