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Best proximity point results for generalization of $\breve{\alpha}$–$\breve{\eta}$ proximal contractive mapping in fuzzy banach spaces

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ABSTRACT

The best proximity point is a generalization of a fixed point that is beneficial when the contraction map is not a self-map. On other hand, best approximation theorems provide an approximate solution to the fixed-point equation $\mathrm{T}x = x$. It is used to solve the problem to determine an approximate solution that is optimum. The main goal of this paper is to present new types of proximal contraction for nonself mappings in a fuzzy Banach space. At first, the notion of the best proximity point is presented. We introduce the notion of $\breve{\alpha}$–$\breve{\eta}$-$\breve{\beta}$ proximal contractive. After that, the best proximity point theorem for such type of mappings in a fuzzy Banach space is proved. In addition, the concept of $\breve{\alpha}$–$\breve{\eta}$-$\breve{\varphi}$ proximal contractive mapping is presented in a fuzzy Banach space and under specific conditions, the best proximity point theorem for such type of mapping is proved. Additionally, some examples are supplied to show the results' applicability.

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1. INTRODUCTION

Numerous problems can be represented by equations of the type $\mathbb{T}x = x$, where $\mathbb{T}$ is a self-mapping defined on a subset of a metric space, a normed linear space, a topological vector space, or another appropriate space. On the other hand, if $\mathbb{T}$ is a nonself-mapping from $\tilde{A}$ to $\tilde{B}$, the aforesaid equation may not accept a solution. In this situation, it is being thought about finding an approximate solution x in $\tilde{A}$ that has the least amount of error $d(x, \mathbb{T}x)$, where d is the distance function. Given that $d(x, \mathbb{T}x)$ is less than $d(\tilde{A}, \tilde{B})$, the best proximity point theorem ensures that $d(x, \mathbb{T}x)$ is minimized globally by requiring that an approximation solution x satisfies the condition $d(x, \mathbb{T}x) = d(\tilde{A}, \tilde{B})$. The best proximity points of the mapping $\mathbb{T}$ are such optimum approximation solutions.

Fan [1] established a fundamental result for the best approximation theorem in 1969, stating that if W represents a Hausdorff locally convex topological vector space and C is a subset of W where C is a nonempty compact convex set and $\mathbb{T}: C \to W$ is a continuous mapping, then there is an element x satisfying the condition $d(x, \mathbb{T}x) = inf\, d(y, \mathbb{T}x): y \in C$, where d represents a metric on W. Following that, several researchers, including Reich [2], Prolla [3], and Sehgal [4] developed the expansions of Fan's theorem in a variety of directions.

On the other hand, Zadeh [5] proposed and investigated the idea of a fuzzy set in his fundamental paper. The study of fuzzy sets led to the fuzzification of a variety of mathematical notions, and it may be used in a variety of fields. Kramosil and Michalek [6] were the first to establish the idea of fuzzy metric spaces. George and Veeramani [7] modified the idea of fuzzy metric spaces. A wide number of works have been published in fuzzy metric spaces; see [8]-[14]. Katsaras [15], [16] was the first to propose the concept of fuzzy norms in linear spaces. Many other mathematicians, such as Felbin [17], Cheng and Mordeson [18], and others, afterward presented the notion of fuzzy normed linear spaces in various ways. A significant number of papers have been published on fuzzy normed linear spaces, for example, see [19]-[23].

In this work, the notion of $\breve{\alpha} - \breve{\eta}$ proximal admissible, $\breve{\alpha}$–$\breve{\eta}$-$\breve{\beta}$ proximal contractive and $\breve{\alpha}$–$\breve{\eta}$-$\breve{\varphi}$ proximal contractive for nonself mappings $\mathbb{T}: \widetilde{A} \to \widetilde{B}$ are introduced and the best proximity point theorem for these kinds of mappings is established in a fuzzy Banach space. Structurally, this paper involves the following: Section 2 is dedicated to reviewing some terms as well as preliminary results that will be utilized in this paper, then in section 3, the definition of $\breve{\alpha}$–$\breve{\eta}$-$\breve{\beta}$ proximal contractive mapping, $\breve{\alpha}$–$\breve{\eta}$-$\breve{\varphi}$ proximal contractive mapping is introduced and the best proximity point theorem for such types of mappings in a fuzzy Banach space is stated and proved. Finally, the paper finished with a conclusion section.

2. PRELIMINARIES

In this section, we introduced the basic notions and results that will be utilized in this paper. At first, the definition of fuzzy normed space and α -admissible mapping is given. Then we state a notion of fuzzy distance in a fuzzy metric space in order to introduce this notion in the setting of fuzzy normed space.

Definition 2.1. [24]: Let L be a vector space over a field R. A fuzzy normed space is a triplet $(L, \widetilde{N}, \circledast)$, where $\circledast$ is a t-norm and $\widetilde{N}$ is a fuzzy set on $L \times R$ that meets the following conditions for all $x, y \in L$:

($\widetilde{N}$1)$\widetilde{N}(x, 0) = 0$,
($\widetilde{N}$2) $\widetilde{N}(x, \tau) = 1, \forall \tau > 0$ if and only if $x = 0$,
($\widetilde{N}$3) $\widetilde{N}(\gamma x, \tau) = \widetilde{N}(x, \tau/|\gamma|), \forall\ (0 \neq)\gamma \in R, \tau \geq 0$
($\widetilde{N}$4) $\widetilde{N}(x, \tau) \circledast \widetilde{N}(y, s) \leq \widetilde{N}(x + y, \tau + s), \forall \tau, s \geq 0$
($\widetilde{N}$5) $\widetilde{N}(x, .)$is left continuous for all $x \in L$, and $\lim_{\tau\to\infty} \widetilde{N}(x, \tau) = 1$.

Definition 2.2. [25]: Let $(L, \widetilde{N}, \circledast)$ be a fuzzy normed space. Then;
(1) a sequence $\{x_n\}$ is termed as a convergent sequence if $\lim_{\tau\to\infty} \widetilde{N}(x_n - x, \tau) = 1$ for each $\tau > 0$ and $x \in L$.
(2) a sequence $\{x_n\}$ is termed as a Cauchy if $\lim_{n\to\infty} \widetilde{N}(x_{n+p} - x_n, \tau) = 1$; for each $\tau > 0$ and $p = 1,2, ...$

Definition 2.3. [25]: Let $(L, \widetilde{N}, \circledast)$ be a fuzzy normed space. Then $(L, \widetilde{N}, \circledast)$ is termed as complete if each Cauchy sequence in L is convergent in L.

