A new properties of fuzzy b-metric spaces

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ABSTRACT

Metric spaces are specific types of topological spaces with pleasing "geometric" characteristics and they have a number of appealing properties and are commonly used in both pure and applied sciences. In this work, the structure of cartesian product space in the setting of a fuzzy b-metric space (Fb-M space) framework is introduced, which is an extension allows to create the large-scale structure for the space of the type fuzzy b-metric. The possibility of transferring some of the results and important features related to Fb-M space to this suggested space are discussed and demonstrated. The Cartesian product of two Fb-M spaces is proved as Fb-M space, this allows, to investigate and prove the topological expectation on the fuzzy product b-metric space. Furthermore, certain specific fuzzy b-metrics on F^2, and fuzzy Euclidean plane are obtained in this way.

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1. INTRODUCTION

Metric spaces theory has been studied in a variety of ways in recent years, one of which is the fuzzy situation. In the paper, Zadeh [1] was the first to present the idea of fuzzy sets. Many eminent authors have introduced the theory of fuzzy sets and its applications to use this term in topology and analysis. A fuzzy metric space is well known to be a useful generalization and is a subject that scientists and mathematicians are very interested in. A number of authors have introduced fuzzy metric spaces in various methods. For example, the continuous t-norm was used to define the fuzzy metric space in [2], In the paper, George and Veeramani [3] modified the definition of fuzzy metric space and established the Hausdorff topology of a fuzzy metric space. Furthermore, in the paper, Dhange [4] tried to generalize the idea of usual metric and introduce the notion of D-metric to convert more results from usual to D-metric. Subsequently, Mustafa and Sims [5] offered the idea of G-metric, in which the tetrahedral inequality was substituted with an inequality using indices recurrence. Following this novel approach, the concept of Q-fuzzy metric and studied some applications in this space were developed by [6]–[9]. As a result of the strong correlation between the theories of fuzzy metric spaces and fuzzy normed spaces, Khan in [10] proposed the concept of generalized call (G-norm). Depending on the notions of Q-fuzzy metric and G-norm, the space of G- fuzzy normed was presented in [11]. The concept of fuzzy norm has been used in developing fuzzy functional analysis and a wide variety of domains see [12]–[16] for reference. Besides, fuzzy metric spaces, there are various extensions of the metric and metric space notions. Some problems, particularly the convergence of measurable functions with respect to measure, lead to a broadening of the concept of metric. In the paper, Czerwik [17] offered a very intriguing expansion of the notion of a metric, termed b-metric. Czerwik discovered some fixed point conclusions in this pioneer paper, including the analog of the Banach contraction principle in the framework of complete b-metric spaces. In the framework of b-metric spaces, the existence (and uniqueness) of (common) fixed points of diverse classes of single-valued and multi-valued operators. has been observed in the following works (see, e.g., [18]–[23] and the related references therein). Following the development of fuzzy b-metric space, the notion of Cartesian product space on fuzzy b-metric spaces is introduced. On the other hand, the concept of a fuzzy b-metric space was presented in [24], which generalizes the fuzzy metric space and b-metric space. Meanwhile, [25]–[29], for example, researchers have taken a keen interest in developing and enhancing various findings in b-metric spaces. The main goal of this research is to study and present a new Cartesian product approach on a fuzzy framework, which has yielded some interesting results.

The following is the structure of the paper. Section 2 is devoted to recalling previously known definitions, information, and preliminary findings that will be used in this work. In section 3, a characterization of the Cartesian product in the context of fuzzy b-metric space (Fb-M space) is presented, as well as some inherited important characteristics related with (Fb-M space). A fuzzy Euclidean plane (FE-plane) are given in section 4, moreover, any FE-plane is complete is proved.

2. PRELIMINARIES AND BASIC RESULTS

First, we recall some basic definitions and notions, which are essential for this work;

Definition 2.1. [16]: A binary operation $\odot: [0,1] \times [0,1] \rightarrow [0,1]$ is called a continuous triangular nor (t-norm) if $([0,1], \leq, \odot)$ is an ordered abelian topological monoid with 1.

Lemma 2.2. [16]: Any continuous triangular norm $\odot$ satisfies that for all $c, 0 < c < 1$ there is a, b, $0 < a, b < 1$ such that $a \odot b = c$

Definition 2.3. [16]: Let $\odot$ and $\otimes$ are two continuous triangular norm then $\otimes$ is said to be dominates $\odot$ and denote by $\otimes \gg \odot$ if for all $a, b, c, d \in [0,1]$ $(a \odot b) \otimes (c \odot d) \geq (a \otimes c) \odot (b \otimes d)$

The definition of a Fb-M space is now given.

Definition 2.4. [23]: A quadruple $(X, M, \odot, k)$ is called a fuzzy b-metric space (Fb-M space) if X is a nonempty set, $k \geq 1$ be a given real number, $\odot$ is continuous triangular norm and M is a fuzzy set in X^2 $\times [0, \infty)$ with the following conditions hold each $x, y, z \in X$

(bM1) $M(x, y, 0) = 0$
(bM2) [For all $t > 0$, $M(x, y, t) = 1$] if and only if $x = y$
(bM3) For all $t \geq 0$ $M(x, y, t) = M(y, x, t)$
(bM4) For all $t, s \geq 0$, $M(x, z, k(t + s)) \geq M(x, y, t) \odot M(y, z, s)$
(bM5) $M(x, y, .): [0, \infty) \rightarrow$ I is left continuous and $\lim_{t \to \infty} M(x, y, t) = 1$

Theorem 2.5. [23]: Let $(X, M, \odot, k)$ be a Fb-M space, then for $x \in X, 0 < б < 1, t > 0$ an open ball is defined by $Ɓ(x, б, t) = \{y \in X : M(x, y, t) > 1 - б\}$. Then $Ţ_M = \{T \subset X : x \in T$ if and only if there is $0 < б < 1, t > 0: Ɓ(x, б, t) \subseteq T\}$ is a topology on X.

