

Scaling Behavior and Phase Change in Complex Network

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Abstract

Scaling behavior is a extremely typical phenomenon in complex system research, as well as it can act that many Macro indicators in system or distribution function of some variables meet exactly power-law behavior, which possesses different kinds of Exponents. In this article, according to Phase Change concept in Physics, it is researched that the nature in critical state of complex network with Seepage model, and it is totally stated that the basic reason of Self-similar behavior, Fractal behavior, and so on, and also Phase Change in complex network in critical state of complex network in accord with power-law distribution.

Keywords: complex network, phase change, no scaling network, seepage model, power-law distribution

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1. Introduction

Scaling behavior is a important area in complex network now, such as no scaling network is just a power-law Distribution network, also, for example, the income in society is in accord with the famous Pareto theorem, which is income density function ($f(x) \sim x^{-1.75}$), belong to power-law Distribution [1]. Another, among English words, the frequency of occurrence of word 'r' in the sequence of frequency of occurrence from largest to smallest is $f(r) \sim r^{-1}$, that is Zipf theorem. In addition, the relation of two variable is power-law, giving an example that organism Metabolism and its Body Size meet $3/4$ power-law ($F \sim M^{3/4}$), called Kleiber theorem. The last example illustrated with is no scaling network as well-known. Many complex network in the real world totally meet power-law distribution ($p(x) \sim x^{-3}$). That is A large class of scaling behavior [2, 3].

Then, another class of scaling behavior is that fractal is familiar to us. Calculating fractal dimension of Fractal images, in fact, there is a kind of power-law relation between Measure Value 'Y' of fractal and Accuracy value 'x' of scale, as $y \sim x^{-D}$, and power-law D is its fractal dimension as talked above. Of course, it is normal that scaling behavior is occurred in random fractal, like Brownian Motion and Levy Flight [4-6]. More and more appearances of phenomenon about scaling, that makes researchers pursue to the nature.

What kind of research is originated in scaling behavior and power-law phenomenon, then? Of course, strictly speaking, in the era of classical mechanics, people had discovered power-law phenomenon, such as the famous Gravitation formula, $F \sim M_1 M_2 / r^2$, is a power-law. However, there are two sources about that the word 'scaling' is really mentioned and a relatively large-scale research have been developed in physics. One is the Turbulence in liquid, when people discovered that Multi-scale phenomena in Turbulence, which was surveyed by different scales, completely showed us the analogy regularity. And the other one is Phase change in Statistical Physics, especially Phase change in critical status [7-8].

With the discovery of researching on complex system in Phase change, like Phase change behavior of a magnet in high temperature, many Accumulation of macro indicators can give a series of scaling behaviors. That is to say, relations among so many indicators can be carved by power-law, another, the system can also show a lot of similar behavior to itself, while close to the phase transition point. So, critical state and scaling behavior is one of the most important branches in classification of Statistical Physics.

2. Percolation Model

Creating a grid chart consists of $L \times L$ lattices, they will be colored by Probability p , which is the color of every lattice decided by one Probability p . When Probability p appears, the lattice will be colored by black. The contrary is white. P is 0.4 and L is 10, arbitrarily. As shown in Figure 1.

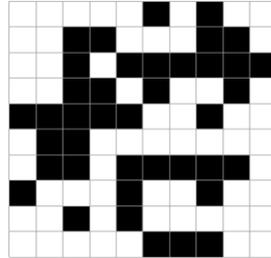


Figure 1. The Grid Chart when $p=0.4$

Next, the black lattices will be colored twice, and some of them communicating with each others must be the same color under constraint, but the opposite is different. For the nice effect, communicating lattices are all colored by black, yet independent lattices is no color. The two lattices communicate with each other means their edges are connected, not points between lattices. The same color lattices connected with the others is called clusters. Figure 2 is obtained by coloring figure above.

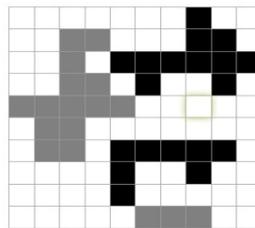


Figure 2. The Communicating Lattices

The figure above is achieved in $L=10$, $p=0.4$. For a better situation, it can be like figure 2.3, 2.4 and 2.5, if while L expand to 100, and p is 0.4, 0.6 and 0.7.

