Indonesian Journal of Electrical Engineering and Computer Science
Vol. 27, No. 1, July 2022, pp. 538~545
ISSN: 2502-4752, DOI: 10.11591/ijeecs.v27.i1.pp538-545

A new direction search of hybrid quasi-Newton

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Article Info

Article history:

Received Nov 29, 2021
Revised May 19, 2022
Accepted Jun 8, 2022

Keywords:

Exact line search
Hybrid search direction
Numerical optimization
Quasi-Newton methods
Unconstrained optimization

ABSTRACT

A new hybrid quasi-Newton search direction (HQN^{EI}) is proposed. It uses the update formula of Broyden–Fletcher–Goldfarb–Shanno (BFGS) with a certain conjugate gradient (CG) parameter by a nested direction. The global convergence analysis and superlinear rate, addtionaly with sufficient descent are proved using exact line search. Finally, the computation comparisons are made with original hybrid parents; BFGS and CG, through the efficiency in terms of iteration numbers and CPU-running time showing the superior of HQN^{EI}. Therefore, the results marked preference of HQN^{EI} from other two producer algorithms.

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1. INTRODUCTION

Technology advancement requairs more efficient methods for solving its problems. For this purpose, scientiests in engineering, physics, chemistry and other branches try to find efficacious algorithms in obtaining optimal solution. Among the problems, there is a minimization unconstrained nonlinear optimization problem.

$$\min f(x), x \in \mathbb{R}^n \tag{1}$$

whenever, $f: \mathbb{R}^n \to \mathbb{R}^m$ is an objective twice continuously differentiable function. For the problem (1), minimum value can be obtained numerically when the analytical methods are stuck. This is done by iterative procedures with line search d_k and step size α_k from (2)

$$x_{k+1} = x_k + \alpha_k d_k \tag{2}$$

where, k is number of iteration with last step x_{k+1} represents minimum of (1) with desirable error in terms of current value x_k. According to the convergence properties and size scales with the algorithms efficiness, two approaches are more pupolar in solving (1); quasi-Newton methods (QNM) and CG methods. =Firstly, many researchers developed QNM as Broyden [1], Fletcher [2], Goldfrab[3], Shanno [4] and Powell [5] after Davidon introduced them [6]. These methods are used for minimizing (1) by employing an approximation Hessian matrix $\mathrm{H_k}$ in search direction $\mathrm{d_k}$. Broyden *et al.* [7] proved the iterative methods of quasi-Newton are locally q-linearly convergence when the generated iterative algorithm sequence of $\{H_k\}$ is bounded, but

Dennis and Moré [8] showed this sequence is convergent to the true Hessian matrix and for this it is required to be q-superlinear convergence rate. Broyden-Fletcher-Goldfrab-Shanno (BFGS) stands at methods . These methods use the direction search having Hessian matrix updated iteratively. It starts from a chosen positive defined matrix H_0, usually identity matrix, with updating it by (3):

$$H_{k+1} = H_k - \frac{H_k s_k s_k' H_k}{s_k' H_k s_k} + \frac{y_k y_k'}{s_k' y_k} \tag{3}$$

where,

$$s_k = x_k - x_{k-1} \, , y_k = g_k - g_{k-1} \tag{4}$$

This belongs to one parameter Broyden family form of rank two (for details see Fletcher [9]). Powell [5] in 1976 gave the Broyden–Fletcher–Goldfarb–Shanno (BFGS) methods global convergence analysis on convex objective function which results $\lim_{k\to\infty} \inf \|g_k\| = 0$ with any initial point and H_0 and it converges to the solution with positive Hessian matrix. This is obtained by superlinear convergent rates [5]. However, for non-convex function; the global convergent property was proved by Li and Fukushima [10]. The methods belong to Broyden class, the class of updating Hessian matrix approximation. Recently, there were many modifications and improvements are done on BFGS methods due to the dimension of problemes and usage memory, and updating the Hessian matrix approximation, for example, Andrei [11] used the divided difference in approximating the diagonial Hessian matrix. Self-scaling is anthor propreties of BFGS methods that some tries to modify, such as, Ali [12] modified a limited-memory of BFGS that are efficient in solving large scaling problems. On the other hand, the CG methods are large iterative methods. The first idea began with a quadratic objective function, which the method terminates at most n iterations along with exact line search. There was attempt to gain more efficient algorithms by combining the steepest descent with conjugate properties [9] (pp. 63). Hybridization is a procedure of combining algorithms to obtain new algorithm having more efficient properties from the parent algorithms.