In the context of fuzzy b-metric space, the convergent sequence and Cauchy sequence can be presented as given in S. Nădăban's notions [24]

Definition 2.6. [23]: In a Fb-M space $(X, M, \odot, k)$, a sequence(x_n) is said to be
(i) convergent if there exists $x \in X$ such that $\lim_{n \to \infty} M(x_n, x, t) = 1$ for all $t > 0$
(ii) Cauchy if $\forall$ б $\in (0,1)$, $\forall\, t > 0$ there is $n_\circ \in \mathbb{N} : M(x_n, x_m, t) > 1 - б$ for all $n, m \geq n_\circ$

Theorem 2.7. [23]: Let the space $(X, M, \odot, k)$ be a fuzzy quasi-pseudo-b-metric and $d_\alpha(x, y) = \inf\{t > 0 : M(x, y, t) > \alpha\}$, where $0 < \alpha < 1$. Then Ɗ$= \{d_\alpha\}_{\alpha \in (0,1)}$ is an ascending family of quasi-pseudo-b-metrics on X.

3. COMPLETENESS OF THE CHARACTERIZATION $(X_1 \times X_2, \widetilde{M}, \otimes, k)$

In his section, a fuzzy b-metric structure defined on the Cartesian product of two Fb-M spaces of type $(X_1 \times X_2, \widetilde{M}, \otimes, k)$ is given. The topological expectation is constructed for such product. The suggested idea $(X_1 \times X_2, \widetilde{M}, \otimes, k)$ is proved a complete Fb-M space. To obtain this completeness, the Cartesian product of two Fb-M spaces is likewise a Fb-M space will be first proved. A description of the Cartesian product in terms of Fb-M space is given the following definition;

Definition 3.1: The application $\widetilde{M} = X_1 \times X_2 \times [0,\infty) \rightarrow [0,1]$ is a Fb-M space on the Cartesian product of the sets X_1 and X_2 defined by $\widetilde{M}((x_1, x_2), (y_1, y_2), t) = M_1(x_1, y_1, t) \otimes M_2(x_2, y_2, t), \forall, t > 0$ and $(x_1, x_2), (y_1, y_2) \in X_1 \times X_2$ named fuzzy product b-metric, where $\otimes$ is continuous t-norm with $\otimes \gg \odot$ then the notion $(\mathbf{X_1} \times \mathbf{X_2}, \widetilde{\mathbf{M}}, \otimes, \mathbf{k})$ is known as the product space.

The Cartesian product of two Fb-M spaces with the same continuous t-norm ($\odot$) a Fb-M space is proved, according to the next theorem.

Theorem 3.2: Suppose that $(X_1, M_1, \odot, k)$ and $(X_2, M_2, \odot, k)$ are two Fb-M spaces, if $\otimes$ is a continuous t-norm with $\otimes \gg \odot$ then the space $(X_1 \times X_2, \widetilde{M}, \otimes, k)$ is a Fb-M.

Proof: For all $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X_1 \times X_2$, we will prove
(bM1) $\widetilde{M}((x_1, x_2), (y_1, y_2), 0) = M_1(x_1, y_1, 0) \otimes M_2(x_2, y_2, 0) = 0$.
Since $M_1(x_1, y_1, 0) = 0$ and $M_2(x_2, y_2, 0) = 0$
(bM2) for all $t > 0$; if $x_1 = y_1$ and $x_2 = y_2$ then
$\widetilde{M}((x_1, x_2), (y_1, y_2), t) = M_1(x_1, y_1, t) \otimes M_2(x_2, y_2, t) = 1 \otimes 1 = 1.$
If $\widetilde{M}((x_1, x_2), (y_1, y_2), t) = 1$ then $M_1(x_1, y_1, t) \otimes M_2(x_2, y_2, t) = 1$. Since we have $\wedge \gg \otimes$ this implies that $1 \leq M_1(x_1, y_1, t) \wedge M_2(x_2, y_2, t) \Longmapsto M_1(x_1, y_1, t) = 1$ and $M_2(x_2, y_2, t) = 1$ it means $x_1 = y_1$ and $x_2 = y_2$.

(bM3) $$\begin{aligned} M((x_1, x_2), (y_1, y_2), t) &= M_1(x_1, y_1, t) \otimes M_2(x_2, y_2, t) \\ &= M_1(y_1, x_1, t) \otimes M_2(y_2, x_2, t) \\ &= \widetilde{M}((y_1, y_2), (x_1, x_2), t) \text{ for all }, t > 0 \end{aligned}$$
(bM4) $$\begin{aligned} \widetilde{M}((x_1, x_2), (z_1, z_2), k(t+s)) &= M_1(x_1, z_1, k(t+s)) \otimes M_2(x_2, z_2, k(t+s)) \\ &\geq [M_1(x_1, y_1, t) \odot M_1(y_1, z_1, s)] \otimes [M_2(x_2, y_2, t) \odot M_2(y_2, z_2, s)] \\ &\geq [M_1(x_1, y_1, t) \otimes M_2(x_2, y_2, t)] \odot [M_1(y_1, z_1, s) \otimes M_2(y_2, z_2, s)] \\ &= \widetilde{M}((x_1, x_2), (y_1, y_2), t) \odot \widetilde{M}((y_1, y_2), (z_1, z_2), s) \text{ for all }, t, s > 0 \end{aligned}$$

