Indonesian Journal of Electrical Engineering and Computer Science
Vol. 27, No. 3, September 2022, pp. 1525~1532
ISSN: 2502-4752, DOI: 10.11591/ijeecs.v27.i3.pp1525-1532 ❐ 1525

Modification of the new conjugate gradient algorithm to solve nonlinear fuzzy equations

Zeyad M. Abdullah[1], Hisham M. Khudhur[2], Amera Khairulla Ahmed[1]
[1]Deparment of Mathematics, College of Computers Sciences and Mathematics, University of Tikrit, Tikrit, Iraq
[2]Deparment of Mathematics, College of Computers Sciences and Mathematics, University of Mosul, Mosul, Iraq

Article Info

Article history:

Received Feb 25, 2021
Revised Jun 15, 2022
Accepted Jul 1, 2022

Keywords:

Algorithm
Conjugate gradient
Fuzzy
Numerical
Optimization

ABSTRACT

The conjugate gradient approach is a powerful tool that is used in a variety of areas to solve problems involving large-scale reduction. In this paper, we propose a new parameter in nonlinear conjugate gradient algorithms to solve nonlinear fuzzy equations based on Polak and Ribiere (PRP) method, where we prove the descent and global convergence properties of the proposed algorithm. In terms of numerical results, the new method has been compared with the methods of Fletcher (CD), Fletcher and Reeves (FR), and Polak and Ribiere (PRP). The proposed algorithm has outperformed the rest of the algorithms in the number of iterations and in finding the best value for the function and the best value for the variables.

Corresponding Author:

Hisham M. Khudhur
Deparment of Mathematics, College of Computers Sciences and Mathematics, University of Mosul
Central Library Building, Al-Majmoa'a Street, Mosul, Ninawa, Iraq
Email: hisham892020@uomosul.edu.iq

1. INTRODUCTION

The numerical solution of a nonlinear algebraic problem like:

$$F(x) = 0 \tag{1}$$

is standard in engineering and natural sciences. Many engineering design issues that must satisfy certain limits can be explained using nonlinear equalities or inequalities. The was the first to present and analyze the concept of fuzzy numbers and arithmetic operations [1]. The solution of nonlinear equations in which fuzzy numbers entirely or partially represent the parameters is one of the most popular applications of fuzzy number arithmetic [2]–[4].

Analytical technology is widely used. Buckley and Qu procedures, for example, are standard analytical techniques [5]–[8] are insufficient for the solution of equations such as:

(i) $ax^5 + bx^4 + cx^3 + dx^2 + ex + f = g$
(ii) $x - \sin x = g$

$x, a, b, c, d, e, f,$ and g are ambiguous numbers. As a result, we must consider Newton's method for solving a fuzzy nonlinear issue while developing numerical approaches for locating the roots of such equations [1].

Once a suitably precise approximation has been determined, the technique proposed by Newton has the benefit of being able to converge quickly. This approach is flawed due to the fact that it requires an accurate starting estimate in order to guarantee convergence. The steepest descent method can only converge to the resolution in a linear fashion, despite the fact that it will typically converge even if the initial approximations are not very accurate [7]. We devised a one-of-a-kind conjugate gradient coefficient and used it in our solution to fuzzy nonlinear equations. This allowed us to successfully solve the equations. In this article, the conjugate gradient method, which is well-known for being both simple and effective, is used to explore how to tackle optimization issues. This method is well-known for being both easy and successful.

2. LITERATURE REVIEW OF CONJUGATE GRADIENT METHODS

The form of the equation for the nonlinear conjugate gradient (CG) technique may be found as (2):

$$x_{k+1} = x_k + \alpha_k d_k, k \geq 1 \tag{2}$$

where x1 is an initial point, $\alpha_k$is a step–length and α_k step size that satisfies the standard Wolfe conditions (SWC) in (4), and where x1 is an initial point (5) [9], [10].

$$f(x_k + \alpha_k d_k) \leq f(x_k) + \delta \alpha_k g_k^T d_k \tag{3}$$

$$d_k^T g(x_k + \alpha_k d_k) \geq \sigma d_k^T g_k \tag{4}$$

The following formula is the strong Wolffe terms in the equation as indicative in (5) and (6) [9], [11].

