Preconditioned successive over relaxation iterative method via semi-approximate approach for Burgers' equation

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ABSTRACT

This paper proposes the combination of a preconditioner applied with successive over relaxation (SOR) iterative method for solving a sparse and huge scale linear system (LS) in which its coefficient matrix is a tridiagonal matrix. The purpose for applying the preconditioner is to enhance the convergence rate of SOR iterative method. Hence, in order to examine the feasibility of the proposed iterative method which is preconditioner SOR (PSOR) iterative method, first we need to derive the approximation equation of one-dimensional (1D) Burgers' equation through the discretization process in which the second-order implicit finite difference (SIFD) scheme together with semi-approximate (SA) approach have been applied to the proposed problem. Then, the generated LS is modified into preconditioned linear system (PLS) to construct the formulation of PSOR iterative method. Furthermore, to analyze the feasibility of PSOR iterative method compared with other point iterative methods, three examples of 1D Burgers' equation are considered. In conclusion, the PSOR iterative method is superior than PGS iterative method. The simulation results showed that our proposed iterative method has low iteration numbers and execution time.

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1. INTRODUCTION

One-dimensional Burgers' equation is a well-known nonlinear partial differential equation (PDE) that has been extensively used in a variety field of mathematics, science and engineering [1]. In 1915, Bateman was the first one to introduce this nonlinear PDE in the field of fluid dynamics. Later in 1948, Burger's also practiced this equation to illustrate the model of mathematic in turbulence and since then this equation is extensively known as Burgers' equation [2]. In fact, this equation has captivated the attention of many researchers to study this equation due to the reason that this equation is actually the simplest form of nonlinear advection and dissipation terms for modelling the physical phenomenon of wave motion. Apart from this, Burgers' equation also emerges in other physical phenomenon such as shock waves, fluid waves and sound waves [3].

In recent years, extensive studies for this equation have been done efficiently to generate the numerical solutions in solving 1D Burgers' equation. There are various numerical approaches that have been presented to attain the approximate solution of 1D Burgers' equation such as finite difference (FD) method, finite element (FE) method, collocation method, differential quadrature method (DQM), semi-implicit FD scheme, Adomian decomposition method (ADM) and compact difference method (CDM). For instance, Mohamed [4] proposed a new fully implicit FD scheme in solving 1D and 2D Burgers' equation without using any transformation for the linearization. Rahman *et. al* [5] obtained the numerical results by using the different semi-implicit FD scheme. In 2018, Chen and Zhang [6] presented numerical solution which is obtained through the implementation of the weak Galerkin FE method. Besides, the collocation method is developed using modified cubic B-splines functions to acquire the solution of Burgers' problem [3]. In addition to deal with Burgers' problem, the application of modified ADM can also be used [2]. Aswin *et al.* [7] proposed the polynomial based DQM (PDQM) to get the approximation equation of Burgers' and quasilinearization to form a linear system. In 2019, Yang *et al.* [8] studied the high-order CDM for solving Burgers' problem.

Thus, this paper considers the second-order IFD scheme with the SA approach to obtain the approximation equation of Burgers' problem. To do this, first we need to derive the formulation of Burgers' equation by using the second-order implicit finite difference (SIFD) scheme in a way to generate an equivalent nonlinear system (NLS). To solve this NLS, there are several methods that can be taken into action such as newton approach in which it is used to construct an array of the equivalent linear system. To develop linear system, Aksan [9] applied the Newton method to transform the NLS of Burgers' equation into an equivalent linear system (LS). Nevertheless, the implementation of Newton approach requires the involvement of inner and outer iterations over the NLS which causes in higher computational complexity. For this reason, we avoid using such approach in order to attain low computational complexity. Therefore, we consider a method which is semi-implicit that is capable to alter any NLS into a sequence of LS [10]–[12].

From the discretization process of Burgers' equation, the semi-approximate implicit (SAI) approximation equation leads to a generated sparse and huge scale LS. Since the LS has these main features for the coefficient matrix, then the iterative methods are the most effective linear solver to seek the approximate solution of the proposed problem. Presently, there are many researchers have proposed and applied numerous iterative family such as Gauss-Seidel (GS) and successive over relaxation (SOR) iterative methods for solving LS [13]–[18]. Actually, these basic iterative methods are low in convergence rate, therefore, several researchers have modified a new variant of basic iterative methods such as preconditioned SOR (PSOR) iterative method in a way to increase the convergence rate of proposed iterative methods [19]–[21]. There has been a great deal of study into preconditioners, $P = (I + S)$ which can be used in accelerating the convergence rate of basic iterative method [22]–[24].