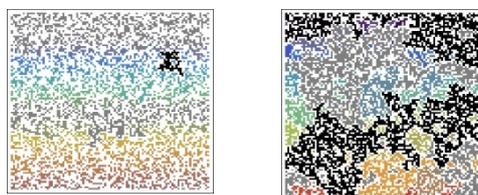


Figure 3. The Grid Chart when $p=0.4$, $p=0.6$



Figure 4. The Grid Chart when $p=0.7$

In these figures, the biggest cluster is marked by black. As we have seen, the smaller p is ($p < 0.6$), the less these clusters are. Moreover, they show a tendency to not connect with, therefore, different grayscale values marked the clusters. When the bigger p is, they trend to a large one. Especially, while p is 0.6, there is the larger cluster of them. At the same time, all kinds of and so many types of clusters are beginning to take shape, and also a large quantity.

3. Seepage and Phase Change

3.1. Seepage

The so-called Seepage is one large cluster in system can get through and permeate left and right, or up and down boundary of these lattices. At that situation of p with three different values as above, the cluster is smaller and no communication with each other, when p is 0.4. That is to say, there is no seepage formation in system. But, when p is 0.6 or 0.7, the phenomenon of that cluster is much larger and interconnected is obvious. In particular, when p is 0.7, the seepage occurs.

Therefore, while p is from 0.4 to 0.7, the cluster turns from small and disconnected to large and interconnected. That is the seepage growing out of nothing. The phenomenon is so-called phase change, which means phase change happens while p is from 0.4 to 0.7 in system. The nature of system has changed during the process of phase change. It is illustrated that the seepage of system must be formed when there is a critical probability P_c is equal or lesser than p .

3.2. Phase Change

To get the value of p_c , when calculating p from 0.1 to 0.9, the biggest size of the cluster is S_{max} . p is as abscissa, and the size is as ordinate. Because system is random, the result is not same to the different value of p . To avoid random disturbance, ensemble average can be to execute. Then, in different size, it shows the S_{max} - p curve, like Figure 5.

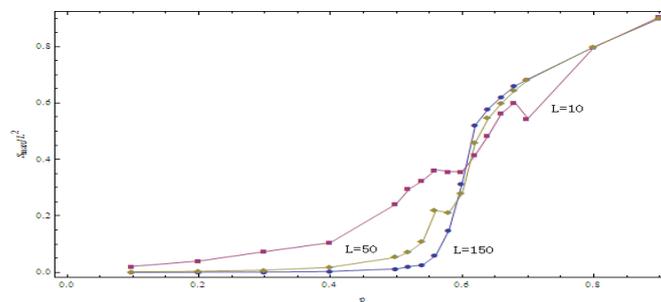


Figure 5. Chart the Biggest Size of Cluster as p Changing

Whatever the p is, all curve is monotonically increasing. And also, the more L is, the steeper the curve is. Especially, when L is bigger ($L = 150$), it is happened that a sudden change is at 0.6. Some macro state of system suddenly turns as one variable has changing, which is called change phrase.

According to the Figure 5, at the value of p about 0.59, there is a sudden change in curve when L is at some point.

4. Scaling Behavior

When p is at about critical point nearby p_c , all kinds of scaling behaviors will be achieved (that is power-law behavior). Firstly, it is surveyed that probability distribution belong to the sizes of every clusters in seepage system. In the clusters in seepage system (every different colors part), the sizes are extremely different. So, to get the diversity, a size of one cluster is seen as a random parameter, and also it can get probability distribution of the parameter. The figure below has shown that different parameters can decide probability distribution of clusters'

size. clusters' size is as abscissa, and p is as ordinate. Figure 6 shows that under $L = 150$, probability distribution of clusters' sizes at different values of p .