$$f(x_k + \alpha_k d_k) \leq f(x) + \delta \alpha_k g_k^T d_k \tag{5}$$

$$|d_k^T g(x_k + \alpha_k d_k)| \leq -\sigma d_k^T g_k \tag{6}$$

$$d_{k+1} = \begin{cases} -g_1, k = 1 \\ -g_{k+1} + \beta_k d_k, k \geq 1 \end{cases} \tag{7}$$

Later, we will discuss some of the different beta parameters. The most famous of these parameters are the Fletcher and Reeves (FR) parameter in 1964 [12], the Fletcher (CD) parameter in 1989 [13], the Polak and Ribiere (PRP) parameter in 1969 [14], the Hestenes-Stiefel (HS) parameter in 1952 [15], the Dai-Yuan (DY) parameter in 1999 [16], and Hisham- Khalil (KH) in 2021 [17]. Here are the formulas for the above parameters.

$$\beta^{FR} = \frac{\| g_{k+1} \|^2}{\| g_k \|^2} \qquad \beta^{CD} = \frac{-\| g_{k+1} \|^2}{g_k^T d_k}$$

$$\beta^{HS} = \frac{g_{k+1}^T y_k}{y_k^T d_k} \qquad \beta^{PRP} = \frac{g_{k+1}^T y_k}{\| g_k \|^2}$$

$$\beta^{DY} = \frac{\| g_{k+1} \|^2}{y_k^T d_k} \qquad \beta^{KH} = \frac{\|g_{k+1}\|_1^2}{\|g_k\|_1^2},$$

Where $g_k = \nabla f(x_k)$, and let $y_k = g_{k+1} - g_k$, for more see [18], [19].

3. THE NEW MODIFIED AZH$_2$-CG ALGORITHM FORMULA

We get the direction of the research of the new modified method from the $d_1 = -g_1$ and,

$$d_{k+1} = -g_{k+1} + \beta^{AZH_2} d_k$$

$$\beta^{AZH_2} = \frac{g_{k+1}^T y_k}{\theta \|g_k\|_1^2 + (1-\theta)\|g_k\|^2}$$

the goal of developing the new algorithm is to obtain a convergent method and achieve the descent property. We utilize the following equation for a new and speedier way:

$$d_{k+1} = -g_{n+1} + \beta^{AZH_2} d_k \quad (8)$$

$$\beta^{AZH_2} = \frac{g_{n+1}^T y_n}{\theta \|g_n\|_1^2 + (1-\theta)\|g_n\|^2} \quad (9)$$

where $0 < \theta < 1$.

3.1. Descent property

In this section we will cover the descent property of the new modified, method to prove the effectiveness of the new method defined in (8) and (9), here is a proof descent property of the new method. Theorem (1): the new modified algorithm fulfills the strong Wolf conditions found in (5) and (6) we put the new algorithm in (2) where the parameter and search direction are calculated from (8) and (9) to be the new search direction are descent for all k provided $g_{n+1}^T d_{k+1} < 0$.

Proof: the prove is by indication, for $k=1$, $d_1 = -g_1 \rightarrow g_1^T d_1 < 0$, .
Now suppose $g_k^T d_k < 0$ or $g_k^T s_k < 0$ $s_k = \alpha_k d_k$ then for $k+1$ we have:

$$g_{n+1}^T d_k = -g_{n+1}^T g_{k+1} + \beta^{AZH_2} g_{n+1}^T d_k$$

$$g_{n+1}^T d_k = -g_{n+1}^T g_{n+1} + \frac{y_n^T g_{n+1}}{\theta \|g_n\|_1^2 + (1-\theta)\|g_n\|^2} g_{n+1}^T d_k$$

assume that $y_n^T g_{n+1} > 0$, $0 < \theta < 1$,

$$g_{n+1}^T d_k \leq g_{n+1}^T g_{n+1}(-1 + \frac{y_k^T g_{n+1}}{\theta \|g_n\|_1^2 \|g_{n+1}\|^2 + (1-\theta)\|g_n\|^2 \|g_{n+1}\|^2} g_{n+1}^T d_k)$$

$$\lambda = \frac{y_k^T g_{n+1}}{\theta \|g_n\|_1^2 \|g_{n+1}\|^2 + (1-\theta)\|g_n\|^2 \|g_{n+1}\|^2}$$

$$g_{n+1}^T d_k \leq g_{n+1}^T g_{n+1}(-1 + \lambda)$$

$$g_{n+1}^T d_k \leq -g_{n+1}^T g_{n+1}(1 - \lambda)$$

where $0 < \lambda < 1$,

$$C = (1 - \lambda)$$

$$g_{n+1}^T d_k \leq -\|g_{n+1}\|^2 C$$

$$g_{n+1}^T d_k \leq -C \|g_{n+1}\|^2$$

by using strong wolfe conditions $g_n^T d_k \leq -c|g_n|$,

$$\therefore g_{n+1}^T d_{k+1} < 0$$

the proof is complete.