On the other hand, the concept of α -admissible mapping was introduced by Samet *et al.* [26] as:

Definition 2.4. [26]: Let L be a nonempty set, $\mathbb{T}: L \to L$, and $\alpha: L \times L \to [0, \infty)$. $\mathbb{T}$ is called α -admissible mapping if for each $x, y \in L$, we have: $\alpha(x, y) \geq 1$ then $\alpha(\mathbb{T}x, \mathbb{T}y) \geq 1$

Next Salimi et al. [27] generalized the notion of α -admissible mappings in the following way**.**

Definition 2.5. [27]: Let L be a nonempty set, $\mathbb{T}: L \to L$, and $\alpha, \eta: L \times L \to [0, \infty)$. Then $\mathbb{T}$ is called α -admissible mapping concerning η if, for each $x, y \in L$, $\alpha(x, y) \geq \eta(x, y)$ then $\alpha(\mathbb{T}x, \mathbb{T}y) \geq \eta(\mathbb{T}x, \mathbb{T}y)$

In [28] Vetro and Salimi introduce the concept of fuzzy distance in fuzzy metric space $(\mathbb{X}, \widetilde{\mathcal{M}}, \circledast)$. Consider $\widetilde{A}$ *and* $\widetilde{B}$ be nonempty subsets of $(\mathbb{X}, \widetilde{\mathcal{M}}, \circledast)$ and $\widetilde{A}_\circ(\tau)$, $\widetilde{B}_\circ(\tau)$ denote the following sets:

$\widetilde{A}_\circ(\tau) = \{x \in \widetilde{A} : \widetilde{\mathcal{M}}(x, y, \tau) = \widetilde{\mathcal{M}}(\widetilde{A}, \widetilde{B}, \tau)$ for some $y \in \widetilde{B}\}$;
$\widetilde{B}_\circ(\tau) = \{y \in \widetilde{B} : \widetilde{\mathcal{M}}(x, y, \tau) = \widetilde{\mathcal{M}}(\widetilde{A}, \widetilde{B}, \tau)$ for some $x \in \widetilde{A}\}$;
where $\widetilde{\mathcal{M}}(\widetilde{A}, \widetilde{B}, \tau) = \sup\{\widetilde{\mathcal{M}}(x, y, \tau): x \in \widetilde{A}, y \in \widetilde{B}\}$.

In this paper, we introduce the above notion in a fuzzy normed space as follows:
Consider $\widetilde{A}$ and $\widetilde{B}$ be nonempty subsets of a fuzzy normed space$(L, \widetilde{N}, \circledast)$. The following sets are indicated by $\widetilde{A}_\circ(\tau)$, $\widetilde{B}_\circ(\tau)$,
$\widetilde{A}_\circ(\tau) = \{x \in \widetilde{A} : \widetilde{N}(x - y, \tau) = N_d(\widetilde{A}, \widetilde{B}, \tau)$ for some $y \in \widetilde{B}\}$;

$\widetilde{B}_\circ(\tau) = \{y \in \widetilde{B} : \widetilde{N}(x - y, \tau) = N_d(\widetilde{A}, \widetilde{B}, \tau)$ for some $x \in \widetilde{A}\}$;
where $N_d(\widetilde{A}, \widetilde{B}, \tau) = sup\{\widetilde{N}(x - y, \tau): x \in \widetilde{A}, y \in \widetilde{B}\}$.

3. MAIN RESULTS

In this section, the concepts of $\breve{\alpha} - \breve{\eta}$ proximal admissible, $\breve{\alpha}$–$\breve{\eta}$-$\breve{\beta}$ proximal contractive and $\breve{\alpha}$–$\breve{\eta}$-$\breve{\varphi}$ proximal contractive mappings are defined, then prove the main results. Saha *et al.* [29] presented the concept of the best proximity point in a fuzzy metric space. In the following, the notion of the best proximity point is introduced in the context of fuzzy normed space.

Definition 3.1: Let $(L, \widetilde{N}, \circledast)$ be a fuzzy Banach space and consider $\widetilde{A}$, $\widetilde{B}$ be nonempty closed subsets of L An element $x^\star \in \widetilde{A}$ is called the best proximity point (BPP) of a mapping $\mathbb{T}: \widetilde{A} \to \widetilde{B}$ if it satisfies the condition that $\widetilde{N}(x^\star - \mathbb{T}x^\star, \tau) = N_d(\widetilde{A}, \widetilde{B}, \tau)$ for all $\tau > 0$.

Definition 3.2: Let $\widetilde{A}$ *and* $\widetilde{B}$ be two nonempty subsets of a fuzzy normed space $(L, \widetilde{N}, \circledast)$. Let $\mathbb{T}: \widetilde{A} \to \widetilde{B}$ be a given non-self mapping. Then $\mathbb{T}$ is called an $\breve{\alpha} - \breve{\eta}$ proximal admissible mapping where $\breve{\alpha}, \breve{\eta}: \widetilde{A} \times \widetilde{A} \times [0, \infty) \to [0, \infty)$ if for each $x, y, u, v \in \widetilde{A}$, and $\tau > 0$,

$$\left.\begin{aligned} \breve{\alpha}(x, y, \tau) &\le \breve{\eta}(x, y, \tau) \\ \widetilde{N}(u - \mathbb{T}x, \tau) &= N_d(\widetilde{A}, \widetilde{B}, \tau) \\ \widetilde{N}(v - \mathbb{T}y, \tau) &= N_d(\widetilde{A}, \widetilde{B}, \tau) \end{aligned}\right\} \Rightarrow \breve{\alpha}(u, v, \tau) \le \breve{\eta}(u, v, \tau) \tag{1}$$

Definition 3.3: Let $\widetilde{A}$ *and* $\widetilde{B}$ be two nonempty subsets of a fuzzy normed space $(L, \widetilde{N}, \circledast)$. Let $\mathbb{T}: \widetilde{A} \to \widetilde{B}$ be a given non-self mapping and $\breve{\alpha}, \breve{\eta}: \widetilde{A} \times \widetilde{A} \times [0, \infty) \to [0, \infty)$ be two functions. $\mathbb{T}$ is called a $\breve{\alpha}$–$\breve{\eta}$-$\breve{\beta}$ proximal contractive mapping if there exists a function $\breve{\beta}: [0,1] \to [1, \infty)$ such that, for any sequence $\{t_n\} \subset [0,1]$, $\breve{\beta}(t_n) \to 1$ implies $t_n \to 1$, for each $x, y, u, v \in \widetilde{A}$, and $\tau > 0$

$$\left.\begin{aligned} \breve{\alpha}(x, \mathbb{T}x, \tau)\breve{\alpha}(y, \mathbb{T}y, \tau) &\le \breve{\eta}(x, \mathbb{T}x, \tau)\breve{\eta}(y, \mathbb{T}y, \tau) \\ \widetilde{N}(u - \mathbb{T}x, \tau) &= N_d(\widetilde{A}, \widetilde{B}, \tau) \\ \widetilde{N}(v - \mathbb{T}y, \tau) &= N_d(\widetilde{A}, \widetilde{B}, \tau) \end{aligned}\right\} \Rightarrow$$
$$\widetilde{N}(u - v, \tau) \ge \breve{\beta}(\widetilde{N}(x - y, \tau))\, \mathcal{L}(x, y, u, v, \tau) \tag{2}$$

where $\mathcal{L}(x, y, u, v, \tau) = \min\{\widetilde{N}(x - y, \tau), \max\{\widetilde{N}(x - u, \tau), \widetilde{N}(y - v, \tau)\}\}$

Theorem 3.4: Suppose that $(L, \widetilde{N}, \circledast)$ be a fuzzy Banach space and let $\widetilde{A}$ and $\widetilde{B}$ nonempty closed subsets of L where $\widetilde{A}_\circ(\tau)$ is nonempty for each $\tau > 0$. Consider $\mathbb{T}: \widetilde{A} \to \widetilde{B}$ is $\breve{\alpha}$–$\breve{\eta}$-$\breve{\beta}$ proximal contractive mapping satisfies the conditions:

