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Adaptive Fuzzy Sliding Mode Stabilization Controller for Inverted Pendulums

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Abstract

This paper addresses the control problem of X-Z inverted pendulum in the presents of system uncertainties and external disturbances, and an adaptive fuzzy sliding mode control approach is proposed. The fuzzy system is used to approximate the system uncertainties and the complicated intermediate control functions in the backstepping control design. To update the parameters of the fuzzy system, a proper proportional-integral adaptation law is introduced. The controller can guanrantee the system states and their derivatives asymptotically converge to zero. Finally, simulation studies are done to show the stabilization of the X-Z inverted pendulum under the proposed method.

Keywords: *adaptive fuzzy control, sliding mode control, X-Z inverted pendulum*

1. Introduction

The controller design for the inverted pendulum is a well known problem in control community and has been studied excessively inthe past two decades [1-4]. The inverted pendulum is a kind of nonlinear and complicated system which provides much challenging problem to the controller design. The X-Z inverted pendulum is free to move in the horizontal and vertical plane with horizontal and vertical forces [5-6]. The traditional inverted pendulum can be treated as the special case of the X-Z inverted pendulum. Compared with the traditional inverted pendulum, the X-Z inverted pendulum has more versatility and is more like the actual control object in reality. In reality, the moving of the high buildings in earthquake as well as the launching of the rocket can be modeled as a X-Z inverted pendulum directly or indirectly. The X-Z inverted pendulum has 3 control freedoms and two control inputs. As a result, the controller design for the X-Z inverted pendulum is more difficult than that of the conventional inverted pendulum. In [5], Maravall *et al.* established a hybrid controller that incorporates a Takagi–Sugeno fuzzy control structure with PD control to stabilize the X–Z inverted pendulum. In [6], a PID controller has been introduced to the tracking control of the X–Z inverted pendulum system. And good tracking control performance has been obtained.

As we know, adaptive fuzzy control is a model-free method. It can take care of ill-defined and complex nonlinear systems, even those with significant uncertainties, unknown external disturbances and unknown dynamics. Fuzzy rule-based control systems have been extensively addressed in a lot of areas, including cluster analysis, the controller design and image processing. The fuzzy control methods have been shown to be effective for systems with uncertainties and external disturbances [7-9]. The stability analysis is always an important domain in the completeness of controller design. With conventional fuzzy control system, the stability of the closed-loop system of a Mamdani fuzzy structure is very difficult to be proved. To ensure the closed-loop stability, some certain hybrid control approach is usually used [7, 10, 11]. Among those schemes, sliding mode control is a kind of robust stabilizing control method by driving the system states into a predefined sliding surface. The main advantages of sliding mode control are the system robustness with structured and unstructured uncertainties and satisfactory transient performance can all be preserved [12-13].

Up to now, many study results show that corporate the sliding mode control with the fuzzy control methods not only can alleviate the chattering effects in sliding mode control but can decrease the complexity of fuzzy controller with reduced number of fuzzy rules [14]. Yet, there are little literatures investigate the control problem of X-Z inverted pendulum by using

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adaptive fuzzy sliding mode control. In this paper, an adaptive fuzzy sliding mode controller is designed to control X-Z inverted pendulum. The fuzzy logic system is used to approximate the system uncertainties and the complicated intermediate control functions in the backstepping procedure.

2. Problem Description and Preliminaries

The X-Z inverted pendulum on a pivot driven by vertical and horizontal control forces can be seen in Figure 1. The control inputs of the system are based on the X-Z vertical and horizontal displacements of the pivot.

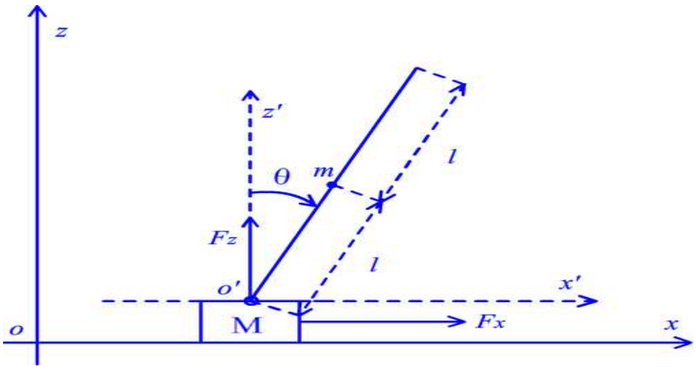

Figure 1. The Structure of the X-Z Inverted Pendulum

The mathematical equations of the X-Z inverted pendulum were given in [1] which can be described by:

$$\begin{aligned}
\ddot{x} &= \frac{Mml\dot{\theta}^2 \sin\theta + (M + m\cos^2\theta)F_x - mF_z \sin\theta\cos\theta}{M(M+m)} \\
\ddot{z} &= \frac{Mml\dot{\theta}^2 \cos\theta + (M + m\sin^2\theta)F_z - mF_x \sin\theta\cos\theta}{M(M+m)} - g \\
\ddot{\theta} &= \frac{-F_x\cos\theta + F_z\sin\theta}{Ml}
\end{aligned} \quad (1)$$