Sofi *et al*. [13], there was the idea to hybrid the quasi-Newton method with steepest descent after modifying the step size given in [14], which presented a new step-size for the method. Sofi *et al*. [13], combined each of Davidon, Fletcher and Powell (DFP) and BFGS with conjugate gradient (CG) under exact line search satisfying Wolfe condition and they saw that there was an improvement in terms of performance. Ibrahim *et al*. [15] employed the new step size that presented by Yuan [14], and they used two stages in line search, firstly, they imposed exact line search then in second iteration; inexact line search with Armijo condition was used. Once more, Sofi *et al*. [16] in 2013 proposed bounded Hessian matrix approximation in search direction containing component of BFGS and fletcher-reeves (FR) conjugate gradient parameter with exact line search and both Wolfe's conditions. They showed an enhancement in computations [16]. Another hybridization was found between BFGS and conjugate descent direction and Armijo line search [17]. Furthermore, FR- parameter of CG is combined with DFP and exact line search [18]. The search directions hybridized of BFGS with CG are classified into two types. The first one is poor hybrid that is a linear combination of projection of Hessian matrix into gradient and parameter of different kinds of conjugate gradient, whereas the second one is combining all direction of CG and finding the optimal combining parameter. As a results, they found the optimal value of direction search hybrid parameter [19]. Also, new hybridization of BFGS with CG [20], that applied to the direction search parameter of Aini, Rivaie, and Mustafa (ARM) [21]. Moreover, there are many researches in hybrid between two CG algorithems to obtain more efficient algorithem. For instance, Jawdow and Al-Naemi [22] presended a convex combination parameter between two given methods of CG using inexact line search satisfying strong Wolfe condtion. Sulaiman *et al*. [23] proposed another hybrid basied on restart condition again with strong Wolfe condition. Hassan. *et al*. [24] employed the idea of convex combination in a hybrid parameter in CG algorithm before use it in a search direction. Finally, colonies such as bee and ant algorithms take apart of hybirtazation procedure [25]-[27]. As it has shown many works in hybrid between two algorithms, it is imperative to seek for more effective hybrid algorithm; accordingly, the HQN^{EI} algorithm is presented, in order to manged problems of continouse global developments.

2. METHOD OF THE PROPOSED HQN^{EI} WITH ITS ALGORITHM

In this work, we suggest HQN^{EI} which is a hybrid direction starts from the gradient projected by approximation Hessian matrix update of quasi-Newton of type BFGS. This will be for the first projection; however, the next projection is composited with gradient by adding preceding direction multiplied by a parameter. It is given as (5):

$$d_k = \begin{cases} -H_k g_k & k=0 \\ -H_k g_k + \eta(-g_k + \beta_k d_{k-1}) & k \geq 1 \end{cases} \tag{5}$$

where, $\eta \in (0,1]$.

$$\beta_k = \frac{\frac{\|g_{k-1}+g_k\|}{\|d_{k-1}\|^2}\|g_k\|^2-(g_k^T g_{k-1})}{\frac{\|g_{k-1}+g_k\|}{\|d_{k-1}\|^2}\|g_{k-1}\|^2} \tag{6}$$

This CG parameter is designed by Mouiyad, Mustafa, and Rivaie (MMR-CG) [28]. However, in this paper, β_k will have different values since it depends on previous direction computed by (5). Moreover, the exact line search is employed in the algorithm with HQN^{EI} direction. The algorithm along with HQN^{EI} direction will be compared to its original component. The following is the steps of $\mathrm{HQN^{EI}}$ algorithm to apply.