3.2. The study of global convergence

Next, we shall demonstrate that the CG technique with $\beta_{k+1}^{AZH_2}$converges globally. Our new method won't work until we make the following assumption:

Assumption (1):

– if f is constrained lower in the level set, we may say that:

$S = \{x \in R^n : f(x) \leq f(x_0)\}$; On some preliminary points.

– in the case of f, the differentiation is continuous, and the gradient is Lipshitz continuous, existing $L > 0$ such that:

$$\| g(x) - g(y) \| \leq L \| x - y \| \ \forall x, y \in N \quad (10)$$

– there exists a constant $\mu > 0$ such that f is a uniformly convex function if and only if:

$$(g(x) - g(y))^T(x - y) \geq \mu \parallel x - y \parallel^2 \text{ for any } x, y \in S \tag{11}$$

or equivalently,

$$y_k^T S_k \geq \mu \|S_k\|^2 \text{ and } \mu\|s_k\|^2 \leq y_k^T s_k \leq L\|s_k\|^2 \tag{12}$$

contrary to this, assumption (1) makes it crystal evident that there are positive constants B that are such that:

$$\parallel x \parallel \leq B, \forall x \in S \tag{13}$$

$$\parallel g(x) \parallel \leq \bar{\gamma}, \forall x \in S \tag{14}$$

Lemma (1): assume that assumption (1) and (13) are true and that any conjugate gradient approach in the range of (1) and (2), where $d_{k+1}$is the descending direction and α_k is the result of a strong Wolfe line search, is taken into consideration. If:

$$\sum_{k>1} \frac{1}{\|d_{k+1}\|^2} = \infty \tag{16}$$

so, we have:

$$\lim_{k\to\infty} inf \|g_k\| = 0$$

more information may be found in the following study [19]–[24].

Theorem (2): let's begin by supposing that the descent condition, (13), and assumption (1) are all correct. Take the following example of a conjugate gradient technique into consideration:

$$d_{k+1} = -g_{n+1} + \beta^{AZH_2} d_k$$

$$\beta^{AZH_2} = \frac{g_{n+1}^T y_n}{\theta\|g_n\|_1^2 + (1-\theta)\|g_n\|^2}$$

where α_k is determined using Wolfe line search conditions (5) and (6), and if the objective function is uniformly distributed over set S, then $\lim_{k\to\infty} inf \|g_k\| = 0$.

Proof: first and foremost, we must substitute our new $\beta_{k+1}^{AZH_2}$in the direction of d_{k+1}, in order to get the desired result:

$$\|d_{n+1}\|^2 = \|-g_{n+1} + \beta^{AZH_2} d_k\|^2$$

$$\|d_{n+1}\|^2 \leq \|-g_{n+1}\|^2 + \| \beta^{AZH_2} d_k\|^2$$

$$\|d_{n+1}\|^2 \leq \|-g_{n+1}\|^2 + \| \beta^{AZH_2} d_k\|^2$$

$$\|d_{n+1}\|^2 \leq \|g_{n+1}\|^2 + \left\| \frac{g_{n+1}^T y_k}{\theta\|g_n\|_1^2 + (1-\theta)\|g_n\|^2} d_k \right\|^2$$

$$\|d_{n+1}\|^2 \leq \|g_{n+1}\|^2 + \frac{\|g_{n+1}\|^2 \|y_k\|^2 \|d_k\|^2}{\theta\|g_n\|_1^2 + (1-\theta)\|g_n\|^2}$$

$$\|d_{n+1}\|^2 \leq \|g_{n+1}\|^2 \left(1 + \frac{\|y_k\|^2 \|d_k\|^2}{\theta\|g_n\|_1^2 + (1-\theta)\|g_n\|^2} \right)$$

$$Iet\ a = 1 + \frac{\|y_k\|^2 \|d_k\|^2}{\theta\|g_n\|_1^2 + (1-\theta)\|g_n\|^2}$$