(bM5) let t_n be a sequence in $[0,\infty)$ with t_n converges to t. Now For all $(x_1, x_2), (y_1, y_2) \in X_1 \times X_2$
$$\begin{aligned} \lim_{n\to\infty} \widetilde{M}((x_1, x_2), (y_1, y_2), t_n) &= \lim_{n\to\infty} M_1(x_1, y_1, t_n) \otimes \lim_{n\to\infty} M_2(x_2, y_2, t_n) \\ &= M_1(x_1, y_1, t) \otimes M_2(x_2, y_2, t) = \widetilde{M}((x_1, x_2), (y_1, y_2), t) \end{aligned}$$
Hence, $\widetilde{M}((x_1, x_2), (y_1, y_2), t_n)$ converges to $\widetilde{M}((x_1, x_2), (y_1, y_2), t)$
That is, $\widetilde{M}((x_1, x_2), (y_1, y_2), .): [0,\infty) \rightarrow I$ is left continuous and
$\lim_{t\to\infty} \widetilde{M}((x_1, x_2), (y_1, y_2), t) = \lim_{t\to\infty} M_1(x_1, y_1, t) \otimes \lim_{t\to\infty} M_2(x_2, y_2, t) = 1 \otimes 1 = 1.$

The next result explains the product topology for two Fb-M spaces.

Theorem 3.3: Let T_{M_1} is a topology on the space $(X_1, M_1, \odot, k)$ of Fb-M, T_{M_2} is a topology on another space $(X_2, M_2, \odot, k)$ of fuzzy b-metric under the same continuous t-norm. If $\widetilde{M}$ is a fuzzy product b-metric, then $\mathrm{T}_{\widetilde{M}}$ is the product topology on $X_1 \times X_2$.

Proof: Firstly, we will prove that for each $0 < б < 1, t > 0$ there exist $0 < б_1 < 1$, $0 < б_2 < 1$ and there exists $t_1 > 0, t_2 > 0$ such that $\mathrm{B}(y_1, б_1, t_1) \times \mathrm{B}(y_2, б_2, t_2) \subset \mathrm{B}((y_1 \times y_2), б, t)\ \forall\ y_1 \times y_2 \in X_1 \times X_2$. Suppose that $0 < б < 1, t > 0$ so From Lemma (2.2) there are $0 < б_1 < 1$, $0 < б_2 < 1$ with $(1 - б_1) \odot (1 - б_2) = 1 - 0.5б$. Putting $t = t_1 = t_2 > 0$, for each $(x_1, x_2) \in \mathrm{B}(y_1, б_1, t_1) \times \mathrm{B}(y_2, б_2, t_2)$, we get $\widetilde{M}((x_1, x_2), (y_1, y_2), t) = M_1(x_1, y_1, t) \odot M_1(x_2, y_2, t)$

$\geq (1 - б_1) \odot (1 - б_2) = 1 - 0.5б > 1 - б$

Conversely, let $y_1 \in X_1, y_2 \in X_2$ we want to prove that for each $0 < б_1 < 1$, $0 < б_2 < 1$ and $t_1 > 0, t_2 > 0$ there are $0 < б < 1$ and $t > 0$ with $B((y_1 \times y_2), б, t) \subset B(y_1, б_1, t_1) \times B(y_2, б_2, t_2)\ \forall y_1 \in X_1, y_2 \in X_2$. Assume that $0 < б_1 < 1$, $0 < б_2 < 1, t_1 > 0, t_2 > 0$ then for $б = \min\{б_1, б_2\}$ and $t = \min\{t_1, t_2\}$, and for each $(x_1, x_2) \in B((y_1 \times y_2), б, t)$, we obtain $x_1 \in B(y_1, б_1, t_1)$ and $x_2 \in B(y_2, б_2, t_2)$.Hence from $M_1(x_1, y_1, t) \odot M_1(x_2, y_2, t) > 1 - б \geq 1 - б_j, j \in \{1,2\}$ the result is $M_1(x_1, y_1, t) > (1 - б_1)$ (because if it isn't we get $M_1(x_1, y_1, t) \odot M_1(x_2, y_2, t) \leq (1 - б_1) \odot 1 = 1 - б_1$) and which implies.

that $M_1(x_2, y_2, t) > (1 - б_2)$ by analogously. Therefore $M_1(x_1, y_1, t) \geq M_1(x_1, y_1, t) > (1 - б_1)$ and $M_2(x_2, y_2, t) \geq M_1(x_2, y_2, t) > (1 - б_2)$

The following result discusses the sequence's limit point attribute in a Fb-M space of type $(X_1 \times X_2, \widetilde{M}, \otimes, k)$.

Proposition 3.4: Let x be a limit point of (x_n) in a Fb-M space $(X_1, M_1, \odot, k)$ and $\hat{x}$ be a limit point of $(\hat{x}_n)$ in a Fb-M space $(X_2, M_2, \odot, k)$. If $\otimes$ is a continuous t-norm with $\otimes \gg \odot$ then $(x, \hat{x})$ be a limit point of the $(x_n, \hat{x}_n)$ in the space $(X_1 \times X_2, \widetilde{M}, \otimes, k)$

Proof: The space $(X_1 \times X_2, \widetilde{M}, \otimes, k)$ is Fb-M by Theorem (3.2). Since x be a limit of (x_n), thus for all $t > 0$, $\lim_{n\to\infty} M_1(x_n, x, t) = 1$ and since $\hat{x}$ be a limit of (x_n), thus $\lim_{n\to\infty} M_2(\hat{x}_n, \hat{x}, t) = 1$. Now for all $t > 0$,
$\lim_{n\to\infty} \widetilde{M}((x_n, \hat{x}_n), (x, \hat{x}), t) = \lim_{n\to\infty} M_1(x_n, x, t) \otimes \lim_{n\to\infty} M_2(\hat{x}_n, \hat{x}, t) = 1 \otimes 1 = 1$.
Hence $\widetilde{M}((x_n, \hat{x}_n), (x, \hat{x}), t) = 1$, it means that $(x, \hat{x})$ is the limit point of $(x_n, \hat{x}_n)$.