Step 1: Initializations: $\mathbf{H_0}$ identity matrix, $\mathbf{X_0} \in \mathbb{R}^\mathbf{n}$, tolerance $\boldsymbol{\varepsilon} = \mathbf{1} \times \mathbf{10^{-7}}$, $\mathbf{d_0} = -\mathbf{H_0 g_0}$, $\mathbf{k} = \mathbf{1}$
Step 2: Termination criteria, if $\|\mathbf{g_k}\| \leq \boldsymbol{\varepsilon}$ or maximum number of iteration reached **stop**
Step 3: Find exact step size $\boldsymbol{\alpha_k}$.
Step 4: Compute $\mathbf{s_k} = \mathbf{x_k} - \mathbf{x_{k-1}}$ and $\mathbf{y_k} = \mathbf{g_k} - \mathbf{g_{k-1}}$
Step 5: Update $\mathbf{H_k}$ by (3) and find $\boldsymbol{\beta_k}$ with (6)
Step 6: evaluate search direction by (5)
Step 7: set $\mathbf{k} = \mathbf{k} + \mathbf{1}$, go to step 2.

3. THE CONVERGENCE ANALYSIS OF HQN^{EI}

In this section, the convergence analysis is showed. The list of the requaried assumptions is assumed and presented. In addition, there are some properties that are important to analyze any suggested algorithm.

3.1. Some assumptions

The required assumptions to obtain desire solutions are assumed as shown in: i) The objective function f is twice continuously differentiable. ii) The Hessian matrix is Lipschitz continuous at x^*, that is, there exist a positive constant c satisfying;

$$\|G(x) - G(x^*)\| \leq c\|x - x^*\|$$

for all x in neighborhood of x^*. iii) If $f \in C^2$ and the level set $L = \{x: f(x) \leq f(x_0)\}$ is convex, there exist positive constants $c_1 and\ c_2$ satisfying;

$$c_1\|z\|^2 \leq z'G(x)\, z \ \leq c_2\|z\|^2 \ \forall z \in \Re^n , x \in L$$

and $G(x)$ is Hessian matrix of f.

3.2. Sufficient descent property of $\mathrm{HQN^{EI}}$.

The sufficient descent is the property which is required to guarantee the minimization process in problem (1). Mathematically, it is means that; $g_k^T d_k \leq -c\|g_k\|^2$, $\forall k \geq 0$. Additionally, this property leads to hold all the above assumptions. It is obvious from (5) and (6), we have:

$$d_k = \begin{cases} -H_k g_k & k=0 \\ -H_k g_k + \eta(-g_k + \beta_k d_{k-1}) & k \geq 1 \end{cases}$$

$$g_0^T d_0 = -g_0^T H_0 g_0$$

$\leq -c\|g_0\|^2$ by assumption 3.1 (iii)

$$g_k^T d_k = g_k^T(-H_k g_k + \eta(-g_k + \beta_k d_{k-1}))$$

$$\leq -k\|g_k\|^2 + \eta(-g_k^T g_k) + \eta g_k^T \beta_k d_{k-1}$$

since it is exact line search, $g_k^T d_{k-1} = 0$, then

$$g_k^T d_k \leq -c\|g_k\|^2$$

where, $c = k + \eta$

Hence, it is sufficient descent. To show the global convergence, the following lemma is needed.

3.2.1. Lemma

If α is generated by exact line search and let part (iii) of assumption 3.1 hold. Then;

$$\lim_{k\to\infty} \|g_k\| = 0$$

Proof: The prove of this lemma, it is referring to lemma 2 in [20]. There is the usage for the mean value theorem for a line search α considered as a solution and with the benfits of Lipschitz continuous property on the line search α. Hence we get the results.

3.2.2. Theore

Suppose the assumptions 3.1 and Lemma 3.2.1 is fulfilled. Then;

$$\lim_{k\to\infty} \inf \|g_k\| = 0$$

Proof: suppose $\|g_k\| > \delta \; \forall \; k \geq 0$;

$$d_k = -H_k g_k + \eta(-g_k + \beta_k d_{k-1})$$
$$\|d_k\| = \|-H_k g_k + \eta(-g_k + \beta_k d_{k-1})\|$$
$$\leq \|H_k\|\|g_k\| + \eta\|g_k\| + \eta|\beta_k|\|d_{k-1}\|$$
$$= (\|H_k\| + \eta)\|g_k\| + \eta|\beta_k|\|d_{k-1}\|$$
$$\|d_k\| \leq \|H_k + \eta\|\|g_k\| + \eta|\beta_k|\|d_{k-1}\|$$
$$\frac{\|d_k\|}{\|g_k\|^2} \leq \frac{\|H_k + \eta\|}{\|g_k\|} + \frac{\eta|\beta_k|\|d_{k-1}\|}{\|g_k\|^2}$$

but, we have:

$$\beta_k \leq \frac{\frac{\|g_{k-1}+g_k\|}{\|d_{k-1}\|^2}\|g_k\|^2}{\frac{\|g_{k-1}+g_k\|}{\|d_{k-1}\|^2}\|g_{k-1}\|^2} = \frac{\|g_k\|^2}{\|g_{k-1}\|^2}$$