$$\|d_{k+1}\|^2 \leq \|g_{n+1}\|^2 a$$

$$\sum_{k\geq 1}\frac{1}{\|d_k\|^2} \geq \frac{1}{a}\frac{1}{\|g_{k+1}\|^2}\sum 1 = \infty$$

$$\lim_{n\to\infty}(inf\|g_n\|) = 0$$

4. NUMERICAL RESULTS AND COMPARISONS

In this part of the paper, we will address the performance of the new algorithm in comparison with Fletcher algorithms CD, FR, and PRP, and the algorithms, and through the numerical results in Table 1, we note the efficiency of the new algorithm compared to other algorithms in the same field. The solutions of the nonlinear fuzzy equations are plotted in graphs (1)-(3). The numerical results and graphics were shown using MATLAB 2021b program on a laptop with a storage capacity of 500 GB, a Core i5 processor and 8 GB RAM.

a. $Y - best$:- best variable
b. $f - best$:- best function value

Table 1 shows the details of the results of the new modified AZH2 algorithm compared with other algorithms such as CD, FR, and PRP. The solutions of the examples are drawn in the Figures 1-3.

Example 1: "Take the nonlinear equation with fuzzy coefficients as an example:

$$(3,4,5)Y^2 + (1,2,3)Y = (1,2,3)$$

Assume that x is positive, and the parametric form of this equation is as follows, with no loss of generality:

$$\begin{cases}(3+\rho)\underline{Y}^2(\rho) + (1+\rho)\underline{Y}(\rho) - (1+\rho) = 0,\\ (5-\rho)\overline{Y}^2(\rho) + (3-\rho)\overline{Y}(\rho) - (3-\rho) = 0.\end{cases}$$

The following settings are required as beginning values for the system described above. For $\rho = 1$:

$$\begin{cases}4\underline{Y}^2(1) + 2\underline{Y}(1) - 2 = 0,\\ 4\overline{Y}^2(1) + 2\overline{Y}(1) - 2 = 0,\end{cases}$$

for $\rho = 0$,

$$\begin{cases}3\underline{Y}^2(0) + \underline{Y}(0) - 1 = 0,\\ 5\overline{Y}^2(0) + \overline{Y}(0) - 3 = 0,\end{cases}$$

with initial values:

$Y_0 = (\underline{Y}(0), \underline{Y}(1), \overline{Y}(1), \overline{Y}(0)) = (0.434, 0.5, 0.5, 0.681)$." [7], [25].

Example 2: "Take the nonlinear equation with fuzzy coefficients as an example:

$$(4,6,8)Y^2 + (2,3,4)\,Y - (8,12,16) = (5,6,7)$$

assume that x is positive, and the parametric form of this equation is as follows, with no loss of generality:

$$\begin{cases}(4+2\rho)\underline{Y}^2(\rho) + (2+\rho)\underline{Y}(\rho) - (3+3\rho) = 0,\\ (8-2\rho)\overline{Y}^2(\rho) + (4-\rho)\overline{Y}(\rho) - (9-3\rho) = 0.\end{cases}$$

the following settings are required as beginning values for the system described above. For $\rho = 1$:

$$\begin{cases}6\underline{Y}^2(1) + 3\underline{Y}(1) - 6 = 0,\\ 6\overline{Y}^2(1) + 3\overline{Y}(1) - 6 = 0,\end{cases}$$

for $\rho = 0$,

$$\begin{cases} 4\underline{Y}^2(0) + 2\underline{Y}(0) - 3 = 0, \\ 8\overline{Y}^2(0) + 4\overline{Y}(0) - 9 = 0, \end{cases}$$

with initial values:

$$Y_0 = (\underline{Y}(0), \underline{Y}(1), \overline{Y}(1), \overline{Y}(0)) = (0.651, 0.7808, 0.7808, 0.8397)$$." [7], [25].