Proposition 3.5
Let the sequence (x_n) is Cauchy in a Fb-M space $(X_1, M_1, \odot, k)$ and the sequence $(\hat{x}_n)$ is Cauchy in a Fb-M space $(X_2, M_2, \odot, k)$. If $\otimes$ is a continuous t-norm with $\otimes \gg \odot$ then $(x_n, \hat{x}_n)$ is Cauchy sequence in the space $(X_1 \times X_2, \widetilde{M}, \otimes, k)$

Proof: Since $(X_1 \times X_2, \widetilde{M}, \otimes, k)$ is the space with a Fb-M notion by Theorem (3.2). Our assumption that (x_n) and $(\hat{x}_n)$ are two Cauchy sequences, thus $\forall\ 0 < б_1, б_2 < 1$ and $t_1, t_2 > 0$, $\exists\, n_\circ \in \mathbb{N}: M_1(x_n, x_m, t_1) > 1 - б_1$ and $M_2(\hat{x}_n, \hat{x}_m, t_2) > 1 - б_2$ for all $n, m \geq n_\circ$. Now, for all $n, m \geq n_\circ$.By taking $t = t_1 = t_2 > 0$. $\widetilde{M}((x_n, \hat{x}_n), (x_m, \hat{x}_m), t) = M_1(x_n, x_m, t) \otimes M_2(\hat{x}_n, \hat{x}_m, t) > (1 - б_1) \otimes (1 - б_2) = 1 - 0.5б > 1 - б$ for all $n, m \in n_\circ$. It means that $\widetilde{M}((x_n, \hat{x}_n), (x_m, \hat{x}_m), t) > 1 - б$. Therefore, $(x_n, \hat{x}_n)$ is Cauchy in $(X_1 \times X_2, \widetilde{M}, \otimes, k)$.

The Cartesian product of two complete Fb-M spaces is a complete Fb-M space, as shown by the following theorem.

Theorem 3.6: Let $(X_1, M_1, \odot, k)$ and $(X_2, M_2, \odot, k)$ are complete Fb-M spaces. If $\otimes$ is a continuous t-norm with $\otimes \gg \odot$ then the space $(X_1 \times X_2, \widetilde{M}, \odot, k)$ is complete Fb-M.

Proof: The space $(X_1 \times X_2, \widetilde{M}, \odot, k)$ is fuzzy b-metric by Theorem (3.2). Suppose that $(x_n, \hat{x}_n)$ is Cauchy in $(X_1 \times X_2)$, that is, for all $б \in (0,1)$, $\forall\, t > 0$ there is $n_\circ \in \mathbb{N}$: $\widetilde{M}((x_n, \hat{x}_n), (x_m, \hat{x}_m), t) > 1 - б$, for all $n, m \geq n_\circ$. Since $б \in (0,1)$, then from Lemma (2.2) there is $0 < б_1, б_2 < 1$ with $(1 - б_1) \otimes (1 - б_2) = 1 - б$, which implies that $\widetilde{M}((x_n, \hat{x}_n), (x_m, \hat{x}_m), t) = M_1(x_n, x_m, t) \otimes M_2(\hat{x}_n, \hat{x}_m, t) > (1 - б_1) \otimes (1 - б_2)$ so that $M_1(x_n, x_m, t) > (1 - б_1)$ and $M_2(\hat{x}_n, \hat{x}_m, t) > (1 - б_2)$. Therefore, (x_n) is Cauchy in $(X_1, M_1, \odot, k)$ and $(\hat{x}_n)$ is Cauchy in $(X_2, M_2, \odot, k)$. But the spaces $(X_1, M_1, \odot, k)$ and $(X_2, M_2, \odot, k)$ are complete Fb-M, hence there is x in X_1 and $\hat{x}$ in X_2 such that x is the limit point of (x_n) and $\hat{x}$ is the limit point of $(\hat{x}_n)$, it means that $\lim_{n\to\infty} M_1(x_n, x, t) = 1$ and $\lim_{n\to\infty} M_2(\hat{x}_n, \hat{x}, t) = 1$ for all $t > 0$.
Now, $\lim_{n\to\infty} \widetilde{M}((x_n, \hat{x}_n), (x, \hat{x}), t) = \lim_{n\to\infty} M_1(x_n, x, t) \otimes \lim_{n\to\infty} M_2(\hat{x}_n, \hat{x}, t) = 1 \otimes 1 = 1$
Thus $(x, \hat{x})$ is the limit point of $(x_n, \hat{x}_n)$. Therefore, $(X_1 \times X_2, \widetilde{M}, \odot, k)$ is complete Fb-M space.

Theorem 3.7 : If $(X_1 \times X_2, \widetilde{M}, \otimes, k)$ is a Fb-M space then $(X_1, M_1, \odot, k)$ and $(X_2, M_2, \odot, k)$ are Fb-M spaces by defining $M_1(x_1, y_1, t) = \widetilde{M}((x_1, 0), (y_1, 0), t)$ and $M_2(x_2, y_2, t) = \widetilde{M}((0, x_2), (0, y_2), t)$ for all $x_1, y_1 \in X_1$, $x_2, y_2 \in X_2, t > 0$.