So,

$$\frac{\|d_k\|}{\|g_k\|^2} \leq \frac{\|H_k + \eta\|}{\|g_k\|} + \frac{\eta\frac{\|g_k\|^2}{\|g_{k-1}\|^2}\|d_{k-1}\|}{\|g_k\|^2}$$
$$= \frac{\|H_k + \eta\|}{\|g_k\|} + \frac{\eta\|d_{k-1}\|}{\|g_{k-1}\|^2}$$
$$= \frac{\|H_k\| + \eta}{\|g_k\|} + \frac{\eta\|d_{k-1}\|}{\|g_{k-1}\|^2}$$

since H_k is bounded, then $H_k < u$,

$$\frac{\|d_k\|}{\|g_k\|^2} \leq \frac{u+\eta}{\|g_k\|} + \frac{\eta\|d_{k-1}\|}{\|g_{k-1}\|^2}$$
$$\leq \frac{u+\eta}{\|g_k\|} + \frac{1}{\|g_{k-1}\|} \leq \sum_{i=1}^{k} \frac{u+\eta}{\|g_i\|} + \frac{1}{\|g_0\|}$$

but $\|g_k\| > \delta \; \forall \; k \geq 0$ by assumption

$$\frac{\|d_k\|}{\|g_k\|^2} \leq \sum_{i=1}^{k} \frac{u+\eta}{\delta} + \frac{1}{\delta}$$
$$\frac{\|d_k\|}{\|g_k\|^2} \leq \frac{(u+\eta)k+1}{\delta}$$
$$\frac{\|g_k\|^4}{\|d_k\|^2} \geq \frac{\delta^2}{((u+\eta)k+1)^2}$$
$$\sum_{k=0}^{\infty} \frac{\|g_k\|^4}{\|d_k\|^2} \geq \sum_{k=0}^{\infty} \frac{\delta^2}{((u+\eta)k+1)^2} = \infty$$
$$\sum_{k=0}^{\infty} \frac{\|g_k\|^4}{\|d_k\|^2} \geq \infty \, , C!$$

Hence, the global convergence is obtained.

3.3. Superlinear convergence of HQN^{EI}

This part is specific for proving the superlinearity convergence of our suggested algorithm. To prove that, we need lemma 4.9 and 4.10 in [29]. Those lemmas state, under the hypotheses 3.1, there exists a sequence $\{\epsilon_k\}$ of numbers such that:

$$\frac{\|y_k - G(x^*)s_k\|}{\|s_k\|} \leq \epsilon_k \tag{7}$$

and

$$\sum_{k=1}^{\infty} \epsilon_k < \infty \tag{8}$$

where, y_k and s_k are defined in (4). These lead to the boundedness of the sequences $\{H_k\}$, $\{H_k^{-1}\}$ and

$$\lim_{k\to\infty} \frac{\left\|\left(H_k^{-1} - G(x^*)\right)s_k\right\|}{\|s_k\|} = 0 \tag{9}$$

3.3.1. Theorem

Suppose assumptions 3.1 (i) and (iii) are holds, $\{x_k\} \to x^*$ and the sequence $\{H_k\}$, $\{H_k^{-1}\}$ are bounded. If $x_{k+1} = x_k + d_k$ holds for all sufficiently large k with,

$$\lim_{k\to\infty} \frac{\left\|\left(H_k^{-1} - G(x^*)\right)d_k\right\|}{\|d_k\|} = 0, \text{ then } \lim_{k\to\infty} \frac{\|x_{k+1} - x^*\|}{\|x_{k+1} - x_k\|} = 0$$