Where $(x,z),(\dot{x},\dot{z}),(\ddot{x},\ddot{z})$ are the position, speed and the acceleration of the pivot respectively. l is the distance form the mass center of the inverted pendulum to the pivot. And g is the acceleration constant of gravity. M,m are the mass of the pivot and the pendulum. F_z is the vertical force, and F_x is the horizontal force.

The fuzzy logic systems that employs singleton fuzzification, sum-product inference and center-off-sets defuzzification can be modeled by [15-16].

$$\alpha(x) = \frac{\sum_{j=1}^{N}\theta_j \Pi_{i=1}^{n}\mu_{F_i^j}(x_i)}{\sum_{j=1}^{N}[\Pi_{i=1}^{n}\mu_{F_i^j}(x_i)]}, \quad (2)$$

Where, $\alpha(x)$ is the output of the fuzzy system, x is the input vector, $\mu_{F_i^j}(x_i)$ is x_i's membership of jth rule and θ_j is the centroid of the jth consequent set. Equation (2) can be rewritten as:

$$\alpha(x) = \vartheta^T\psi(x) \quad (3)$$

with $\vartheta = [\vartheta_1, \cdots, \vartheta_N]^T$, $\psi(x) = [p_1(x), p_2(x), \cdots, p_N(x)]^T$, and the fuzzy basis function can be expressed as: $p_j(x) = \dfrac{\Pi_{i=1}^{n} \mu_{F_i^j}(x_i)}{\sum_{j=1}^{N} [\Pi_{i=1}^{n} \mu_{F_i^j}(x_i)]}$.

3. Controller Design for the X-Z Inverted Pendulum

Let us define the following transform:

$$x_1 = x + l\sin\theta, z_1 = z + l\cos\theta - l. \tag{4}$$

Based on system (1), we have:

$$\begin{aligned} \ddot{x}_1 &= \frac{F_x \sin^2\theta + F_z \sin\theta\cos\theta - Mml\dot{\theta}^2 \sin\theta}{M+m} \\ \ddot{z}_1 &= \frac{F_x \sin\theta\cos\theta + F_z \cos^2\theta - Mml\dot{\theta}^2 \cos\theta}{M+m} - g \\ \ddot{\theta} &= \frac{-F_x \cos\theta + F_z \sin\theta}{Ml} \end{aligned} \tag{5}$$

Let $F_1 = (F_x \sin\theta + F_z \cos\theta - Mml\dot{\theta}^2)/(M+m)$, $F_2 = (-F_x \cos\theta + F_z \sin\theta)/Ml$, we can obtain:

$$\begin{aligned} \ddot{x}_1 &= F_1 \sin\theta \\ \ddot{z}_1 &= F_1 \cos\theta - g \\ \ddot{\theta} &= F_2 \end{aligned} \tag{6}$$

And.

$$\begin{aligned} F_x &= (M+m)F_1 \sin\theta - MlF_2 \cos\theta, \\ F_z &= (M+m)F_1 \cos\theta + MlF_2 \sin\theta. \end{aligned} \tag{7}$$

If we define the change of coordinates as $\tan\theta = y$, $u_1 = F_1\cos\theta$, $u_2 = (F_2 + 2\dot{\theta}^2 \tan\theta)\sec^2\theta$, and take the system uncertainties into consideration, yields:

$$\begin{aligned} \ddot{x}_1 &= u_1 y \\ \ddot{z}_1 &= u_1 - g \\ \ddot{y} &= u_2 + f(\xi) \end{aligned} \tag{8}$$

Where $\xi = [x_1, \dot{x}_1, z_1, \dot{z}_1, y, \dot{y}]$, and $f(\xi)$ is unknown system uncertainty with unknown bound.

Let us define the sliding mode surface as:

$$\begin{aligned} s_1 &= \dot{x}_1 + \lambda_1 x_1, \\ s_2 &= \dot{z}_1 + \lambda_2 z_1, \end{aligned} \tag{9}$$

With $\lambda_i > 0, i = 1,2$ such that the roots of the polynomial $H_i(s) = \lambda_i + s_i$ related to the characteristic equation of $H_i(s) = 0$ are all in the open-half plane.