Proof: For each $x_1, y_1 \in X_1, x_2, y_2 \in X_2$
(bM1) $M_1(x_1, y_1, t) = \widetilde{M}((x_1, 0), (y_1, 0), 0) = 0$
(bM2) for all $t > 0$, $M_1(x_1, y_1, t) = 1$ if and only if for all $t > 0$, $\widetilde{M}((x_1, 0), (y_1, 0), t) = 1 \Leftrightarrow$
$(x_1, 0) = (y_1, 0)$ if and only if $x_1 = y_1$
(bM3) For all $t \geq 0$ $M_1(x_1, y_1, t) = \widetilde{M}((x_1, 0), (y_1, 0), t) = \widetilde{M}((y_1, 0), (x_1, 0), t) = M_1(y_1, x_1, t)$
(bM4) For all $t, s \geq 0$, $x_1, y_1, z_1 \in X_1$

$$\begin{aligned} M_1(x_1, z_1, k(t+s)) &= \widetilde{M}((x_1, 0), (z_1, 0), k(t+s)) \\ &\geq \widetilde{M}((x_1, 0), (y_1, 0)t) \odot \widetilde{M}((y_1, 0), (z_1, 0), s) \\ &= M_1(x_1, y_1, t) \odot M_1(y_1, z_1, s) \end{aligned}$$

(bM5) $M_1(x_1, y_1, .) = \widetilde{M}((x_1, 0), (y_1, 0), .): [0, \infty) \rightarrow$ I is left continuous and $\lim_{t \to \infty} M_1(x_1, y_1, t) = 1$
As a result, $(X_1, M_1, \odot, k)$ is a Fb-M space.
Similarly, we can also demonstrate that $(X_2, M_2, \odot, k)$ is a Fb-M space.

Theorem 3.8: If $(X_1 \times X_2, \widetilde{M}, \otimes, k)$ is a complete Fb-M space then $(X_1, M_1, \odot, k)$ and $(X_2, M_2, \odot, k)$ are complete Fb-M spaces where $M_1(x_1, y_1, t) = \widetilde{M}((x_1, 0), (y_1, 0), t)$ and $M_2(x_2, y_2, t) = \widetilde{M}((0, x_2), (0, y_2), t)$ for all $x_1, y_1 \in X_1$, $x_2, y_2 \in X_2$, $t > 0$.

Proof: $(X_1, M_1, \odot, k)$ and $(X_2, M_2, \odot, k)$ are Fb-M spaces according to Theorem (3.7). Let (x_{1n}) be a Cauchy sequence in $(X_1, M_1, \odot, k)$ that is if $\forall\, б \in (0,1)$, $\forall\, t > 0$ there is $n_\circ \in \mathbb{N}: M_1(x_{1n}, x_{1m}, t) > 1 - б$ for all $n, m \geq n_\circ$. Now, for $б \in (0,1)$, $t > 0$ there is $n_\circ \in \mathbb{N}$: $\widetilde{M}((x_{1n}, 0), (x_{1m}, 0), t) > 1 - б$ for all $n, m \geq n_\circ$, this implies that $(x_{1n}, 0)$ is Cauchy sequence in $X_1 \times X_2$. But $(X_1 \times X_2, \widetilde{M}, \otimes, k)$ is complete so $(x_{1n}, 0)$ converges to $(x_1, 0)$ in $X_1 \times X_2$, that is that $\lim_{n \to \infty} \widetilde{M}((x_{1n}, 0), (x_1, 0), t) = 1$ for all $t > 0$. Hence $\lim_{n \to \infty} M_1(x_{1n}, x_1, t) = \lim_{n \to \infty} \widetilde{M}((x_{1n}, 0), (x_1, 0), t) = 1$ for all $t > 0$, it means that (x_{1n}) converge to x_1 in X_1. As a result, $(X_1, M_1, \odot, k)$ is complete. Similarly, we can also demonstrate that $(X_2, M_2, \odot, k)$ is complete.

4. A FUZZY B-METRIC ON EUCLIDEAN PLANE

An application of our result to Fb-M space s is derived. The following constructive example demonstrates that $\mathbb{F}$ (the field of real or complex numbers) with some conditions is a Fb-M space.

Example 4.1: Let $M: \mathbb{F}^2 \times [0, \infty) \rightarrow [0,1]$ be a fuzzy set defined by $M(x, y, t) = (t - d(x, y)/t + d(x, y))$ for $t > d(x, y)$ and $M(x, y, t) = 0$ for $t \leq d(x, y)$. Then $(\mathbb{F}, M, \odot, k)$ is a Fb-M space, where $a \odot b = min\{a, b\}$

Proof: According to Definition 2.4, the conditions (bM1), (bM2) and (bM3) are directly satisfied.
Now, to verify the condition (bM4) for all $x, y, z \in \mathbb{F}$ and $t, s \in [0, \infty)$ the following cases it follows
a- If $t \leq d(x, y)$ or $s \leq d(y, z)$ or both then,

$M(x, z, k(t+s) \geq M(x, y, t) \odot M(y, z, s)$ Satisfied obviously

b- If $t > d(x, y)$ and $s > d(y, z)$ then,

$$\begin{aligned} M(x, z, k(t+s)) = \frac{k(t+s) - d(x, z)}{k(t+s) + d(x, z)} &\geq \frac{k(t+s) - k[d(x, y) + d(y, z)]}{k(t+s) + k[d(x, y) + d(y, z)]} \\ &= \frac{(t+s) - d(x, y) - d(y, z)}{(t+s) + d(x, y) + d(y, z)} \\ &\geq \min\left\{\frac{t - d(x, y)}{t + d(x, y)}, \frac{s - d(y, z)}{s + d(y, z)}\right\} \\ &= M(x, y, t) \odot M(y, z, s) \end{aligned}$$