Proof:

$$\begin{aligned}
[H_k^{-1} - G(x^*)]d_k &= H_k^{-1}d_k - G(x^*)d_k \\
&= H_k^{-1}[-H_k g_k + \eta(-g_k + \beta_k d_{k-1})] - G(x^*)d_k \\
&= -g_k - \eta H_k^{-1} g_k + \eta\beta_k d_{k-1} H_k^{-1} - G(x^*)d_k \\
&= (-I - \eta H_k^{-1})(g_k + G(x^*)d_k) + \eta\beta_k d_{k-1} H_k^{-1} + \eta H_k^{-1} G(x^*)d_k \\
&= (-I - \eta H_k^{-1})g_{k+1} + \eta\beta_k d_{k-1} H_k^{-1} + \eta H_k^{-1} G(x^*)d_k + o(\|d_k\|)
\end{aligned}$$

$\{\|H_k\|\}$, $\{\|H_k^{-1}\|\}$ are bounded sequence by (7), (8) and this leads to β_k be coverage. Therefore, we get:

$$\left\|\left(H_k^{-1} - G(x^*)\right)d_k\right\| = [1 + O(1)]\|g_{k+1}\| + o(\|d_k\|)$$

since it is given that,

$$\lim_{k\to\infty} \frac{\left\|\left(H_k^{-1} - G(x^*)\right)d_k\right\|}{\|d_k\|} = 0$$

which is implies that,

$$\lim_{k\to\infty} \frac{\|g_{k+1}\|}{\|d_k\|} = \lim_{k\to\infty} \frac{\|g_{k+1}\|}{\|x_{k+1} - x_k\|} = 0$$

On the other hand,

$$g_{k+1} - G(x^*) - G(x^*)(x_{k+1} - x^*) = o(\|x_{k+1} - x^*\|), \text{ with } G(x^*) = 0,$$

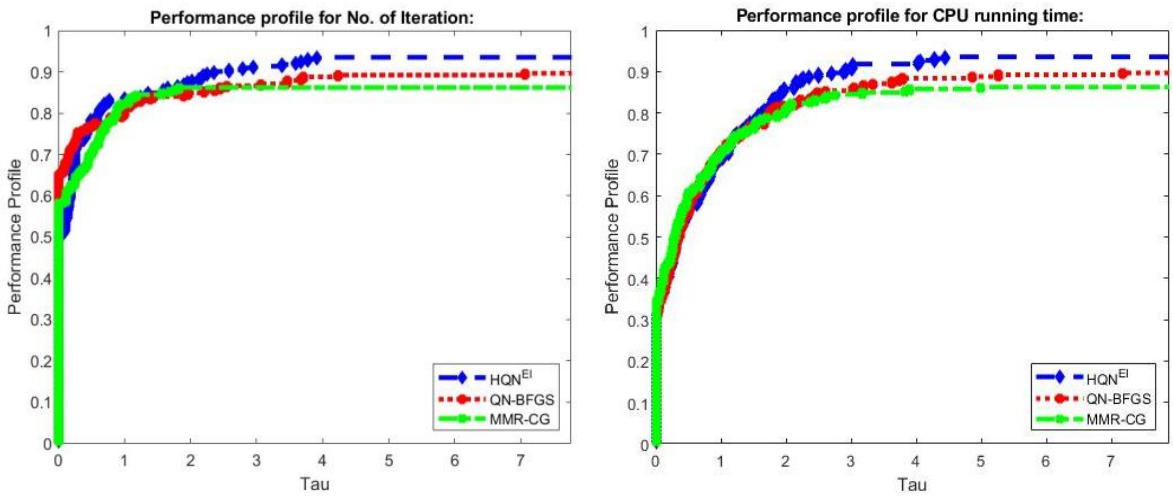
Hence, $\lim_{k\to\infty} \frac{\|x_{k+1} - x^*\|}{\|x_{k+1} - x_k\|} = 0.$