From (8) and (9) we have:

$$\begin{aligned} \dot{s}_2 &= u_1 - g + \lambda_2 \dot{z}_1, \\ \dot{s}_1 &= u_1 y + \lambda_1 \dot{x}_1. \end{aligned} \tag{10}$$

Then the transformed control input u_1 can be constructed as:

$$u_1 = g - \lambda_2 \dot{z}_1 - k_2 s_2, \quad (11)$$

Where $k_2 > 0$ is a design parameter. If we choose λ_2 and k_2 small enough then we can get $u_1 > 0$.

Let treat $y = y^*$ as an intermediate control function, and form the second equation of (10) we have:

$$u_1 y^* = -\lambda_1 \dot{x}_1 - k_1 s_1, \quad (12)$$

Where $k_1 > 0$ is controller design parameter. Note $u_1 > 0$, the intermediate control input y^* can be described as:

$$y^* = \frac{-\lambda_1 \dot{x}_1 - k_1 s_1}{u_1}. \quad (13)$$

To realize y converges to y^*, define $e = y - y^*$ and the sliding surface.

$$s_3 = \lambda_3 e + \dot{e}, \quad (14)$$

then we have:

$$\dot{s}_3 = \lambda_3 \dot{y} - \lambda_3 \dot{y}^* + u_2 + f(\xi) - \ddot{y}^*. \quad (15)$$

Since $\dot{y}$ and $\ddot{y}^*$ have complicated structure, in this paper, we employ the fuzzy logic system to approximate the unknown function $f(\xi)$ incorporated with $\dot{y}$ and $\ddot{y}^*$. Let us define:

$$\alpha(\xi, u_1) = -\lambda_3 \dot{y}^* + f(\xi) - \ddot{y}^*, \quad (16)$$

then we can approximate the unknown nonlinear function $\alpha(\xi, u_1)$, through the fuzzy logic system (3), as:

$$\hat{\alpha}(\xi, u_1, \vartheta) = \vartheta^T \psi(x, u_1). \quad (17)$$

Let us define the ideal parameter of ϑ as:

$$\vartheta^* = \arg\min_{\vartheta} \left[\sup \left| \alpha(\xi, u_1) - \hat{\alpha}(\xi, u_1, \vartheta) \right| \right], \quad (18)$$

and define the parameter estimation error and the fuzzy system approximation error as:

$$\begin{aligned} &\tilde{\vartheta} = \vartheta - \vartheta^*, \\ &\varepsilon(\xi, u_1) = \alpha(\xi, u_1) - \hat{\alpha}(\xi, u_1, \vartheta^*). \end{aligned} \quad (19)$$

As in literature [8, 9], it is reasonable for us to assume that the fuzzy logic system approximation error is bounded, i.e., there exists some positive constant $\bar{\varepsilon}$, such that:

$$\left| \varepsilon(\xi, u_1) \right| < \bar{\varepsilon}. \quad (20)$$

From above analysis, we can obtain:

$$\hat{\alpha}(\xi,u_1,\vartheta)-\alpha(\xi,u_1)= \\ \hat{\alpha}(\xi,u_1,\vartheta)-\hat{\alpha}(\xi,u_1,\vartheta^*)+\hat{\alpha}(\xi,u_1,\vartheta^*)-\alpha(\xi,u_1) \\ =\tilde{\vartheta}^T\psi(\xi,u_1)-\varepsilon(\xi,u_1). \tag{21}$$

Then the controller u_2 can be chosen as:

$$u_2=-\lambda_3\dot{y}-\hat{\alpha}(\xi,u_1,\vartheta)-k_3s_3-k_4 sign(s_3), \tag{22}$$

Where $k_3,k_4>0$ are design parameters. The fuzzy system parameter is updated by the following adaptation PI law:

$$\vartheta=\int_0^t[\sigma\gamma_1\,|\,s_3\,|\,\vartheta+\gamma_1 s_3\psi(\xi,u_1)]d\tau-\gamma_2\delta \tag{23}$$

With,

$$\delta=\sigma\,|\,s_3\,|\,\vartheta-s_3\psi(\xi,u_1), \tag{24}$$

Where $\sigma,\gamma_1,\gamma_2>0$ are design parameters. `The update law (24) this paper designed has a nice performance as the statement of the following theorem.

Theorem 1. The update law (24) can guarantee that the fuzzy system parameter $\vartheta\in L_\infty$ for bounded initial $\vartheta(0)$.