For instance, Milaszewicz [25] introduced a new preconditioner $P = (I + S')$ to enhance the convergence rate in which:

$$S' = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ -a_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ -a_{n1} & 0 & \cdots & 0 \end{bmatrix}.$$

Gunawerdana *et al.* [26] consider the classic preconditioner $P = (I + S)$ in solving the LS $Ax = b.$ The authors attempted to apply this preconditioner, P into the M-matrix in which it can increase the rate of convergence for Jacobi and GS iterations with:

$$S = \begin{bmatrix} 0 & -a_{12} & 0 & \cdots & 0 \\ 0 & 0 & -a_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -a_{n-1n} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

in addition, Cheng *et al.* [27] also presented the new preconditioner $P = (I + \beta S)$ in which β is a real number for solving the Z-matrix LS as:

$$S = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & 0 & \cdots & 0 \end{bmatrix}.$$

later, Sunarto and Sulaiman [28] applied the classic preconditioner $P = (I + S)$ into the approximate time-fractional equation with:

$$S = \begin{bmatrix} 0 & -r_1 a_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & -r_2 a_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & -r_3 a_{34} & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & -r_{m-1} a_{m-1m} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{(m-1)(m-1)} .$$

Therefore, to enhance the convergence rate of basic iterative method, we propose this preconditioner, $P = (I + S)$ together with SOR iterative method to acquire approximate solution of proposed problem. Then, the primary concept in this research is to evaluate the feasibility of PSOR iterative method where it is among the preconditioned iterative method to resolving LS generated via discretization process using SIFD scheme and SAI approach to obtain the SAI approximation equation of 1D Burgers' equation. The following general equation of 1D Burgers' equation are considered to examine the feasibility of PSOR iterative method together with SAI approximation:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial w} = \gamma \frac{\partial^2 v}{\partial w^2}, \tag{1}$$

with initial condition $v(w,0) = v(w), k \le w \le l$ and boundary conditions $v(k,t) = f_a(t), v(l,t) = f_b(t), t > 0$ where γ is viscosity and $v\frac{\partial v}{\partial w}$ is the term of nonlinear.

2. APPROXIMATION OF BURGERS' EQUATION

As mentioned in the previous section, in an effort to attain the corresponding NL approximation equation of problem (1), the SIFD scheme is imposed into the proposed problem (1). Then, the SA approach is imposed to derive the NL approximation equation to develop a LS. Before we proceed to discretization process, let problem (1) be defined which is used to:

$$\frac{\partial v}{\partial t} + F(z,t,v)\frac{\partial v}{\partial w} = \gamma \frac{\partial^2 v}{\partial w^2}. \tag{2}$$

then, we consider the segmentation of solution domain pointed out as $y_i, i = 0,1,2,\dots,m-1,m$ and $t_j = 0,1,2,\dots$. Then, the SIFD scheme is used to discretize over (2) to get the equivalent NL approximation equation expressed as [29], [30]:

$$\frac{v_{i.j+1} - v_{i.j}}{\Delta t} + f_{i.j}\left(v_{1.j+1}, v_{2.j+1}, \dots, v_{m-1.j+1}\right) = \frac{\gamma}{(\Delta h)^2}\left(v_{i-1.j+1} - 2v_{i.j+1} + v_{i+1.j+1}\right), \tag{3}$$

where,

$$f_{i.j}\left(y_{1.j+1}, y_{2.j+1}, \dots, y_{m-1.j+1}\right) = d\left(w_i, t_{j+1}, v_{i.j+1}, \frac{v_{i+1.j+1} + v_{i-1.j+1}}{2}\right) \tag{4}$$

Since there is NL term in (4), we require to get rid the NL term by using the SA approach [10]–[12] for the sake of developing a LS of Burgers' problem (1). To do this, the term $y_{i.j+1}$ in (4) is approximated by $y_{i.j}$ since the value of Δt is significantly low value. Thus, (4) can be converted to (5).

$$f_{i.j}\left(v_{1.j+1}, v_{2.j+1}, \dots, v_{m-1.j+1}\right) = d\left(w_i, t_{j+1}, v_{i.j}, \frac{v_{i+1.j+1} - v_{i-1.j+1}}{2\Delta h}\right) \tag{5}$$

For the simplicity purpose, we get the second-order SAI approximation equation for problem (1) expressed as:

$$-p_i v_{i-1.j+1} + s_i v_{i.j+1} - v_{i+1.j+1} = H_{i.j},\ i = 1,2,3,\dots,m-1, \tag{6}$$

where,

$$p_i = \frac{\left(\frac{1}{2(\Delta h)}\right)D_{i.j} + \frac{\gamma}{(\Delta h)^2}}{\beta_i},\ \ s_i = \frac{1 + \frac{2\gamma\Delta t}{(\Delta h)^2}}{\beta_i},\ \beta_i = \frac{\gamma}{(\Delta h)^2} - \left(\frac{1}{2(\Delta h)}\right)D_{i.j},\ \ H_{i.j} = \frac{v_{i.j}}{\Delta t.\beta_i}.$$