(a) $\mathbb{T}$ is $\breve{\alpha}$–$\breve{\eta}$ proximal admissible mapping and $\mathbb{T}(\widetilde{A}_\circ(\tau)) \subseteq \widetilde{B}_\circ(\tau)$;
(b)There exist elements $x_\circ$ and x_1 in $\widetilde{A}_\circ(\tau)$ such that
$\widetilde{N}(x_1 - \mathbb{T}x_\circ, \tau) = N_d(\widetilde{A}, \widetilde{B}, \tau)$; $\breve{\alpha}(x_\circ, x_1, \tau) \le \breve{\eta}(x_\circ, x_1, \tau)$ for each $\tau > 0$;

(c) If $\{y_n\}$ is a sequence in $\widetilde{B}_\circ(\tau)$ and $x \in \widetilde{A}$ is such that $\widetilde{N}(x - y_n, \tau) = N_d(\widetilde{A}, \widetilde{B}, \tau)$ as $n \to \infty$, then $x \in \widetilde{A}_\circ(\tau)$ for each $\tau > 0$.

(d) If $\{x_n\}$ is a sequence in L such that $\tilde{\alpha}(x_n, x_{n+1}, \tau) \le \tilde{\eta}(x_n, x_{n+1}, \tau)$ for each $n \ge 1$ and $x_n \to x$ as $n \to \infty$, then $\tilde{\alpha}(x_n, x, \tau) \le \tilde{\eta}(x_n, x, \tau), \forall n \ge 1$ and $\tau > 0$.
Then $\mathbb{T}$ has BPP.

Proof: According to condition (b), there are elements**,** say $x_\circ$, x_1 in $\widetilde{A}_\circ(\tau)$ such that
$\widetilde{N}(x_1 - \mathbb{T}x_\circ, \tau) = N_d(\widetilde{A}, \widetilde{B}, \tau)$; $\tilde{\alpha}(x_\circ, x_1, \tau) \le \tilde{\eta}(x_\circ, x_1, \tau)$ for each $\tau > 0$
On the other hand, since $\mathbb{T}(\widetilde{A}_\circ(\tau)) \subseteq \widetilde{B}_\circ(\tau)$, there exists $x_2 \in \widetilde{A}_\circ(\tau)$ such that

$$\widetilde{N}(x_2 - \mathbb{T}x_1, \tau) = N_d(\widetilde{A}, \widetilde{B}, \tau)$$

Now, since $\mathbb{T}$ is an $\tilde{\alpha}$- $\tilde{\eta}$ proximal admissible mapping, then $\tilde{\alpha}(x_1, x_2, \tau) \le \tilde{\eta}(x_1, x_2, \tau)$, Again, since $\mathbb{T}(\widetilde{A}_\circ(\tau)) \subseteq \widetilde{B}_\circ(\tau)$, there exists $x_3 \in \widetilde{A}_\circ(\tau)$ such that:

$$\widetilde{N}(x_3 - \mathbb{T}x_2, \tau) = N_d(\widetilde{A}, \widetilde{B}, \tau)$$

Thus,

$$\widetilde{N}(x_2 - \mathbb{T}x_1, \tau) = N_d(\widetilde{A}, \widetilde{B}, \tau);$$
$$\widetilde{N}(x_3 - \mathbb{T}x_2, \tau) = N_d(\widetilde{A}, \widetilde{B}, \tau)$$
$$\tilde{\alpha}(x_1, x_2, \tau) \leq \tilde{\eta}(x_1, x_2, \tau)$$

Again, since $\mathbb{T}$ is an $\tilde{\alpha}$–$\tilde{\eta}$ proximal admissible mapping, then $\tilde{\alpha}(x_2, x_3, \tau) \leq \tilde{\eta}(x_2, x_3, \tau)$, hence it follows that:

$$\widetilde{N}(x_3 - \mathbb{T}x_2, \tau) = N_d(\widetilde{A}, \widetilde{B}, \tau);\ \tilde{\alpha}(x_2, x_3, \tau) \leq \tilde{\eta}(x_2, x_3, \tau)$$

If we keep going this way, we will obtain:

$$\widetilde{N}(x_{n+1} - \mathbb{T}x_n, \tau) = N_d(\widetilde{A}, \widetilde{B}, \tau);\ \tilde{\alpha}(x_n, x_m, \tau)) \leq \tilde{\eta}(x_n, x_m, \tau) \tag{3}$$

for each $n, m \geq 1$, and any $n \geq 0$.
Now using (3) and applying the inequality (2) with $u = y = x_n$, $v = x_{n+1}$ and $x = x_{n-1}$ obtain:

$$\widetilde{N}(x_n - x_{n+1}, \tau) \geq \breve{\beta}\left(\widetilde{N}(x_{n-1} - x_n, \tau)\right)\mathcal{L}(x_{n-1}, x_n, x_n, x_{n+1}, \tau) \tag{4}$$

where
$\mathcal{L}(x_{n-1}, x_n, x_n, x_{n+1}, \tau) = \min\{\widetilde{N}(x_{n-1} - x_n, \tau), \max\{\widetilde{N}(x_{n-1} - x_n, \tau), \widetilde{N}(x_n - x_{n+1}, \tau)\}\}$
for each $n \in N$ and $\tau > 0$.

If we have $\widetilde{N}(x_{n-1} - x_n, \tau) \leq \widetilde{N}(x_n - x_{n+1}, \tau)$ for some $n \in$ N, then obtain:

$$\min\{\widetilde{N}(x_{n-1} - x_n, \tau), \max\{\widetilde{N}(x_{n-1} - x_n, \tau), \widetilde{N}(x_n - x_{n+1}, \tau)\} = \widetilde{N}(x_{n-1} - x_n, \tau)$$

Also if $\widetilde{N}(x_n - x_{n+1}, \tau) < \widetilde{N}(x_{n-1} - x_n, \tau)$ for some $n \in N$, then:

$$\min\{\widetilde{N}(x_{n-1} - x_n, \tau), \max\{\widetilde{N}(x_{n-1} - x_n, \tau), \widetilde{N}(x_n - x_{n+1}, \tau)\} = \widetilde{N}(x_{n-1} - x_n, \tau)$$

That is, for each $n \in N$ and $\tau > 0$,

$$\min\{\widetilde{N}(x_{n-1} - x_n, \tau), \max\{\widetilde{N}(x_{n-1} - x_n, \tau), \widetilde{N}(x_n - x_{n+1}, \tau)\} = \widetilde{N}(x_{n-1} - x_n, \tau)$$

Hence,
$$\widetilde{N}(x_n - x_{n+1}, \tau) \geq \breve{\beta}\left(\widetilde{N}(x_{n-1} - x_n, \tau)\right)\widetilde{N}(x_{n-1} - x_n, \tau) \geq \widetilde{N}(x_{n-1} - x_n, \tau) \ \ldots(5)$$

and hence $\{\widetilde{N}(x_n - x_{n+1}, \tau)\}$ in (0,1] is an increasing sequence, consequently, there is $\gamma(\tau) \in (0,1]$ such that $\lim_{n\to\infty} \widetilde{N}(x_n - x_{n+1}, \tau) = \gamma(\tau)$ for each $\tau > 0$. Now to show that $\gamma(\tau) = 1$ for each $\tau > 0$. Assume that there is $\tau_\circ > 0$ such that $0 < \gamma(\tau_\circ) < 1$.