It means that $M(x, z, k(t+s) \geq M(x, y, t) \odot M(y, z, s)$.
For (bM5), assume that $(t_n)$is a sequence in $[0, \infty)$ such that (t_n) converge to t, then $\forall\, x, y \in \mathbb{F}$ $\lim_{n \to \infty} M(x, y, t_n) = \lim_{n \to \infty} \frac{t_n - d(x, y)}{t_n + d(x, y)} = \frac{\lim_{n \to \infty} t_n - \lim_{n \to \infty} d(x, y)}{\lim_{n \to \infty} t_n + \lim_{n \to \infty} d(x, y)} = \frac{t - d(x, y)}{t + d(x, y)} = M(x, y, t)$.So $M(x, y, t_n)$ converge to

$M(x, y, t)$. Hence $M(x, y, .): [0, \infty) \to I$ is left continuous and $\lim_{t\to\infty} M(x, y, t) = \lim_{t\to\infty} \frac{t-d(x,y)}{t+d(x,y)} = \lim_{t\to\infty} \frac{\frac{t}{t}-\frac{d(x,y)}{t}}{\frac{t}{t}+\frac{d(x,y)}{t}} = 1$

The following theorem gives a new characterization of a convergence sequence that belongs to X in a Fb-M space.

Theorem 4.2: Let $(\mathbb{F}, M, \odot, k)$ be a Fb-M space and (x_n) be a sequence in $\mathbb{F}$. Then (x_n) is converges to x if and only if $\lim_{n\to\infty} d_\alpha(x_n, x) = 0$ for all $\alpha, 0 < \alpha < 1$.

Proof: Assume that (x_n) be a sequence in $\mathbb{F}$ with (x_n) is converges to x, this means that $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$.we take $\alpha \in (0,1)$ this implies that $\lim_{n\to\infty} M(x_n, x, t) > \alpha$ for all $t > 0$ so there is a positive integer $n_\circ(\alpha, t)$ with $M(x_n, x, t) > \alpha$ for all $n \geq n_\circ(\alpha, t) \to d_\alpha(x_n, x) < t$ *for all* $n \geq n_\circ(\alpha, t) \to \lim_{n\to\infty} d_\alpha(x_n, x) < t$ for all $t > 0 \to \lim_{n\to\infty} d_\alpha(x_n, x) = 0$ for all $0 < \alpha < 1$.

Conversely, let $\lim_{n\to\infty} d_\alpha(x_n, x) = 0$ for all $\alpha, 0 < \alpha < 1$ then corresponding to any $t > 0$ there is a positive integer $n_\circ(\alpha, t)$ with $d_\alpha(x_n, x) < t$ for all $n \geq n_\circ(\alpha, t)$ this implies that $M(x_n, x, t) > \alpha$ for all $n \geq n_\circ(\alpha, t)$ it means $\lim_{n\to\infty} M(x_n, x, t) > \alpha$ for all $0 < \alpha < 1 \to \lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$.

Lemma 4.3: Let $(\mathbb{F}, M, \odot, k)$ be a Fb-M space. Then there exists $0 < \alpha < 1$ with $d_\alpha(1, 1) \neq 0$

Proof: $d_\alpha(1, 1) = inf\,\{t > 0 : M(1, 1, t) > \alpha\}$. Our assumption that $d_\alpha(1, 1) = 0$ for all $0 < \alpha < 1$. Since $M(1, 1, t) > \alpha$ for all $0 < \alpha < 1$ and $t > 0$ then $M(1, 1, t) = 1$ for all $t > 0$, it means $1 = 0$ and this is contradiction.

Proposition 4.4: In a Fb-M space$(\mathbb{F}, M, \odot, k)$, a sequence (x_n) is convergent if and only if (x_n) is convergent in $(\mathbb{F}, |\cdot|)$.

Proof: A sequence (x_n) is convergent to x in Fb-M space $(\mathbb{F}, M, \odot, k) \leftrightarrow \lim_{n\to\infty} d_\alpha(x_n, x) = 0$ for all $\alpha, 0 < \alpha < 1 \leftrightarrow \lim_{n\to\infty} |x_n - x|\, d_\alpha(1, 1) = 0$ for all $\alpha, 0 < \alpha < 1 \leftrightarrow \lim_{n\to\infty} |x_n - x| = 0 \leftrightarrow (x_n)$ is convergent in $(\mathbb{F}, |\cdot|)$.

Definition 4.5: The quadruple $(\mathbb{F}^2, M, \odot, k)$ is called fuzzy Euclidean plane if $\odot$ is continuous t-norm, $k \in \boldsymbol{R}$ where$k \geq 1$ and $M: \mathbb{F}^2 \times \mathbb{F}^2 \times [0, \infty) \to [0,1]$ is a fuzzy b-metric defined by,

$$\widetilde{M}\left((x_1, x_2), (y_1, y_2), t\right) = M_1(x_1, y_1, t) \wedge M_2(x_2, y_2, t)$$

Where M_1 and M_2 are fuzzy b-metrics on $\mathbb{F}$ and note that for all fuzzy b-metrics $M_1$and M_2 are satisfied (bM4) with respect to a t-norm $(\odot)$.

Remark 4.6: Theorem 3.2 and the fact that $\wedge \gg \odot$ for any continuous t-norm Θ assure the precision of the previous definition, implying that M is certainly Fb-M space on $\mathbb{F}^2$.