Example 3: "Take the nonlinear equation with fuzzy coefficients as an example

$(1,2,3)Y^3 + (2,3,4)Y^2 + (3,4,5) = (5,8,13)$

Assume that x is positive, and the parametric form of this equation is as follows, with no loss of generality:

$$\begin{cases} (1+\rho)\underline{Y}^3(\rho) + (2+\rho)\underline{Y}^2(\rho) - (2+2\rho) = 0, \\ (3-\rho)\overline{Y}^3(\rho) + (4-\rho)\overline{Y}^2(\rho) - (8-4\rho) = 0. \end{cases}$$

The following settings are required as beginning values for the system described above. For $\rho = 1$

$$\begin{cases} 2\underline{Y}^3(1) + 3\underline{Y}^2(1) - 4 = 0, \\ 2\overline{Y}^3(1) + 3\overline{Y}^2(1) - 4 = 0, \end{cases}$$

for $\rho = 0$,

$$\begin{cases} \underline{Y}^3(0) + 2\underline{Y}^2(0) - 2 = 0, \\ 3\overline{Y}^3(0) + 4\overline{Y}^2(0) - 8 = 0. \end{cases}$$

with initial values:

$$Y_0 = (\underline{Y}(0), \underline{Y}(1), \overline{Y}(1), \overline{Y}(0)) = (0.76, 0.91, 0.91, 1.06)$$." [7], [25].

Table 1. Numerical comparison of the modified method with other methods

Problems	Algorithms	Iterations	Y-best	f-best
1	FR-CG	8	[0.4343,0.5,0.5,0.5307]	5.2505e-014
	PRP-CG	6		1.2314e-016
	CD-CG	8		8.1709e-014
	AZH2-CG	6		2.2529e-16
2	FR-CG	12	[0.6514,0.7808,0.7808,0.8397]	2.3998e-010
	PRP-CG	12		7.9887e-012
	CD-CG	14		5.9506e-010
	AZH2-CG	8		1.4357e-12
3	FR-CG	19	[0.8393,0.9108,0.9108,1.0564]	2.6011e-011
	PRP-CG	16		4.6595e-012
	CD-CG	135		1.4978e-008
	AZH2-CG	12		1.9792e-12

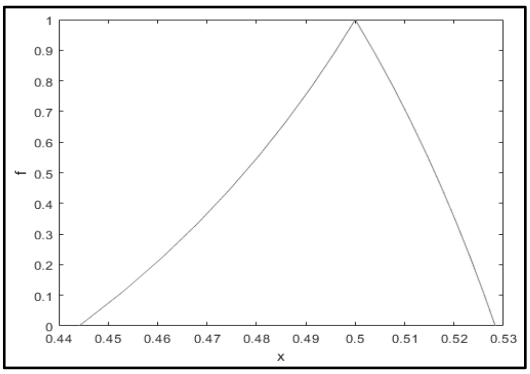

Figure 1. Diagram of the solution to the first problem using the AZH1 method

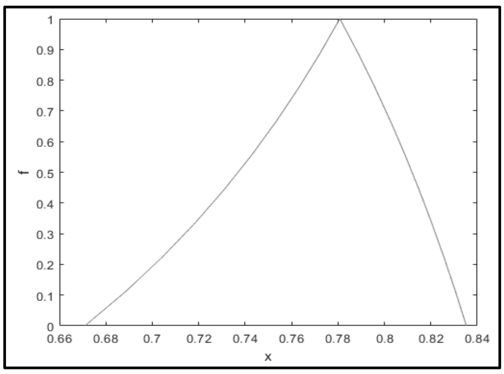

Figure 2. Diagram of the solution to the second problem using the AZH1 method

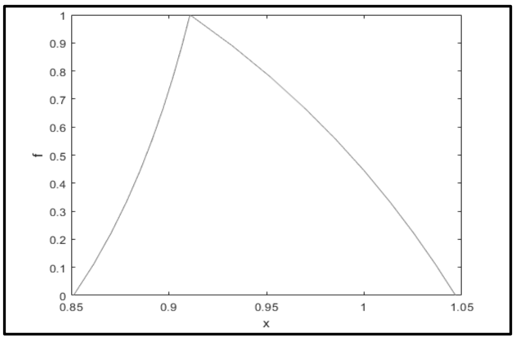

Figure 3. Diagram of the solution to the third problem using the AZH1 method

5. CONCLUSION

We conclude from this paper that using the new modified AZH_2 algorithm to solve nonlinear fuzzy equations gives us high efficiency in solving with fewer iterations and with higher accuracy compared with other algorithms in the same field, where other algorithms can be developed in the future that may be more efficient in solving non-fuzzy problems linear.