From (5),

$$\frac{\widetilde{N}(x_n - x_{n+1}, \tau)}{\widetilde{N}(x_{n-1} - x_n, \tau)} \geq \breve{\beta}\left(\widetilde{N}(x_{n-1} - x_n, \tau)\right) \geq 1$$

which implies that $\lim_{n\to\infty} \breve{\beta}\left(\widetilde{N}(x_{n-1} - x_n, \tau)\right) = 1$. In terms of $\breve{\beta}$'s property which indicates that $\gamma = 1$, we deduce:

$$\lim_{n\to\infty} \widetilde{N}(x_n - x_{n+1}, \tau) = 1 \tag{6}$$

Following that, we show that $\{x_n\}$ is a Cauchy sequence. Assume that $\{x_n\}$ is not Cauchy. Then there is $\mathfrak{z} \in (0,1)$ such that for each $\kappa \geq 1$, there are $\mathfrak{m}(\kappa), \mathfrak{n}(\kappa) \in N$ with $\mathfrak{m}(\kappa) > \mathfrak{n}(\kappa) \geq \kappa$ and

$$\widetilde{N}(x_{m(\kappa)} - x_{n(\kappa)}, \tau_\circ) \leq 1 - \mathfrak{z}\ , \tau_\circ > 0$$

Assume that $\mathfrak{m}(\kappa)$ is the smallest integer greater than $\mathfrak{n}(\kappa)$, meeting the condition above:

$$\widetilde{N}(x_{m(\kappa)-1} - x_{n(\kappa)}, \tau_\circ) > 1 - \mathfrak{z}$$

and for each κ,

$$\begin{aligned} 1 - \mathfrak{z} &\geq \widetilde{N}(x_{m(\kappa)} - x_{n(\kappa)}, \tau_\circ) \\ &\geq \widetilde{N}(x_{m(\kappa)} - x_{m(\kappa)-1}, \tau_\circ) \circledast \widetilde{N}(x_{m(\kappa)-1} - x_{n(\kappa)}, \tau_\circ) \\ &> \widetilde{N}(x_{m(\kappa)} - x_{m(\kappa)-1}, \tau_\circ) \circledast 1 - \mathfrak{z} \end{aligned}$$

In the previous inequality, if use limit as $\kappa \to \infty$ and using (6), obtain:

$$\lim_{n\to\infty} \widetilde{N}(x_{m(\kappa)} - x_{n(\kappa)}, \tau_\circ) = 1 - \mathfrak{z} \tag{7}$$

Now from,

$$\widetilde{N}(x_{m(\kappa)+1} - x_{n(\kappa)+1}, \tau_\circ) \geq \widetilde{N}(x_{m(\kappa)+1} - x_{m(\kappa)}, \tau_\circ) \circledast \widetilde{N}(x_{m(\kappa)} - x_{n(\kappa)}, \tau_\circ) \circledast \widetilde{N}(x_{n(\kappa)} - x_{n(\kappa)+1}, \tau_\circ)$$

and

$$\widetilde{N}(x_{m(\kappa)} - x_{n(\kappa)}, \tau_\circ) \geq \widetilde{N}(x_{m(\kappa)} - x_{m(\kappa)+1}, \tau_\circ) \circledast \widetilde{N}(x_{m(\kappa)+1} - x_{n(\kappa)+1}, \tau_\circ) \circledast \widetilde{N}(x_{n(\kappa)+1} - x_{n(\kappa)}, \tau_\circ)$$

it follows that:

$$\lim_{n\to\infty} \widetilde{N}(x_{m(\kappa)+1} - x_{n(\kappa)+1}, \tau_\circ) = 1 - \mathfrak{z} \tag{8}$$

From (3),

$$\begin{cases} \alpha(x_{n(\kappa)}, x_{m(\kappa)}, \tau_\circ) \leq \tilde{\eta}(x_{n(\kappa)}, x_{m(\kappa)}, \tau_\circ) \\ \widetilde{N}(x_{m(\kappa)+1} - \mathbb{T}x_{m(\kappa)}, \tau_\circ) = N_d(\widetilde{A}, \widetilde{B}, \tau_\circ) \\ \widetilde{N}(x_{n(\kappa)+1} - \mathbb{T}x_{n(\kappa)}, \tau_\circ) = N_d(\widetilde{A}, \widetilde{B}, \tau_\circ) \end{cases} \tag{9}$$

Hence, by (2) and (9).

$$\widetilde{N}(x_{m(\kappa)+1} - x_{n(\kappa)+1}, \tau_\circ) \geq \breve{\beta}(\widetilde{N}(x_{m(\kappa)} - x_{n(\kappa)}, \tau_\circ))\mathcal{L}(x_{m(\kappa)}, x_{n(\kappa)}, x_{m(\kappa)+1}, x_{n(\kappa)+1}, \tau_\circ)$$

Where,

$$\mathcal{L}(x_{m(\kappa)}, x_{n(\kappa)}, x_{m(\kappa)+1}, x_{n(\kappa)+1}, \tau_\circ) = \min\{\widetilde{N}(x_{m(\kappa)} - x_{n(\kappa)}, \tau_\circ), \max\{\widetilde{N}(x_{m(\kappa)} - x_{m(\kappa)+1}, \tau_\circ), \widetilde{N}(x_{n(\kappa)} - x_{n(\kappa)+1}, \tau_\circ)\}\}$$

Hence,

$$\frac{\widetilde{N}(x_{m(\kappa)+1} - x_{n(\kappa)+1}, \tau_\circ)}{\mathcal{L}(x_{m(\kappa)}, x_{n(\kappa)}, x_{m(\kappa)+1}, x_{n(\kappa)+1}, \tau_\circ)} \geq \breve{\beta}(\widetilde{N}(x_{m(\kappa)} - x_{n(\kappa)}, \tau_\circ)) \geq 1$$

passing to limit as $\kappa \to \infty$ in the above inequality;

$$\lim_{k\to\infty} \breve{\beta}(\widetilde{N}(x_{m(\kappa)} - x_{n(\kappa)}, \tau_\circ)) = 1$$

It follows that:

$$1 - \mathfrak{z} = \lim_{k\to\infty} \widetilde{N}\big(x_{m(\kappa)} - x_{n(\kappa)}, \tau_\circ\big) = 1$$

and so $\mathfrak{z} = 0$, but this is a contradiction, thus $\{x_n\}$ is a Cauchy sequence. Since $(L, \widetilde{N}, \circledast)$ is complete then $\{x_n\}$ converges to some $x^* \in L$,

$$\lim_{n\to\infty} \widetilde{N}(x_n - x^\star, \tau) = 1 \text{ for each } \tau > 0.$$