Next, based on the SAI approximation (6), it is clear that a series of LS at each time level $(j+1)$ can be expressed in the form of matrix as (7):

$$Lv_{j+1} = H_j, \quad (7)$$

where:

$$L = \begin{bmatrix} s_1 & -1 & & & & \\ -p_2 & s_2 & -1 & & & \\ & -p_3 & s_3 & -1 & & \\ & \ddots & \ddots & \ddots & & \\ & & -p_{m-2} & s_{m-2} & -1 & \\ & & & -p_{m-1} & s_{m-1} \end{bmatrix},$$

$$\underline{v}_{j+1} = \left[v_{1.j+1}, v_{2.j+1}, v_{3.j+1}, \dots, v_{m-1.j+1}\right]^T,$$

$$\underline{H}_j = \left[H_{1.j} + p_i v_{0.j+1}, H_{2.j}, H_{3.j}, \dots, H_{m-2.j}, H_{m-1.j+1} + v_{m.j+1}\right]^T$$

3. NUMERICAL METHODS

For instance, referring to the generated LS (7) which constructed through the combination of SIFD scheme and the SA approach in the section 2, indicates that the primary features of the coefficient tridiagonal matrix of the LS can be identified as sparse and huge scale. Previously, the first section describe that iteration families are the most effective linear solver for LS (7). PSOR iterative method is among the most effective point iterative methods that can be applied to enhance the convergence rate in obtaining the approximate solution compared with PGS iterative method [31]–[33]. Considering the benefit of PSOR iterative method, this paper proposes the PSOR iterative method by imposing the SA approach to solve the LS (7). Before applying the PSOR iterative method, let us reconstruct the original LS (7) into this PLS as follows [34], [35]:

$$L^* v_{j+1} = H_j^*, \quad (8)$$

where:

$$L^* = PL, H_j^* = PH_j.$$

In fact, the tridiagonal matrix P is recognized as the preconditioned matrix that can be represented as [26].

$$P = I + S, \quad (9)$$

Where:

$$S = \begin{bmatrix} 0 & \frac{1}{s_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{s_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{s_3} & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{s_{m-1}} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{(m-1)\times(m-1)},$$

and I is known as identical matrix of order $(m-1)$. To start in deriving the formulation of PSOR iterative method, let the coefficient matrix PL in (7) be rewritten into the summation of three matrices as:

$$L^* = X - T - G, \quad (10)$$

where C is lower triangular, W is diagonal and H is upper triangular matrices. Then, we consider (8) and (10) to construct the formulation of PSOR iterative method as [28]:

$$v_{j+1}^{(k+1)} = (X - \omega T)^{-1}[(1-\omega)X + G\omega]v_{j+1}^{(k)} + (X - \omega T)^{-1} H_j^*. \quad (11)$$

Algorithm 1. Depicts the execution of PSOR iterative method

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Algorithm 1: PSOR iterative method
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i. Set the starting value $P, B, F_j, v_0^{(0)}$ and $\varepsilon \leftarrow 10^{-10}$.

ii. For $j = 0, 1, 2, \dots, n-1$, execute

a) Set $v_{j+1}^{(0)} = 0$and boundary conditions.

b) Compute vector $B^* = PB$ and $F_j^* = PF_j$.

c) Compute new value of $v_{j+1}^{(k+1)}$ by applying

i. $i = 1$:

$v_1 = (1-\omega)v_{i,j+1}^{(k)} + \frac{\omega}{L_{1,1}}\left(H_1^* - L_{1,3}^* v_3\right)$.

ii. For $i = 2, 3, \dots, m-3$,

$v_i = (1-\omega)v_{i,j+1}^{(k)} + \frac{\omega}{L_{i,i}^*}\left(H_i^* - L_{i,i-1}^* v_{i-1} - L_{i,i+2}^* v_{i+2}\right)$.

iii. $i = m-2, m-1$:

$v_1 = (1-\omega)v_{i,j+1}^{(k)} + \frac{\omega}{L_{i,i}^*}\left(H_i^* - L_{i,i-1}^* v_{i-1}\right)$.

d) Execute the test of convergence, $\left|v_{i,j+1}^{(k+1)} - v_{i,j+1}^{(k)}\right| \le \varepsilon = 10^{-10}$. If satisfied, move to step e. If not return to step c.

e) Obtain the recent value, $v_j^{(k+1)}$.

iii. Present the approximate solution.