Proposition 4.7: Let $(\mathbb{F}^2, M, \odot, k)$ be a fuzzy Euclidean plane. If (x_n) be a sequence in $\mathbb{F}^2$, then (x_n) converge to x in $(\mathbb{F}^2, M, \odot, k)$ if and only if (x_n) converge to x in $(\mathbb{F}^2, d)$ where d denotes the Euclidean metric on $\mathbb{F}^2$

Proof: Let (x_n) be a sequence in $\mathbb{F}^2$ such that (x_n) converge to x in $(\mathbb{F}^2, M, \odot, k)$ if and only if $M(x_n, x, t) \to 1$ *as* $n \to \infty$ for all $t > 0$ if and only if $M_1({x_n}^1, x^1, t) \wedge M_2({x_n}^2, x^2, t) \to 1$ *as* $n \to \infty$ if and only if $M_j({x_n}^j, x^j, t) \to 1$ *as* $n \to \infty$ for all $j = 1,2$ if and only if $|{x_n}^j - x^j| \to 0$ all $j = 1,2$ if and only if $d({x_n}^j, x^j) \to 0$

The behavior of a fuzzy Euclidean plane $(\mathbb{F}^2, M, \odot, k)$ is described by the following theorem.

Theorem 4.8: Any fuzzy Euclidean plane $(\mathbb{F}^2, M, \odot, k)$ is complete

Proof: Suppose that (x_n) is a Cauchy sequence in $(\mathbb{F}^2, M, \odot, k)$. Then,

$$\begin{aligned} \mathrm{d}_{\alpha,M}(x_n, x_m) &= inf\{t > 0: M(x_n, x_m, t) > \alpha\} \\ &= inf\{t > 0: M_1({x_n}^1, {x_m}^1, t) \wedge M_2({x_n}^2, {x_m}^2, t) > \alpha\} \\ &= inf\{t > 0: M_1({x_n}^1, {x_m}^1, t) > \alpha, M_2({x_n}^2, {x_m}^2, t) > \alpha\} \\ &\geq inf\{t > 0: M_j({x_n}^j, {x_m}^j, t) > \alpha\}, \text{ for all } j = 1,2 \end{aligned}$$

Hence $\mathrm{d}_{\alpha,M}(x_n, x_m) \geq d_{\alpha,M_j}({x_n}^j, {x_m}^j) = |{x_n}^j - {x_m}^j|\, d_{\alpha,M_j}(1, 1)$, for all $j = 1,2$. When Lemma (4.3) is applied to a fuzzy b-metric M_j, it follows that there exists $0 < \alpha_j < 1$with $d_{\alpha_j,M_j}(1, 1) \neq 0$. So $\mathrm{d}_{\alpha_j,M}(x_n, x_m) \geq |{x_n}^j - {x_m}^j|\, d_{\alpha,M_j}(1, 1)$, for all $j = 1,2$. Since (x_n) is a Cauchy in $(\mathbb{F}^2, M, \odot, k)$, then the $({x_n}^j)$ is a Cauchy in $(\mathbb{F}, d)$ is obtained for all $j = 1,2$. Hence $({x_n}^j)$ is converge to x^j all $j = 1,2$. Therefore, (x_n) is converge to $x = (x^1, x^2)$ in $(\mathbb{F}^2, d)$ and by Proposition (4.7), (x_n) is converge to x in $(\mathbb{F}^2, M, \odot, k)$ is obtained.

5. CONCLUSION

We have defined the product of two Fb-M spaces and proved that the product of two Fb-M spaces is also Fb-M space up to this point. The implication of two Fb-M spaces with the product topology is established and proven. We've also given a description of fuzzy b-metric space, the fuzzy Euclidean b-metric plane, based on this new structure.

REFERENCES

[1] L. A. Zadeh, “Fuzzy sets,” *Information and Control*, vol. 8, no. 3, pp. 338–353, Jun. 1965, doi: 10.1016/S0019-9958(65)90241-X.

[2] I. Kramosil and J. Michalek, “Fuzzy metrics and statistical metric spaces,” *Kybernetika*, vol. 11, no. 5, pp. 336–344, 1975, [Online]. Available: http://eudml.org/doc/28711.

[3] A. George and P. Veeramani, “On some results in fuzzy metric spaces,” *Fuzzy Sets and Systems*, vol. 64, no. 3, pp. 395–399, Jun. 1994, doi: 10.1016/0165-0114(94)90162-7.

[4] B. C. Dhage, “Generalised metric space and mappings with fixed point,” *Bull.Cal.Math.Soc*, vol. 82, pp. 329–336, 1992.

[5] Z. Mustafa and B. Sims, “A new approach to generalized metric spaces,” *Journal of Nonlinear and convex Analysis*, vol. 7, no. 2, pp. 289–297, 2006.

[6] S. Qasim Hasan and T. Shihab Ahmed, “Some common random fixed point theorems in s-metric space with new type of rational contractive conditions,” *Eng. &Tech.Journal*, vol. 3, no. 1, p. 3, 2014.

[7] A. S. Al-Fhaid and S. A. Mohiuddine, “On the Ulam stability of mixed type QA mappings in IFN-spaces,” *Advances in Difference Equations*, vol. 2013, no. 1, Jul. 2013, doi: 10.1186/1687-1847-2013-203.

[8] M. Jleli and B. Samet, “A generalized metric space and related fixed point theorems,” *Fixed Point Theory and Applications*, vol. 2015, no. 1, Apr. 2015, doi: 10.1186/s13663-015-0312-7.

[9] S. Sedghi, A. Gholidahneh, and K. P. R. Rao, “Common fixed point of two r-weakly commuting mappings in Sb-metric spaces,” *Mathematical Sciences Letters*, vol. 6, no. 3, pp. 249–253, Sep. 2017, doi: 10.18576/msl/060305.