Furthermore,

$$\begin{aligned} N_d(\widetilde{A}, \widetilde{B}, \tau) &= \widetilde{N}(x_{n+1} - \mathbb{T}x_n, \tau) \\ &\geq \widetilde{N}(x_{n+1} - x^\star, \tau) \circledast \widetilde{N}(x^\star - \mathbb{T}x_n, \tau) \\ &\geq \widetilde{N}(x_{n+1} - x^\star, \tau) \circledast \widetilde{N}(x^\star - x_{n+1}, \tau) \circledast \widetilde{N}(x_{n+1} - \mathbb{T}x_n, \tau) \\ &= \widetilde{N}(x_{n+1} - x^\star, \tau) \circledast \widetilde{N}(x^\star - x_{n+1}, \tau) \circledast N_d(\widetilde{A}, \widetilde{B}, \tau) \end{aligned}$$

which implies

$$N_d(\widetilde{A}, \widetilde{B}, \tau) \geq \widetilde{N}(x_{n+1} - x^\star, \tau) \circledast \widetilde{N}(x^\star - \mathbb{T}x_n, \tau)$$
$$\geq \widetilde{N}(x_{n+1} - x^\star, \tau) \circledast \widetilde{N}(x^\star - x_{n+1}, \tau) \circledast N_d(\widetilde{A}, \widetilde{B}, \tau)$$

In the previous inequality, if use limit as $n \to \infty$, obtain:

$$N_d(\widetilde{A}, \widetilde{B}, \tau) \geq 1 \circledast \widetilde{N}(x^\star - \mathbb{T}x_n, \tau)$$
$$\geq 1 \circledast 1 \circledast N_d(\widetilde{A}, \widetilde{B}, \tau)$$

that is,

$$\lim_{n\to\infty} \widetilde{N}(x^\star - \mathbb{T}x_n, \tau) = N_d(\widetilde{A}, \widetilde{B}, \tau)$$

and so, by condition (c), $x^\star \in \widetilde{A}_\circ(\tau)$. Since $\mathbb{T}(\widetilde{A}_\circ(\tau)) \subseteq \widetilde{B}_\circ(\tau)$, there exists $z \in \widetilde{A}_\circ(\tau)$ such that $\widetilde{N}(z - \mathbb{T}x^\star, \tau) = N_d(\widetilde{A}, \widetilde{B}, \tau)$. Consequently, it follows from condition (d) and inequality (2) with $u = x_{n+1}, v = z, x = x_n$ and $y = x^\star$ that

$$\widetilde{N}(x_{n+1} - z, \tau) \geq \check{\beta}(\widetilde{N}(x_n - x^\star, \tau))\, \mathcal{L}(x_n, x^\star, x_{n+1}, z, \tau)$$

where $\mathcal{L}(x_n, x^\star, x_{n+1}, z, \tau) = \min\{\widetilde{N}(x_n - x^\star, \tau), \max\{\widetilde{N}(x_n - x_{n+1}, \tau), \widetilde{N}(x^\star - z, \tau)\}\}$

In the previous inequality, if we use limit as $n \to \infty$, obtain:

$$\lim_{n\to\infty} \mathcal{L}(x_n, x^\star, x_{n+1}, z, \tau) = 1$$

Now,

$$\widetilde{N}(x^\star - z, \tau) \geq \widetilde{N}(x^\star - x_n, \tau) * \widetilde{N}(x_n - x_{n+1}, \tau) * \widetilde{N}(x_{n+1} - z, \tau)$$
$$\geq \widetilde{N}(x^\star - x_n, \tau) * \widetilde{N}(x_n - x_{n+1}, \tau) * \check{\beta}(\widetilde{N}(x_n - x^\star, \tau))\, \mathcal{L}(x_n, x^\star, x_{n+1}, z, \tau)$$

Letting $n \to \infty$ in the previous inequality, get:

$N(x^* - z, \tau) = 1$, that is $x^\star = z$ and $\widetilde{N}(x^\star - \mathbb{T}x^\star, \tau) = \widetilde{N}(z - \mathbb{T}x^\star, \tau) = N_d(\widetilde{A}, \widetilde{B}, \tau)$. Thus $x^\star$ is BPP of $\mathbb{T}$.

Example 3.5: Let $L = \mathbb{R} \times \mathbb{R}$ with the fuzzy norm, $\widetilde{N}: L \times \mathbb{R} \to [0,1]$ defined by $\widetilde{N}(x, \tau) = (\frac{\tau}{\tau+1})^{\|x\|}$ for each $x \in L$ and $\tau > 0$, where $\|x\|: \mathbb{R} \to [0, \infty)$ is the standard norm.

$$\|x - y\| = |x_1 - x_2| + |y_1 - y_2|$$

for each $x = (x_1, x_2), y = (y_1, y_2) \in L$.

Let $\widetilde{A} = \{(0,x): x \in \mathbb{R}\}$ and $\widetilde{B} = \{(1,x): x \in \mathbb{R}\}$

So that $N_d(\widetilde{A},\widetilde{B},\tau) = sup\{\widetilde{N}(x - y,\tau): x \in \widetilde{A}, y \in \widetilde{B}\} = \frac{\tau}{\tau+1}$
Also, define $\mathbb{T}: \widetilde{A} \to \widetilde{B}$ by:

$$\mathbb{T}(x_1,x_2) = \begin{cases} (1,2\pi) \text{ if } (x_1,x_2) \in \widetilde{A} \setminus W \\ \left(1,\frac{1}{m}\right) \text{ if } (x_1,x_2) = \left(0,\frac{1}{m}\right) \text{ for all } m \geq 1 \\ (1,0) \text{ if } (x_1,x_2) = (0,0) \end{cases}$$

where,

$$W = \left\{\left(0,\frac{1}{m}\right): m \geq 1\right\} \cup \{(0,0)\}$$

Notice that $\widetilde{A}_\circ(\tau) = \widetilde{A}$ and $\widetilde{B}_\circ(\tau) = \widetilde{B}$, $T(\widetilde{A}_\circ(\tau)) \subseteq \widetilde{B}_\circ(\tau)$.
Also, define $\breve{\alpha},\breve{\eta}: \widetilde{A} \times \widetilde{A} \times (0,\infty) \to [0,+\infty)$ by:

$$\breve{\alpha}((0,x),(0,y),\tau) = \begin{cases} 1 \text{ if } (0,x),(0,y) \in W \\ \frac{1}{2} \text{ otherwise} \end{cases}$$

and

$$\breve{\eta}\big((0,x),(0,y),\tau\big) = \begin{cases} 2 \text{ if if } (0,x),(0,y) \in W \\ -2 \text{ otherwise} \end{cases}$$

Also, assume that:

$$\begin{cases} \breve{\alpha}(x,y,\tau) \leq \breve{\eta}(x,y,\tau) \\ \widetilde{N}(u - \mathbb{T}x,\tau) = N_d(\widetilde{A},\widetilde{B},\tau) \\ \widetilde{N}(v - \mathbb{T}y,\tau) = N_d(\widetilde{A},\widetilde{B},\tau) \end{cases}$$

then,

$$\begin{cases} (x,y) \in W \\ \widetilde{N}(u - \mathbb{T}x,\tau) = N_d(\widetilde{A},\widetilde{B},\tau) \\ \widetilde{N}(v - \mathbb{T}y,\tau) = N_d(\widetilde{A},\widetilde{B},\tau) \end{cases}$$

then,

$$(u,x),(v,y) \in \left\{\big((0,0),(0,0)\big),\left(\left(0,\frac{1}{2m}\right),\left(0,\frac{1}{m}\right)\right)\right\},$$

We conclude $\breve{\alpha}(u,v,\tau) \leq \breve{\eta}(u,v,\tau)$ that is means $\mathbb{T}$ is an $\breve{\alpha} - \breve{\eta}$ proximal admissible mapping.