4. NUMERICAL EXPERIMENTS

By referring to the second-order SAI approximation equation (6), we deal with three examples of the proposed problem (1) to test the efficiency of PGS and PSOR iterative methods. To do this, we conducted the comparative analysis among the proposed iterations and PGS iterative method is appointed as a control method. The numerical results obtained by implementing both iterations in which PGS and PSOR have been evaluated according to three aspects seek as iteration number, computation time and maximum norm. Based on (12), we used this maximum norm as a stopping criterion and defined as:

$$\text{Max-norm} = \max_{n-1}^{i=1}\left|S\left(x_i, t_{j+1}\right) - v\left(x_i, t_{j+1}\right)\right|. \tag{12}$$

the three numerical examples of proposed problem (1) with their exact solutions are presented here. Example 1 [36], for this problem, we consider the initial value (IV) equation of proposed problem (1) as:

$$v(w, 0) = 2w, \text{ for } t > 0, \tag{13}$$

and the analytical solution of equation (13) is provided as:

$$v(w, t) = \frac{2w}{1+2t}. \tag{14}$$

Example 2 [37], let us consider proposed problem (1) with IV equation are taken from the analytical solution [38]:

$$\boldsymbol{v(w,t) = \frac{\gamma}{1+\gamma t}\left(w + tan\left(\frac{w}{2+2\gamma t}\right)\right),\ t \ge 0.} \tag{15}$$

Example 3 [39], for this problem, we consider the IV equation of proposed problem (1) as:

$$v(w, 0) = \frac{1}{2} - \frac{1}{2}tanh\left(\frac{1}{4}\left[w - \frac{1}{2}\right]t\right), \text{ for } t > 0, \tag{16}$$

and the analytical solution of (16) is provided as:

$$v(w, t) = \frac{1}{2} - \frac{1}{2}tanh\left(\frac{w}{4}\right). \tag{17}$$

5. DISCUSSION

This section is intended to describe the approximate solutions resulting through the implementation of PGS and PSOR iterative method via SA approach for all proposed examples 1, 2 and 3 differing in grid sizes, m=256, 512, 1024, 2048, and 4096. All the numerical results for PGS and PSOR iterative methods for all examples are recorded in Table 1 which represent the iteration number, computation time and maximum norm respectively. Based on the numerical results by implementing PGS and PSOR iterative methods as in Table 1, it indicates that PSOR iterative method have declined tremendously in aspect of iteration number. Meanwhile, PSOR iterative method required less computation time than PGS iterative method. The maximum norm for PGS and PSOR iterative methods are in a good agreement and the value for each proposed examples is proximate to its exact solution.

Table 1. Numerical results for all examples

Example	M	Iteration number		Computation time (second)		Maximum norm	
		PGS	PSOR	PGS	PSOR	PGS	PSOR
1	256	3372	239	3.35	0.30	2.12E-07	5.32E-09
	512	12354	472	24.29	1.06	8.51E-07	8.34E-09
	1024	449510	878	175.79	3.91	3.41E-06	8.83E-09
	2048	162056	1724	1279.47	14.72	1.36E-05	9.59E-09
	4096	577420	3763	9274.96	64.76	5.43E-05	3.03E-08
2	256	14	12	0.13	0.11	1.55E-09	4.61E-10
	512	36	23	0.14	0.12	9.62E-09	4.90E-09
	1024	113	46	0.48	0.26	5.73E-08	1.17E-08
	2048	393	92	3.12	0.86	2.50E-05	2.11E-08
	4096	1399	182	21.70	3.21	1.10E-06	3.58E-08
3	256	3436	236	3.39	0.29	9.86E-06	9.64E-06
	512	12633	422	24.50	0.95	1.05E-06	9.63E-06
	1024	46126	855	178.68	3.66	1.32E-05	9.64E-06
	2048	166895	1702	1283.47	14.43	2.38E-05	9.64E-06
	4096	597125	3695	9176.99	62.57	6.64E-05	9.65E-06

6. CONCLUSION

In this paper, we considered a preconditioned matrix of the type $P = I + S$ and we applied it into the LS (7) to transform into the PLS (8). Then, by considering the PLS (8), we successfully constructed the formulation of PSOR iterative method which derived by using the second-order SAI approximation equation. The comparative analysis between the PGS and PSOR iterative methods have been illustrated to examine the feasibility of PSOR iterative method. Based on the numerical result obtained by solving three examples of proposed problem (1), it clearly shows that the PSOR iterative method is more superior than PGS iterative method in terms of iteration numbers and computation time. Meanwhile, the accuracy for proposed iterative method is in a good agreement with PGS iterative method. In the future, this paper should be extended by implementing the same discretization scheme to solve the nonlinear problem (1) via the block iteration families.

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