[10] K. A. Khan, “Generalized normed spaces and fixed point theorems,” *Journal of Mathematics and Computer Science*, vol. 13, no. 02, pp. 157–167, Nov. 2014, doi: 10.22436/jmcs.013.02.07.

[11] S. Chatterjee, T. Bag, and S. K. Samanta, Schauder-type fixed point theorem in generalized fuzzy normed linear spaces.,” *Mathematics,* vol. 120, no. 5, pp. 1295–1320, Sep. 2020, doi: 10.3390/MATH8101643.

[12] Mami Sharma and Debajit Hazarika, “On linear operators in Felbin’s fuzzy normed linear space,” *Annals of Fuzzy Mathematics and Informatics*, vol. 13, no. 6, pp. 749–758, Jun. 2017, doi: 10.30948/afmi.2017.13.6.749.

[13] I. N. Ahmed, “Controllability of nonlinear boundary value control systems in banach spaces using schauder fixed point theorem,” *Eng. &Tech.Journal*, vol. 33, no. 4, pp. 640–653, 2015, Accessed: Feb. 22, 2022. [Online]. Available: www.pdffactory.com

[14] M. N. Mohammedali, “Fuzzy real pre-Hilbert space and some of their properties,” *Baghdad Science Journal*, vol. 19, no. 2, pp. 313–320, Sep. 2022, doi: 10.21123/BSJ.2022.19.2.0313.

[15] S. Abed and Z. Mohamed Hasan, “Weak convergence of two iteration schemes in banach spaces,” *Engineering and Technology Journal*, vol. 37, no. 2B, pp. 32–40, May 2019, doi: 10.30684/etj.37.2b.1.

[16] T. Bînzar, F. Pater, and S. Nădăban, “On fuzzy normed algebras,” *Journal of Nonlinear Science and Applications*, vol. 9, no. 9, pp. 5488–5496, Sep. 2016, doi: 10.22436/jnsa.009.09.16.

[17] S. Czerwik, “Contraction mappings in b-metric spaces,” *Ostraviensis*, vol. 1, no. 1, pp. 5–115, 1993, Accessed: Feb. 22, 2022. [Online]. Available: http://dml.cz/dmlcz/120469%0Ahttp://project.dml.cz

[18] H. Aydi, M. F. Bota, E. Karapinar, and S. Mitrović, “A fixed point theorem for set-valued quasi-contractions in b-metric spaces,” *Fixed Point Theory and Applications*, vol. 2012, no. 1, May 2012, doi: 10.1186/1687-1812-2012-88.

[19] A. F. Roldán-López-De-Hierro, E. Karapinar, and S. Manro, “Some new fixed point theorems in fuzzy metric spaces,” *Journal of Intelligent and Fuzzy Systems*, vol. 27, no. 5, pp. 2257–2264, 2014, doi: 10.3233/IFS-141189.

[20] A. F. Roldán López de Hierro, A. Fulga, E. Karapınar, and N. Shahzad, “Proinov-type fixed-point results in non-Archimedean fuzzy metric spaces,” *Mathematics*, vol. 9, no. 14, p. 1594, Jul. 2021, doi: 10.3390/math9141594.

[21] A. F. Roldán-López-de-Hierro, E. Karapinar, and P. Kumam, “Irremissible stimulate on ‘Unified fixed point theorems in fuzzy metric spaces via common limit range property,’” *Journal of Inequalities and Applications*, vol. 2014, no. 1, Jul. 2014, doi: 10.1186/1029-242X-2014-257.

[22] H. Aydi, M. F. Bota, E. Karapinar, and S. Mitrović, "A common fixed point for weak øcontractions on b-metric spaces," *Fixed Point Theory*, vol. 13, no. 2, pp. 337–346, 2012.
[23] E. Karapinar and A. Fulga, "Fixed point on convex b-metric space via admissible mappings," *Twms Journal of Pure and Applied Mathematics*, vol. 12, no. 2, pp. 254–264, 2021.
[24] S. Nădăban, "Fuzzy b-metric spaces," *International Journal of Computers, Communications and Control*, vol. 11, no. 2, pp. 273–281, Jan. 2016, doi: 10.15837/ijccc.2016.2.2443.
[25] H. Afshari, H. Aydi, and E. Karaplnar, "On generalized α-ψ-Geraghty contractions on b-metric spaces," *Georgian Mathematical Journal*, vol. 27, no. 1, pp. 9–21, Mar. 2020, doi: 10.1515/gmj-2017-0063.
[26] E. Karapinar, A. Fulga, and A. Petrusel, "On istratescu type contractions in b-metric spaces," *Mathematics*, vol. 8, no. 3, p. 388, Mar. 2020, doi: 10.3390/math8030388.
[27] M. Jleli, E. Karapinar, and B. Samet, "On cyclic (ψ, φ)-contractions in Kaleva-Seikkala's type fuzzy metric spaces," *Journal of Intelligent and Fuzzy Systems*, vol. 27, no. 4, pp. 2045–2053, 2014, doi: 10.3233/IFS-141170.
[28] E. Karapinar and C. Chifu, "Results in wt-distance over b-metric spaces," *Mathematics*, vol. 8, no. 2, p. 220, Feb. 2020, doi: 10.3390/math8020220.
[29] M. A. Alghamdi, S. Gulyaz-Ozyurt, and E. Karapinar, "A note on extended Z-contraction," *Mathematics*, vol. 8, no. 2, p. 195, Feb. 2020, doi: 10.3390/math8020195.

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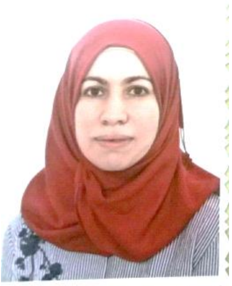

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