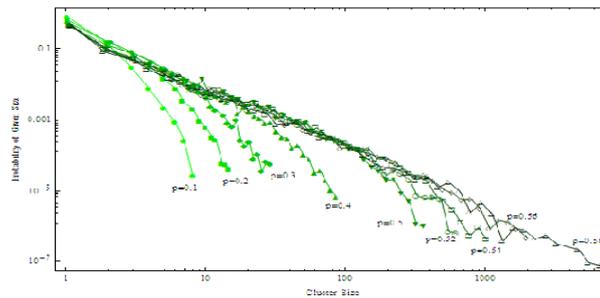


Figure 6. The Probability Different Clusters' Sizes

In figure 6, every data point represents, under a fixed size x , there are some proportion of clusters in Inter-cell $x+dx$. As we shown, as the p is more and more to be approaching the critical point 0.59, the distribution curve is to be a line eventually. Double logarithmic coordinates that is abscissa and ordinate are both logarithmic. Thus, the line means that two variables are to meet the power-law relationship. Under $p=0.58$, the distribution density function of its size can be marked by $p(x) = 0.37 * x^{-1.72}$.

In a word, the distribution of system's size is power-law at critical point as a conclusion, called a scaling behavior.

Nearby critical state, the larger cluster in seepage system is just the similar fractal object. For instance, it can be the biggest one under $L=150$, $p=0.6$. As Figure 7.

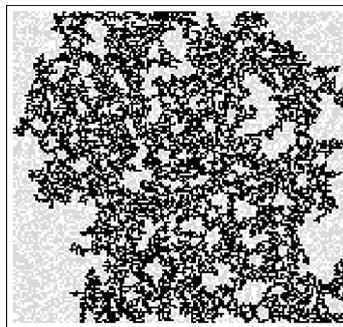


Figure 7. Grid Figure under $L=150$, $p=0.6$

The black cluster is the biggest one in seepage. To shown clearly, others is colored by light gray. According to the cluster, it is extremely similar to random trajectory of common Brownian motion. In fact, it is a random fractal geometry, which can calculate the black biggest fractal dimension by box covering.

So-called box covering is so simple, that is the dimension is covered by different resolutions boxes. Then at a fixed resolution, it will be tested that the number of boxes $l(s)$ needed is as approximate area of this dimension. Next, some smaller boxes s' are used by covering these lattices and a new approximate area is achieved. Repeating the process, it can get a curve depicted the relation between $l(s)$ and s as change of different s . For two-dimensional geometry, like a circle, the curve got by this action is also a power-law one. That is $l(s) \sim s^{-D}$, and $D = 2$. However, for fractal geometry, although a power-law can be achieved, D is less than 2 as usual. So it is a fractal.

As below, the fractal dimension of the red cluster is got by Box Covering, result shown by the Figure 8.

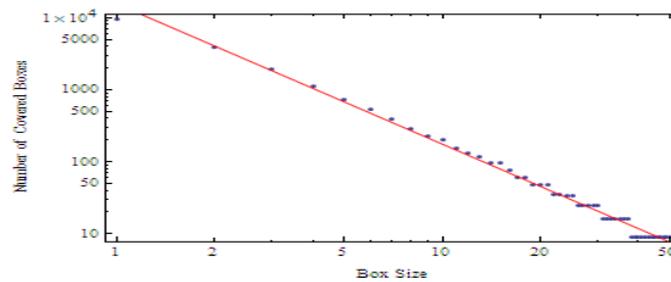


Figure 8. The Fractal Dimension

It is the line under power-law that the whole figure is covered by changing the size of boxes, and also the slope of line, that is fractal dimension, is 1.95, less than 2. It is asserted that it can get a smaller clump of fractal dimension when L is tested as a bigger value. It will reflect that Complexity of Clusters can deviate far from conventional geometry.

Thus, it can be concluded that clusters is a similar fractal structure under critical state, and that is a scaling behavior.