Also, assume that $\breve{\alpha}(x,\mathbb{T}x,\tau) \leq \breve{\eta}(x,\mathbb{T}x,\tau)$ and $(y,\mathbb{T}y,\tau) \leq \breve{\eta}(y,\mathbb{T}y,\tau)$, get:

$$\breve{\alpha}(x,\mathbb{T}x,\tau)\breve{\alpha}(y,\mathbb{T}y,\tau) \leq \breve{\eta}(x,\mathbb{T}x,\tau)\breve{\eta}(y,\mathbb{T}y,\tau)$$

Now we define $\breve{\beta}: [0,1] \to [0,1]$ by $\breve{\beta}(s) = 1$ for each $s \in [0,1]$ and differentiate between the following cases:
Case 1: If $(u,x) = \left(\left(0,\frac{1}{2n}\right),\left(0,\frac{1}{n}\right)\right)$ and $(v,y) = \left(\left(0,\frac{1}{2m}\right),\left(0,\frac{1}{m}\right)\right)$ $\forall n, m \geq 1$
Then,

$$\widetilde{N}(u - v,\tau) = \frac{\tau}{\tau+\|u-v\|}$$
$$= \frac{\tau}{\tau+\left|\frac{1}{2n}-\frac{1}{2m}\right|}$$

$$\geq \breve{\beta}\left(\frac{\tau}{\tau+\left|\frac{1}{n}-\frac{1}{m}\right|}\right)\left(\frac{\tau}{\tau+\left|\frac{1}{n}-\frac{1}{m}\right|}\right)=\breve{\beta}(\widetilde{N}(x-y,\tau))\left(\widetilde{N}(x-y,\tau)\right)$$

Case 2: If $(u,x)=\left((0,0),(0,0)\right)$ and $(v,y)=\left(\left(0,\frac{1}{2m}\right),\left(0,\frac{1}{m}\right)\right)$ for each $m\geq 1$
Then,

$$\widetilde{N}(u-v,\tau)=\frac{\tau}{\tau+\|u-v\|}$$
$$=\frac{\tau}{\tau+\left|\frac{1}{2m}\right|}$$
$$\geq \breve{\beta}\left(\frac{\tau}{\tau+\left|\frac{1}{m}\right|}\right)\left(\frac{\tau}{\tau+\left|\frac{1}{m}\right|}\right)=\breve{\beta}(\widetilde{N}(x-y,\tau))\left(\widetilde{N}(x-y,\tau)\right)$$

Case 3: If $(u,x)=(v,y)=\left((0,0),(0,0)\right)$. Then,

$$\widetilde{N}(u-v,\tau)=\frac{\tau}{\tau+\|u-v\|}=\frac{\tau}{\tau}$$
$$=1$$
$$\geq \breve{\beta}(1).1=\breve{\beta}(\widetilde{N}(x-y,\tau))\left(\widetilde{N}(x-y,\tau)\right)$$

Thus, all hypotheses of Theorem 3.4 is fulfilled. As a result, $\mathbb{T}$ has a unique BPP. In this example $x^{\star}=(0,0)$ is BPP. In the following, the definition of $\breve{\alpha}$–$\breve{\eta}$-$\breve{\varphi}$ proximal contractive for mappings $\mathbb{T}:\widetilde{A}\rightarrow\widetilde{B}$ is presented and the best proximity point theorem for this type of mapping is introduced. Let $\breve{\Phi}$ be the class of all mappings $\breve{\varphi}:[0,1]\rightarrow[0,1]$ such that $\breve{\varphi}$ is continuous, nondecreasing and $\breve{\varphi}(s)>s$ for each $s\in[0,1]$.

Definition 3.6: Let $(L,\widetilde{N},\circledast)$ be a fuzzy normed space and let $\widetilde{A}$, $\widetilde{B}$ be two nonempty subsets of L**.** Assume that $\mathbb{T}:\widetilde{A}\rightarrow\widetilde{B}$ be a given non-self mapping and $\breve{\alpha},\breve{\eta}:\widetilde{A}\times\widetilde{A}\times[0,\infty)\rightarrow[0,\infty)$ be two functions. $\mathbb{T}$ is called $\breve{\alpha}$–$\breve{\eta}$-$\breve{\varphi}$ proximal contractive mapping if for each $x,y,u,v\in\widetilde{A}$, and $\tau>0$**,**

$$\left.\begin{array}{r}\breve{\alpha}(x,\mathbb{T}x,\tau)\breve{\alpha}(y,\mathbb{T}y,\tau)\leq\breve{\eta}(x,\mathbb{T}x,\tau)\breve{\eta}(y,\mathbb{T}y,\tau)\\ \widetilde{N}(u-\mathbb{T}x,\tau)=N_d(\widetilde{A},\widetilde{B},\tau)\\ \widetilde{N}(v-\mathbb{T}y,\tau)=N_d(\widetilde{A},\widetilde{B},\tau)\end{array}\right\}\Rightarrow\widetilde{N}(u-v,\tau)\geq\breve{\varphi}(\mathcal{L}(x,y,u,v,\tau)\qquad(10)$$

where $\mathcal{L}(x,y,u,v,\tau)=\min\{\widetilde{N}(x-y,\tau),\max\{\widetilde{N}(x-u,\tau),\widetilde{N}(y-v,\tau)\}\}$
Next, the best proximate point theorem for $\breve{\alpha}$–$\breve{\eta}$-$\breve{\varphi}$ proximal contractive mapping will be proved.

Theorem 3.7: Suppose that $(L,\widetilde{N},\circledast)$ be a fuzzy Banach space and $\widetilde{A}$, $\widetilde{B}$ be nonempty closed subsets of L where $\widetilde{A}_{\circ}(\tau)$ is nonempty for each $\tau>0$. Consider $\mathbb{T}:\widetilde{A}\rightarrow\widetilde{B}$ be $\breve{\alpha}$–$\breve{\eta}$-$\breve{\varphi}$ proximal contractive mapping meeting the conditions:
a. $\mathbb{T}$ is $\breve{\alpha}$–$\breve{\eta}$ proximal admissible mapping and $\mathbb{T}(\widetilde{A}_{\circ}(\tau))\subseteq\widetilde{B}_{\circ}(\tau)$ for each $\tau>0$;
b. (b)There exist elements $x_{\circ}$ and x_1 in $\widetilde{A}_{\circ}(\tau)$ such that
c. $\widetilde{N}(x_1-\mathbb{T}x_{\circ},\tau)=N_d(\widetilde{A},\widetilde{B},\tau)$; $\breve{\alpha}(x_{\circ},x_1,\tau)\leq\breve{\eta}(x_{\circ},x_1,\tau)$ for each $\tau>0$;
d. If $\{y_n\}$ is a sequence in $\widetilde{B}_{\circ}(\tau)$ and $x\in\widetilde{A}$ such that $\widetilde{N}(x-y_n,\tau)=N_d(\widetilde{A},\widetilde{B},\tau)$ as $n\rightarrow\infty$, then $x\in\widetilde{A}_{\circ}(\tau)$ for each $\tau>0$.
e. If $\{x_n\}$ is a sequence in L such that $\tilde{\alpha}(x_n,x_{n+1},\tau)\leq\tilde{\eta}(x_n,x_{n+1},\tau)$ for each $n\geq 1$ and $x_n\rightarrow x$ as $n\rightarrow\infty$, then $\tilde{\alpha}(x_n,x,\tau)\leq\tilde{\eta}(x_n,x,\tau)$, $\forall n\geq 1$ and $\tau>0$.