ACKNOWLEDGMENTS

I extend my sincere thanks and greetings to the faculties of Computer Science and Mathematics at the Universities of Mosul and Tikrit for providing the necessary resources to complete this paper and show it in this way.

REFERENCES

[1] L. A. Zadeh, "The concept of a linguistic variable and its application to approximate reasoning-III," *Information Sciences*, vol. 9, no. 1, pp. 43–80, Jan. 1975, doi: 10.1016/0020-0255(75)90017-1.

[2] R. Badard, "The law of large numbers for fuzzy processes and the estimation problem," *Information Sciences*, vol. 28, no. 3, pp. 161–178, Dec. 1982, doi: 10.1016/0020-0255(82)90046-9.

[3] P. Diamond, "Fuzzy least squares," *Information Sciences*, vol. 46, no. 3, pp. 141–157, Dec. 1988, doi: 10.1016/0020-0255(88)90047-3.

[4] M. Friedman, M. Ming, and A. Kandel, "Fuzzy linear systems," *Fuzzy Sets and Systems*, vol. 96, no. 2, pp. 201–209, Jun. 1998, doi: 10.1016/S0165-0114(96)00270-9.

[5] J. J. Buckley and Y. Qu, "Solving linear and quadratic fuzzy equations," *Fuzzy Sets and Systems*, vol. 38, no. 1, pp. 43–59, Oct. 1990, doi: 10.1016/0165-0114(90)90099-R.

[6] J. J. Buckley and Y. Qu, "On using α-cuts to evaluate fuzzy equations," *Fuzzy Sets and Systems*, vol. 43, no. 1, p. 125, Sep. 1991, doi: 10.1016/0165-0114(91)90026-M.

[7] J. J. Buckley and Y. Qu, "Solving fuzzy equations: a new solution concept," *Fuzzy Sets and Systems*, vol. 39, no. 3, pp. 291–301, Feb. 1991, doi: 10.1016/0165-0114(91)90099-C.

[8] J. J. Buckley and Y. Qu, "Solving systems of linear fuzzy equations," *Fuzzy Sets and Systems*, vol. 43, no. 1, pp. 33–43, Sep. 1991, doi: 10.1016/0165-0114(91)90019-M.

[9] B. A. Hassan, K. Muangchoo, F. Alfara, A. H. Ibrahim, and A. B. Abubakar, "An improved quasi-Newton equation on the quasi-Newton methods for unconstrained optimizations," *Indonesian Journal of Electrical Engineering and Computer Science (IJEECS)*, vol. 22, no. 2, pp. 389–397, 2020, doi: 10.11591/ijeecs.v22.i2.pp389-397.

[10] B. A. Hassan and M. W. Taha, "A new variants of quasi-newton equation based on the quadratic function for unconstrained optimization," *Indonesian Journal of Electrical Engineering and Computer Science*, vol. 19, no. 2, pp. 701–708, Aug. 2020, doi: 10.11591/ijeecs.v19.i2.pp701-708.

[11] B. Hassan, H. Jabbar, and A. Al-Bayati, "A new class of nonlinear conjugate gradient method for solving unconstrained minimization problems," in *2019 International Conference on Computing and Information Science and Technology and Their Applications (ICCISTA)*, Mar. 2019, pp. 1–4, doi: 10.1109/ICCISTA.2019.8830657.

[12] R. Fletcher and C. M. Reeves, "Function minimization by conjugate gradients," *The Computer Journal*, vol. 7, no. 2, pp. 149–154, Feb. 1964, doi: 10.1093/comjnl/7.2.149.

[13] C. Witzgall and R. Fletcher, "Practical methods of optimization," *Mathematics of Computation*, vol. 53, no. 188, p. 768, Oct. 1989, doi: 10.2307/2008742.

[14] E. Polak and G. Ribiere, "Note on the convergence of conjugate direction methods (in France)," *Revue française d'informatique et de recherche opérationnelle. Série rouge*, vol. 3, no. 16, pp. 35–43, May 1969, doi: 10.1051/m2an/196903R100351.

[15] M. R. Hestenes and E. Stiefel, "Methods of conjugate gradients for solving linear systems," *Journal of Research of the National Bureau of Standards*, vol. 49, no. 6, p. 409, Dec. 1952, doi: 10.6028/jres.049.044.

[16] Y. H. Dai and Y. Yuan, "A nonlinear conjugate gradient method with a strong global convergence property," *SIAM Journal on Optimization*, vol. 10, no. 1, pp. 177–182, Jan. 1999, doi: 10.1137/S1052623497318992.