Proof. Define the Lyapunov candidate function as:

$$V_1=\frac{1}{2\gamma_1}\left(\vartheta+\gamma_2\delta\right)^T\left(\vartheta+\gamma_2\delta\right). \tag{25}$$

Then we have:

$$\dot{V}_1=\frac{1}{\gamma_1}\left(\vartheta+\gamma_2\delta\right)^T\left(\dot{\vartheta}+\gamma_2\dot{\delta}\right) \\ =\left(\vartheta+\gamma_2\delta\right)^T\left(-\sigma|s_3|\vartheta+s_3\psi(\xi,u_1)\right). \tag{26}$$

If δ is chosen as (24), one can obtain:

$$\dot{V}_1\le-\sigma|s_3|\|\vartheta\|^2+|s_3|\|\vartheta\|\|\psi(\xi,u_1)\|-\gamma_2\|\delta\|^2 \\ \le-\sigma|s_3|\|\vartheta\|\left(\|\vartheta\|-c/\sigma\right). \tag{27}$$

Noting $c=\sup\|\psi(\xi,u_1)\|$. Then we can conclude that if $\|\vartheta\|>c/\sigma$, $\dot{V}_1<0$. Thus we know that $\vartheta\in L_\infty$. This ends the proof of theorem 1.

From above discussion, now we are ready to give the following results.

Theorem 2. Consider system (1) or the equivalent system (8). If the sliding mode surface is chosen as (10) and (14), the parameters adaptation laws are defined as (23) and (24) and the controller is constructed by (11)-(13) and (22), we have the following results:

I. All the signals in the closed-loop system will remain bounded.

II. The system states and their derivatives asymptotically converge to zero.

Proof. Let define the following Lypunov function:

$$V=\frac{1}{2}\sum_{i=1}^{3}s_i^2+\frac{1}{2\gamma_1}\left(\tilde{\vartheta}+\gamma_2\delta\right)^T\left(\tilde{\vartheta}+\gamma_2\delta\right). \tag{28}$$

From (10) and (11), we have:

$$s_2\dot{s}_2 = -k_2 s_2^2.$$

According to (15) and (22) we get:

$$s_3\dot{s}_3 = -\tilde{\vartheta}^T\psi(\xi,u_1)s_3 + \varepsilon(\xi,u_1)s_3 \text{-} k_3 s_3^2 - k_4 \mid s_3 \mid . \tag{29}$$

Let $V_2 = \frac{1}{2\gamma_1}\left(\tilde{\vartheta}+\gamma_2\delta\right)^T\left(\tilde{\vartheta}+\gamma_2\delta\right)$, substituting the fuzzy system adaptation law (23) and (24) into (25) and by using the inequality $-2\tilde{\vartheta}^T\vartheta \le -\left\|\tilde{\vartheta}\right\|^2 + \left\|\vartheta^*\right\|^2$, one can get:

$$\begin{aligned}\dot{V}_1 &= \left(\vartheta+\gamma_2\delta\right)^T\left(-\sigma\left|s_3\right|\vartheta + s_3\psi(\xi,u_1)\right)\\ &\le \tilde{\vartheta}\psi(\xi,u_1)s_3 - \gamma_2\left\|\delta\right\|^2 \text{-} \frac{1}{2}\sigma\left|s_3\right|\left\|\tilde{\vartheta}\right\|^2 + \frac{1}{2}\sigma\left|s_3\right|\left\|\tilde{\vartheta}^*\right\|^2\end{aligned} \tag{30}$$

Then if we choose that $k_4 > \overline{\varepsilon} + 0.5\sigma\left\|\tilde{\vartheta}^*\right\|^2$, form (30) and (31) we know $s_3\dot{s}_3 + \dot{V}_2 \le -k_3 s_3^2$ which means that s_3 converges to zero, i.e. $e \to 0$ as time $t \to \infty$. Then from (10)-(12), we know:

$$\dot{s}_1 = u_1(y^* + e) + \lambda_1\dot{x}_1. \tag{31}$$

According to (13) and $e \to 0$, one can obtain that:

$$s_1\dot{s}_1 = -k_1 s_1^2 \tag{32}$$

From above discussion, we can obtain that:

$$\dot{V} \le -k_1 s_1^2 - k_2 s_2^2 - k_3 s_3^2. \tag{33}$$

So, $\dot{V}$ is always negative, which means that the signals $s_i, i = 1,2,3$ and $\tilde{\vartheta} + \gamma_2\delta$ are bounded. Then from (23), we can easily know $\vartheta, \tilde{\vartheta} \in L_\infty$. Integrating (34) yields:

$$\int_0^\infty \sum_{i=1}^{3} k_i s_i^2 dt \le V(0) - V(\infty) < \infty \tag{34}$$

Which implies that $s_i \in L_2$. Then form (10) and (15), we can easily conclude that $\dot{s}_i \in L_\infty$. At last by using Barbalat's lemma [8], we know $s_i \to 0$, and all the signals in the closed-loop system is bounded, and the states of the system converge to zero as $t \to \infty$. This ends the proof of the theorem.