Then $\mathbb{T}$ has BPP.

Proof: By using a similar approach as in proving Theorem 3.4, we may construct a sequence $\{x_n\}$ in $\widetilde{A}_{\circ}(\tau)$ such that:

$$\widetilde{N}(x_{n+1}-\mathbb{T}x_n,\tau)=N_d(\widetilde{A},\widetilde{B},\tau);\ \tilde{\alpha}(x_n,x_m,\tau))\leq\tilde{\eta}(x_n,x_m,\tau)\qquad(11)$$

for each $n,m\geq 1$, and any $n\geq 0$.
Now using (11) and applying the inequality (10) with $u=y=x_n$, $v=x_{n+1}$ and $x=x_{n-1}$ obtain:

$$\widetilde{N}(x_n-x_{n+1},\tau)\geq\breve{\varphi}\left(\mathcal{L}(x_{n-1},x_n,x_n,x_{n+1},\tau)\right)\qquad(12)$$

where,

$\mathcal{L}(x_{n-1}, x_n, x_n, x_{n+1}, \tau) = \min\{\widetilde{N}(x_{n-1} - x_n, \tau), \max\{\widetilde{N}(x_{n-1} - x_n, \tau), \widetilde{N}(x_n - x_{n+1}, \tau)\}\}$
$=\widetilde{N}(x_{n-1} - x_n, \tau)$

Hence,

$$\widetilde{N}(x_n - x_{n+1}, \tau) \geq \breve{\varphi}\left(\widetilde{N}(x_{n-1} - x_n, \tau)\right) > \widetilde{N}(x_{n-1} - x_n, \tau) \quad (13)$$

and hence $\{\widetilde{N}(x_n - x_{n+1}, \tau)\}$ is an increasing sequence in (0,1]. Consequently, there is $\gamma(\tau) \in (0,1]$ such that $\lim_{n\to\infty} \widetilde{N}(x_n - x_{n+1}, \tau) = \ell(\tau)$ for each $\tau > 0$. We shall prove that $\gamma(\tau) = 1$ for each $\tau > 0$.

From (12),

$$\widetilde{N}(x_n - x_{n+1}, \tau) \geq \breve{\varphi}\left(\widetilde{N}(x_{n-1} - x_n, \tau)\right)$$

Since $\breve{\varphi}$ is continuous, $\gamma \geq \breve{\varphi}(\gamma)$. This implies that $\gamma = 1$ and therefore:

$$\lim_{n\to\infty} \widetilde{N}(x_n - x_{n+1}, \tau) = 1 \quad (14)$$

Following that, we prove that $\{x_n\}$ is Cauchy sequence. Suppose that this is not true and proceed as in Theorem 3.4's proof there exists $\mathfrak{z} \in (0,1)$ and $\tau_\circ > 0$ such that, $\forall \kappa \geq 1$, $\exists \mathfrak{m}(\kappa)$; $\mathfrak{n}(\kappa) \in N$ with $\mathfrak{m}(\kappa) > \mathfrak{n}(\kappa) \geq \kappa$ such that:

$$\lim_{n\to\infty} \widetilde{N}\left(x_{\mathfrak{m}(\kappa)} - x_{\mathfrak{n}(\kappa)}, \tau_\circ\right) = 1 - \mathfrak{z}$$

and

$$\lim_{n\to\infty} \widetilde{N}\left(x_{\mathfrak{m}(\kappa)+1} - x_{\mathfrak{n}(\kappa)+1}, \tau_\circ\right) = 1 - \mathfrak{z}$$

Applying (10) and (9), obtain:

$$\widetilde{N}\left(x_{\mathfrak{m}(\kappa)+1} - x_{\mathfrak{n}(\kappa)+1}, \tau_\circ\right) \geq \breve{\varphi}\left(\mathcal{L}\left(x_{\mathfrak{m}(\kappa)}, x_{\mathfrak{n}(\kappa)}, x_{\mathfrak{m}(\kappa)+1}, x_{\mathfrak{n}(\kappa)+1}, \tau_\circ\right)\right)$$

where,

$\mathcal{L}\left(x_{\mathfrak{m}(\kappa)}, x_{\mathfrak{n}(\kappa)}, x_{\mathfrak{m}(\kappa)+1}, x_{\mathfrak{n}(\kappa)+1}, \tau_\circ\right) = \min\{\widetilde{N}\left(x_{\mathfrak{m}(\kappa)} - x_{\mathfrak{n}(\kappa)}, \tau_\circ\right), \max\{\widetilde{N}\left(x_{\mathfrak{m}(\kappa)} - x_{\mathfrak{m}(\kappa)+1}, \tau_\circ\right), \widetilde{N}\left(x_{\mathfrak{n}(\kappa)} - x_{\mathfrak{n}(\kappa)+1}, \tau_\circ\right)\}\}$

Taking the limit as $\kappa \to \infty$ in the inequality above, get:

$$1 - \mathfrak{z} \geq \varphi(1 - \mathfrak{z}) > 1 - \mathfrak{z}$$

but this is a contradiction, hence $\{x_n\}$ is a Cauchy sequence. Since $(L, \widetilde{N}, \circledast)$ is complete, therefore the sequence $\{x_n\}$ converges to some $x^\star \in L$,

$$\lim_{n\to\infty} \widetilde{N}(x_n - x^\star, \tau) = 1 \text{ for each } \tau > 0.$$

In addition,
$N_d(\widetilde{A}, \widetilde{B}, \tau) = \widetilde{N}(x_{n+1} - \mathbb{T}x_n, \tau)$
$\geq \widetilde{N}(x_{n+1} - x^\star, \tau) \circledast \widetilde{N}(x^\star - \mathbb{T}x_n, \tau)$
$\geq \widetilde{N}(x_{n+1} - x^\star, \tau) \circledast \widetilde{N}(x^\star - x_{n+1}, \tau) \circledast \widetilde{N}(x_{n+1} - \mathbb{T}x_n, \tau)$
$= \widetilde{N}(x_{n+1} - x^\star, \tau) \circledast \widetilde{N}(x^\star - x_{n+1}, \tau) \circledast N_d(\widetilde{A}, \widetilde{B}, \tau)$
which implies

$$N_d(\widetilde{A}, \widetilde{B}, \tau) \geq \widetilde{N}(x_{n+1} - x^\star, \tau) \circledast \widetilde{N}(x^\star - \mathbb{T}x_n, \tau)$$

$$\geq \tilde{N}(x_{n+1} - x^\star, \tau) \circledast \tilde{N}(x^\star - x_{n+1}, \tau) \circledast N_d(\tilde{A}, \tilde{B}, \tau)$$