4. RESULTS AND NUMERICAL DISCUSSION

In this part, The performance profile plots; suggested in [30], is displayed. It is a good way to highlight the differences among approaches; in case, if there is a significant difference in the area of interests. In other words, the performance profile plot does not show small variant among algorithms. In this paper, this way is used to reveal the comparison of different algorithms based on central processing unit (CPU) running time and number of iterations to obtain the solution with desired error. A list of 40 test functions collected in each of [31] and [32] using different dimensions for problems, as showed in Table 1, is run. The results presented in Figure 1, by ploting the performance profile plot. The Figure 1(a) shows the performance of $\mathrm{HQN}^{\mathrm{EI}}$, BFGS and MMR-CG in terms of number of iteration; that counted to get the wanted solution. It is obvious that $\mathrm{HQN}^{\mathrm{EI}}$ reached to the solution with less number of iteration, followed by BFGS and at last MMR-CG comes. However, the powful of running time of CPU is demonstrated in Figure 1(b). It is clear that HQN^{EI} dominated by the others, which means that in this comparsion criterion of our algorithm is also better than BFGS and MMR-CG.

Table 1. List of test functions with some dimension

ID	Function Name	Dimension	ID	Function Name	Dimension
1	Zettl	2	21	Diagonal 2	2,4,6,8,20,30
2	Camel Six Hump	2	22	Diagonal 3	2,4,6,8,20,30
3	Camel Three Hump	2	23	Diagonal 4	2,4,6,8,20,30
4	Brent	2	24	Diagonal 5	2,4,6,8,20,30
5	Quartic	2	25	Diagonal 6	2,4,6,8,20,30
6	Sphere	2,4,6,8,20,30	26	Diagonal 7	2,4,6,8,20,30
7	Schwefel_2_4	2,4,6,8,20,30	27	Hager	2,4,6,8,20,30
8	Zakharov	2,4,6,8,20,30	28	Ex. TET	2,4,6,8,20,30
9	Dixon & Price	4,6,8,20,30	29	ARWHEAD	2,4,6,8,20,30
10	Gen. Rosenbrock	2,4,6,8,20,30	30	Ex. DENSCHNB	2,4,6,8,20,30
11	Ex. Rosenbrock	2,4,6,8,20,30	31	COSINE	2,4,6,8,20,30
12	Raydan 1	2,4,6,8,20,30	32	BIGGSB1	2,4,6,8,20,30
13	Raydan 2	2,4,6,8,20,30	33	DIXON3DQ	2,4,6,8,20,30
14	Ex. BD1 Block Diagonal	2,4,6,8,20,30	34	EX. Penalty	2,4,6,8,20,30
15	Gen. Quartic	2,4,6,8,20,30	35	ENGVAL 1	2,4,6,8,20,30
16	Gen. Tridiagonal 1	2,4,6,8,20,30	36	Almost Perturbed Quadratic	2,4,6,8,20,30
17	Ex. Tridiagonal 1	2,4,6,8,20,30	37	POWER	2,4,6,8,20,30
18	Ex. Freudenstein Roth	2,4,6,8,20,30	38	DQDRTIC	2,4,6,8,20,30
19	Diagonal 1	2,4,6,8,20,30	39	Cliff	2,4,6,8,20,30
20	Ex. White & Holst	2,4,6,8,20,30,100,200,400	40	Ex. Beale	2,4,6,8,20,30,100,200,1000



(a) (b)

Figure 1. Performance profile for BFGS, CG-MMR and $\mathrm{HQN}^{\mathrm{EI}}$ (a) Number of iterations and (b) CPU running time

Now, more details about running three programs are givin. MATLAB 2018a codes is used with an exact line search using secant method to approximate the step size with combination parameter $\eta = 0.01$ and $\varepsilon = 1 \times 10^{-7}$ or maximum number of iterations which is the number of variable times 1000. Also, MATLAB codes for comparison of algorithm performance profiles were written in [33] and they are utilized here. Moreover, the recommended initial points for function in [31] and the border possible value for those in [32] is run; for all test function named in Table 1.

5. CONCLUSION

In this paper, a new direct search HQN^{EI} is introduced. It is a type of line combination of projection of BFGS Hessian matrix update together with the direction parameter β_k of a specific designed CG method, MMR-CG. After running many test functions, the powerful of the HQN^{EI} algorithm is obtained. This means that, there is an improvement of the performance of algorithm with less number of iteration and CPU running time in comparison with its originator components, BFGS and MMR-CG algorithms. This entire conclusion is along with sufficient descent, global convergence and superlinear rate property of the proposed algorithm.

ACKNOWLEDGEMENTS

The authors thank their University to open the opertunity for conducting this research.

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