4. Simulation Results

The parameters of the X-Z inverted pendulum are chosen as in Table 1.

Table 1. The Parameters of the X-Z Inverted Pendulum

M (kg)	m (kg)	l (m)	g (m/s^2)
1	0.1	0.5	9.8

The parameters of the sliding mode control are chosen as $\lambda_1 = \lambda_3 = 2, \lambda_2 = 0.5, k_1 = k_3 = 1, \quad k_2 = 0.2$. In the Simulation, the discontinuous function $sign(\cdot)$ has been replaced by smooth function $\arctan(20\cdot)$.

The fuzzy logic system uses ξ and u_1 as the inputs. For each variable of ξ, we define three Gaussian membership functions uniformly distributed on the interval $[-1,1]$. And with respect to u_1, we use five Gaussian membership functions uniformly distributed on the interval $[-40, 40]$. The Gaussian membership functions of ξ are shown in Figure 2.

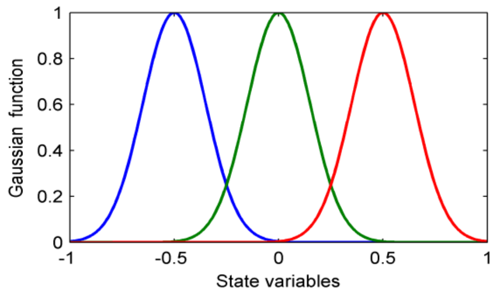

Figure 2. Gaussian Membership Functions of ξ

The initial values of the system are chosen as $x(0) = 0.5, \quad \dot{x}(0) = 0, \quad z(0) = 0.7,$ $\dot{z}(0) = 0.2, \quad \theta(0) = \pi / 4, \quad \dot{\theta}(0) = 0.$ The initial values of the fuzzy system are chosen as $\vartheta(0) = 0$. The system uncertainties in (8) are assumed to be:

$$f(\xi) = 0.1\theta + \sin t. \tag{35}$$

The simulation results are shown in Figure 3-Figure 6. From the simulation results we can conclude that the stabilization of the X-Z inverted system is achieved and the system performance is good.

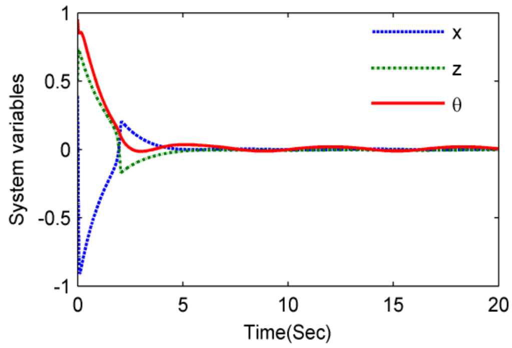

Figure 3. Stabilization of the X-Z Inverted Pendulum

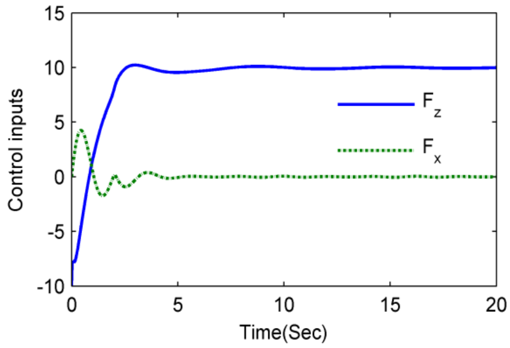

Figure 4. The Control Inputs

5. Conclusion

In this paper, an equivalent transform of the X-Z inverted pendulum is introduced and the adaptive fuzzy sliding mode controller is constructed for X-Z inverted pendulum. The major contributions of our work can be summarized as the following aspects. Firstly, we give an equivalent transform of the inverted pendulum. Then, fuzzy systems are used to approximate the unknown system uncertainties as well as the intermediate control input function. The controller we designed can ensure the stabilization of the system and all the signals in the closed-loop system remain bounded. The simulation results show that good control performance has been achieved.

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