In the previous inequality, if use limit as $n \to \infty$, obtain:

$$N_d(\tilde{A}, \tilde{B}, \tau) \geq 1 \circledast \tilde{N}(x^\star - \mathbb{T}x_n, \tau) \geq 1 \circledast 1 \circledast N_d(\tilde{A}, \tilde{B}, \tau)$$

that is,

$$\lim_{n\to\infty} \tilde{N}(x^\star - \mathbb{T}x_n, \tau) = N_d(\tilde{A}, \tilde{B}, \tau)$$

and so, by condition (c), $x^\star \in \tilde{A}_\circ(\tau)$. Since $\mathbb{T}(\tilde{A}_\circ(\tau)) \subseteq \tilde{B}_\circ(\tau)$, there exists $z \in \tilde{A}_\circ(\tau)$ such that $\tilde{N}(z - \mathbb{T}x^\star, \tau) = N_d(\tilde{A}, \tilde{B}, \tau)$. Consequently, it follows from condition (d) and inequality (10) with $u = x_{n+1}, v = z, x = x_n$ and $y = x^\star$ that:

$$\tilde{N}(x_{n+1} - z, \tau) \geq \breve{\varphi}(\mathcal{L}(x_n\, x^\star, x_{n+1}, z, \tau)$$

where,

$$\mathcal{L}(x_n\, x^\star, x_{n+1}, z, \tau) = \min\{\tilde{N}(x_n - x^\star, \tau), \max\{\tilde{N}(x_n - x_{n+1}, \tau), \tilde{N}(x^\star - z, \tau)\}$$

We have:

$$\tilde{N}(x^\star - z, \tau) \geq \tilde{N}(x^\star - x_n, \tau) * \tilde{N}(x_n - x_{n+1}, \tau) * \tilde{N}(x_{n+1} - z, \tau) \geq \tilde{N}(x^\star - x_n, \tau) * \tilde{N}(x_n - x_{n+1}, \tau) * \breve{\varphi}(\mathcal{L}(x_n\, x^\star, x_{n+1}, z, \tau)$$

In the previous inequality, if taking the limit as $n \to \infty$, get:

$\tilde{N}(x^\star - z, \tau) = 1$, which means $x^\star = z$, that is $\tilde{N}(x^\star - \mathbb{T}x^\star, \tau) = \tilde{N}(z - \mathbb{T}x^\star, \tau) = N_d(\tilde{A}, \tilde{B}, \tau)$, thus $x^\star$ is BPP of $\mathbb{T}$.

Example 3.8: Let $L = \mathbb{R}$ with the fuzzy norm, $\tilde{N}: L \times \mathbb{R} \to [0,1]$ defined by $\tilde{N}(x, \tau) = \frac{\tau}{\tau + \|x\|}$ $\forall x \in L$ and $\tau > 0$, where $\|x\|: \mathbb{R} \to [0, \infty)$ such that $\|x\| = |x|$

Let $\tilde{A} = \{1,2,3,4,5\}$ and $\tilde{B} = \{6,7,8,9,10\}$

So that $N_d(\tilde{A}, \tilde{B}, \tau) = sup\{\tilde{N}(x - y, \tau): x \in \tilde{A}, y \in \tilde{B}\} = \frac{\tau}{\tau+1}$
Also, define $\mathbb{T} : \tilde{A} \to \tilde{B}$ by

$$\mathbb{T}(x) = \begin{cases} 6, if\ x = 5 \\ x + 5, otherwise \end{cases}$$

Notice that $\tilde{A}_\circ(\tau) = 5$ and $\tilde{B}_\circ(\tau) = 6$, $T(\tilde{A}_\circ(\tau)) \subseteq \tilde{B}_\circ(\tau)$.

Also, define $\tilde{\alpha}, \breve{\eta} : \tilde{A} \times \tilde{A} \times (0, \infty) \to [0, +\infty)$ by:

$$\tilde{\alpha}(x, y, \tau) = 1\ \forall\ x, y \in \tilde{A}$$

and

$$\breve{\eta}(x, y, \tau) = 2 \text{ for each } x, y \in \tilde{A}$$

Also, assume that:

$$\begin{cases} \breve{\alpha}(x, y, \tau) \leq \breve{\eta}(x, y, \tau) \\ \tilde{N}(u - \mathbb{T}x, \tau) = N_d(\tilde{A}, \tilde{B}, \tau) \\ \tilde{N}(v - \mathbb{T}y, \tau) = N_d(\tilde{A}, \tilde{B}, \tau) \end{cases}$$

then,

$$\begin{cases} \text{x}, \mathcal{y} \in \widetilde{A} \\ \widetilde{\text{N}}(\mathcal{u} - \mathbb{T}\text{x}, \tau) = N_d(\widetilde{A}, \widetilde{B}, \tau) \\ \widetilde{\text{N}}(\mathcal{v} - \mathbb{T}\mathcal{y}, \tau) = N_d(\widetilde{A}, \widetilde{B}, \tau) \end{cases}$$

then,

$(\mathcal{u}, \text{x}) = (5,5)$ or $(\mathcal{u}, \text{x}) = (5,1)$. Putting $(\mathcal{u}, \text{x}) = (5,5)$ and $(\mathcal{v}, \text{x}) = (5,1)$. Then conclude $\breve{\alpha}(\mathcal{u}, \mathcal{v}, \tau) \leq \breve{\eta}(\mathcal{u}, \mathcal{v}, \tau)$ that is means $\mathbb{T}$ is an $\breve{\alpha} - \breve{\eta}$ proximal admissible mapping.

Also, assume that $\breve{\alpha}(\text{x}, \mathbb{T}\text{x}, \tau) \leq \breve{\eta}(\text{x}, \mathbb{T}\text{x}, \tau)$ and $(\mathcal{y}, \mathbb{T}\mathcal{y}, \tau) \leq \breve{\eta}(\mathcal{y}, \mathbb{T}\mathcal{y}, \tau)$, get:

$$\breve{\alpha}(\text{x}, \mathbb{T}\text{x}, \tau)\breve{\alpha}(\mathcal{y}, \mathbb{T}\mathcal{y}, \tau) \leq \breve{\eta}(\text{x}, \mathbb{T}\text{x}, \tau)\breve{\eta}(\mathcal{y}, \mathbb{T}\mathcal{y}, \tau)$$

Now we define $\breve{\varphi} : [0,1] \to [0,1]$ by $\breve{\varphi}(s) = \sqrt{s}$,$\forall\, s \in [0, 1]$ then from (10),

$$\widetilde{\text{N}}(\mathcal{u} - \mathcal{v}, \tau) = \frac{\tau}{\tau + \|u - v\|}$$

$$= \frac{\tau}{\tau + |5 - 5|}$$

$= 1 \geq \breve{\varphi}\left(\widetilde{\text{N}}(\text{x} - \mathcal{y}, \tau)\right)$ for each $\tau > 0$.

4. CONCLUSION

In this paper, we introduced the notions of $\breve{\alpha}$–$\breve{\eta}$-$\breve{\beta}$ proximal contractive and $\breve{\alpha}$–$\breve{\eta}$-$\breve{\beta}$ proximal contractive mappings in a fuzzy Banach space. After that, the existence of the best proximity point for these types of mappings is proved. Some examples are provided to demonstrate the applicability of the results obtained. This work lays the groundwork for further research on other new types of contraction functions in fuzzy Banach space and to study the applications for these types of mappings in a fuzzy Banach space.

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