[17] H. M. Khudhur and K. K. Abbo, "A new type of conjugate gradient technique for solving fuzzy nonlinear Algebraic equations," *Journal of Physics: Conference Series*, vol. 1879, no. 2, p. 022111, May 2021, doi: 10.1088/1742-6596/1879/2/022111.

[18] H. M. Khudhur and K. I. Ibraheem, "Metaheuristic optimization algorithm based on the two-step Adams-Bashforth method in training multi-layer perceptrons," *Eastern-European Journal of Enterprise Technologies*, vol. 2, no. 4 (116), pp. 6–13, Apr. 2022, doi: 10.15587/1729-4061.2022.254023.

[19] A. S. Ahmed, H. M. Khudhur, and M. S. Najmuldeen, "A new parameter in three-term conjugate gradient algorithms for unconstrained optimization," *Indonesian Journal of Electrical Engineering and Computer Science (IJEECS)*, vol. 23, no. 1, pp. 338–344, Jul. 2021, doi: 10.11591/ijeecs.v23.i1.pp338-344.

[20] M. M. Abed, U. Öztürk, and H. M. Khudhur, "Spectral CG algorithm for solving fuzzy non-linear equations," *Iraqi Journal for Computer Science and Mathematics*, vol. 3, no. 1, pp. 1–10, Jan. 2022, doi: 10.52866/ijcsm.2022.01.01.001.

[21] A. Sahiner, N. Yilmaz, and S. A. Ibrahem, "Smoothing approximations to non-smooth functions," *Journal of Multidisciplinary Modeling and Optimization*, vol. 1, no. 2, pp. 69–74, 2019.

[22] A. Sahiner and S. A. Ibrahem, "A new global optimization technique by auxiliary function method in a directional search," *Optimization Letters*, vol. 13, no. 2, pp. 309–323, Mar. 2019, doi: 10.1007/s11590-018-1315-1.

[23] X. Jiang, W. Liao, J. Yin, and J. Jian, "A new family of hybrid three-term conjugate gradient methods with applications in image restoration," *Numerical Algorithms*, Mar. 2022, doi: 10.1007/s11075-022-01258-2.

[24] A. Sahiner, S. A. Ibrahem, and N. Yilmaz, "Increasing the effects of auxiliary function by multiple extrema in global optimization," in *Numerical Solutions of Realistic Nonlinear Phenomena*, J. A. T. Machado, N. Özdemir, and D. Baleanu, Eds. Springer, 2020, pp. 125–143.

[25] K. I. Ibraheem and H. M. Khudhur, "Optimization algorithm based on the Euler method for solving fuzzy nonlinear equations," *Eastern-European Journal of Enterprise Technologies*, vol. 1, no. 4 (115), pp. 13–19, Feb. 2022, doi: 10.15587/1729-4061.2022.252014.

BIOGRAPHIES OF AUTHORS

Assis. Prof. Dr. Zeyad Mohammed Abdullah is Assistant Professor in the Department of Mathematics, College of Computer Science and Mathematics, University of Tikrit, Iraq. He holds a Bachelor's degree in University Mathematics, a Master's degree in Mathematics from the University of Mosul, and a Ph.D. in Computational Mathematics. Specialization: Computational numerical optimization. He can be contacted at email: zeyaemoh1978@tu.edu.iq.

Dr. Hisham Mohammed Khudhur is Instructor in Department of Mathematics, College of Computer Science and Mathematics, University of Mosul, Iraq. He holds a Bachelor's degree in Mathematics, a Master's degree in Mathematics from the University of Mosul, and a Ph.D. degree in computational mathematics. Specialization: Intelligent Numerical Algorithms. He can be contacted at email: hisham892020@uomosul.edu.iq.

Amera Khairulla Ahmed received the bachelor's degree in mathematics in applied mathematics from the Tikrit University - College of Computer Science and Mathematics - Department of Mathematics. she is currently a master student in the Department of Mathematical Sciences, Faculty of Tikrit University - College of Computer Science and Mathematics - Department of Mathematics, Iraq, she researches interest in applied mathematics, with a field of concentration of optimization include conjugate gradient, steepest descent methods, Broyden's family and Quasi Newton methods. She can be contacted at email: amera@st.tu.edu.iq.