5. Renormalization Equation

To any percolation model, the parameter that decides its nature is P, and every renormalization operation make P change a time to a percolation model. It assumes that in original scale S probability of black lattices is P(s), and after renormalization operation, the scale S becomes S', which is larger. Also, the P(s) change into P(s'). The problem is that what relation is between P(s) and P(s').

$$P(s') = f(P(s)) \tag{1}$$

That f, the function nature is lay in Coarse Graining rule, as shown Figure 3, that is what is provision of ignoring information. Attention is that the left of these rules is the situation of many black lattices occupied partly under original scale S_n in fact. The right is the situation of new scale S_{n+1} occupied. Then, the relation of between $P(s_{n+1})$ and $P(s_n)$ can be calculated by the equation as follow:

$$P(s_{n+1}) = P(s_n)^4 + 4P(s_n)^3(1 - P(s_n)) + 2P(s_n)^2(1 - P(s_n))^2 \tag{2}$$

4 power items of $P(s_n)$ is the last rule in corresponds with ones (that means black lattices appear consecutively four times and the probability is obviously $P(s_n)^4$ in original scale), 3 power item is the situation that there are three black lattices on left of rule, which is totally four rules.

So the coefficient is four. The probability of consecutively appearing three times and the last is white lattice. That is $P(s_n)^3(1 - P(s_n))$. 2 power items corresponds with that two kinds of two black vertical rules connected, and the coefficient is 2.

By comparing the three situation: when $P = 0.5, 0.6$ and 0.7 , there is much larger influence in renormalization especially $P = 0.5$ and 0.7 . For instance, while $P = 0.5$, renormalization make probability turn smaller and the colored lattices can be more and more sparse. When $P = 0.7$, the situation is opposite. Otherwise, these two situations can affect the proportion of the biggest cluster. When $P = 0.5$, the proportion of black lattices turn smaller quickly, and when 0.7 rapidly increase nearly to 1.

However, in critical status, even if $P = 0.6$ is similar to P_c , there is hardly influence in the biggest cluster to renormalization operation to the density of black lattices. As Figure 8.

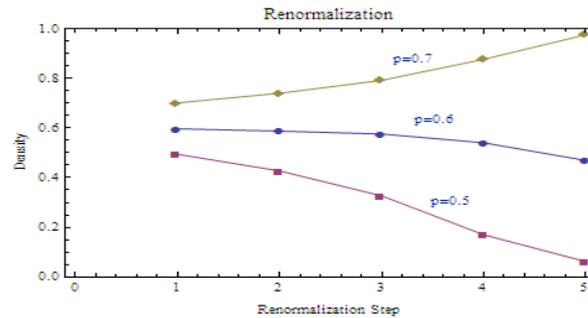


Figure 8. The Curve of the Density of Colored Lattices from Renormalization Steps

In Figure 8, the density of colored lattices (original black ones) change from renormalization steps. In these three statuses, when only in $P = 0.6$, the curve still stay same, and in other status, the density is either larger or smaller.

Therefore, in different status, the result of renormalization of percolation model is got, then, and how to calculate probability P_c in the critical status by renormalization?

6. Renormalization and Fixed Point

While the scale of model turns from S_n to S_{n+1} to every operation of Renormalization, and the probability is from $P(s_n)$ to $P(s_{n+1})$, which is according refer to (2). Then, iterative equation can be achieved, which shows every renormalization operation make black lattices change. Different initial probability can decide the evolutionary tracks of this iterative equation. Also Figure 7 shows track of renormalization equation in different initial point $P(s_0) = 0.5, 0.6, 0.7$. It is seen that the curve gradually declines when $P(s_0) = 0.5$. If original lattice is infinity and renormalization keeps continued, then, the curve approaches to 0. When $P(s_0) = 0.7$, the curve is near 1. Otherwise, when $P(s_0) = 0.6$, although it decline so slowly, it may be near 0. In fact, the behaviours of these curves are completely decided by fixed points of renormalization groups. So-called the fixed point is a special $P(s^*)$.

$$P(s^*) = P(s^*)^4 + 4P(s^*)^3(1 - P(s^*)) + 2P(s^*)^2(1 - P(s^*))^2 \tag{3}$$

When N approaches infinity, we can get the equation on the top, and solving this equation, we can get four answers of $P(s^*)$.

$$\left\{ 0, 1, \frac{1}{2(-1 - \sqrt{5})}, \frac{1}{2(-1 + \sqrt{5})} \right\} \tag{4}$$

Among these answers, the third is omitted because of negativity. Therefore, there are three answers: $\{0, 1, 0.618\}$, which are thought as the fixed points of renormalization equation. For the case of 0 and 1, they are called as trivial fixed points. They are corresponded to the density 0 (no lattice) and 1 (all are lattices) of last lattices after operation renormalization. Both are stable attractor, which is $P(s_0)$ initial. After renormalization unlimited operation, they will converge to the two attractors. And once they converge, there is no escape. For the last answer 0.618, it is non-trivial fixed point. The situation is exactly fractal. That is to say, when $P(s_0)$ is about 0.618, renormalization operation does not affluent the seepage graphics.

So, the attractor $P(s^*)$ is the critical point P_c that we need to find. Under this

probability of critical point, seepage system is with scale invariance. However it zooms (renormalization), it can be got a similar system.

7. Conclusion

Percolation model is a kind of simple rule model, but its behavior is so complex and includes many critical phase transition phenomena and all kinds of scaling behaviors. One of the Methods is used in natures and details of so many complex network, thereby, it can be overall grasped the fundamental nature of complex network.

According to calculating exactly, now, it is widely recognized that $P_c=0.593$, but it is 0.618 by means of renormalization equation. Although it is similar, it is yet not the same. The reason is that calculating renormalization equation is an act of approximate operating, and non-trivial fixed point is similar to P_c . The sources of error mainly happen in Coarse Graining rule. In that rule, it is seen approximately as a black lattice when the number of black lattices is larger than and equal to three. And when it is two, we only consider that it is a black lattice through Up and down. Ever, the fact is that the approximate operating may destroy status of original clusters. If two clusters are nearby on same level, Coarse Graining rule may ignore it. Therefore, the error appears. If the operating much rougher, only in accordance with the majority principle, when the number of original black lattices is larger than 3, it can be mapped a black lattice, otherwise, white. At that situation, we can get a probability deviate from P_c far. In opposite, if Coarse Graining is much finer, we can get better result.

Seepage model is a kind of simple rules, but its behaviour is so complex, even includes phase transition and critical phenomena, and also all kind of scaling behaviours. In two-dimensional percolation model, many acts can expand in complex network. Although Physicist can calculate analytical solutions of two-dimensional percolation problems, people realize hardly to lots of scaling phenomenon from a traditional point of view of time and space. In addition, renormalization is a similar act, but its entry point is very deep. It can grasp the nature of scaling behaviour, scale invariance.

That is also fractal characteristics and scale invariance of system. Whatever initial dynamics rules of system is, whatever seepage model or Ising model is, as long as system turns to the critical status, when it can produce all kinds of scaling behaviours and omit some restrictions of dynamics rules, it can completely portray scaling behaviour newly from the point of view of the renormalization equation. So, renormalization method is likely to be a new starting point, rather than a simple technical means.

References

- [1] Barabasi AL, Bonabeau E. Scal-freenetworks. *Scientific American*. 2003; 50-59.
- [2] Newman MEJ. The structure and function of networks. *Computer Physics Communications*. 2002; 147: 40-45.
- [3] Faloutsos M Faloutsos C. On power-law relationships of the Internet topology. *ACM SIGCOMM Computer Communication Review*. 1999; 29(4): 251-262.
- [4] Pastor-Satorras R Vespignani A. Epidemics dynamics and endynamic states in complex network. *Phys Rev. E*. 2001; 63: 066117.
- [5] Pastor-Satorras R. Vespignani A. Epidemics and immunization in scal-free networks. In *Handbook of Graphs and Networks*, Bornholdt S. Schuster H.G. (etc.), WILEY-VCH publisher. 2003.
- [6] Pastor-Satorras R. Vespignani A. Epidemics dynamics in finite size scale-free networks. *Phys. Rev. E*. 2003; 68:035103.
- [7] Cohen R, Havlin 5. Probabilistic Prediction in Scale- Free Networks: Diameter Changes. *Phys. Rev. Lett*. 2003; 90: 058701.
- [8] Andrea C, Alessio C, Irene G, Giorgio P, S Raffaele, Fabio S, MassimilianoV. Scale-free correlations in starling flocks, *PNAS*. 2010; 107(26): 11865.