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Mathematics Approaches in Compressed Sensing

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Abstract

Mathematical approaches refer to make quantitative descriptions, deductions and calculations through the use of mathematics concepts, approaches and techniques, then draw some new conclusions and foresee by mathematical analysis and judgement. In recent years, Compressed Sensing theory (CS) provides solutions in alleviating the huge amount of information demand in the pressure of signal sampling, transmission and storage pressure. It is a novel signal sampling theory under the condition that the signal is compressible or sparse.In this case, the signal can be reconstructed accurately from the small amount of signal values if the signal is sparse or compressible. This paper introduces the CS theory framework and key technical issues, and focus on the analysis of the application of mathematical approaches in three aspects of the signal sparse representation, signal sparse transformation and reconstruction.In the end, Some mathematics problems of compressed sensing to solve are given and further development is pointed out.

Keywords: *mathematical approaches, compressed sensing, sparse transformation, observation matrix, signal reconstruction*

1. Introduction

Mathematics method is the most important one in scientific and rational approaches, it consists of all kinds of pure mathematics and applied mathematics method, such as digital calculation, mathematical reasoning, establishing and solving equation, logistic, statistics, computer simulation and inference, etc. Mathematics method can form interpretation, judgment and prediction by mathematical language expressing the state of things, relationships, and process, then derivation, calculus and analysis. Mathematics method has the following three basic characteristics, one is highly abstract and general, the second is logical rigor and conclusions accuracy, the third is the universality and operability in the application.

Mathematical approaches are divided into the following categories: constant mathematics approaches, basically consist of arithmetic method, algebraic method, geometric method and trigonometry method; Variable mathematics approaches, most basically consist of analytic geometry method and differential and integral calculus; The necessity mathematical approaches, generally consist of equation and equations (algebraic equations, the function equation, differential equations, partial differential equation and difference equation, etc.), Random mathematical approaches, mainly consist of theory of probability and mathematical statistics, The fuzzy mathematics method generally refer to mutation mathematical approaches, in aspect of mutations problem under no more than four control factor conditions, there are seven kinds of discontinuous process mutation type (folding transformation, Angle type, split type, butterfly type, hyperbolic umbilical point type, elliptical umbilical point type and parabolic umbilical point type).

Mathematical approaches own several important characteristics of highly abstract, high accuracy and strict logical, provide concise and accurate formal languages, quantitative analysis and theoretical calculation approaches, tools of logical reasoning for many scientific research. Cs is a promising mathematical strategy for sampling signals characterized by large numbers of nominal samples and yet exhibiting high compressibility by known mathemactical transforms like the fourier transform or wavelet transform [1]. The limited experiments conducted here already show gains by factors of more in moderate-size problem sizes, and these can be enhanced by deployment in a multiscale fashion and by applying denoising to reconstructions.

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the CS framework is able to save significantly over traditional sampling by mathematics approaches, and there are many useful extensions of the basic idea.

2. Mathematical Approaches in Compressed Sensing
2.1. Compressed Sensing Theory

In the traditional sampling process, in order to avoid signal distortion, sampling frequency shall not be lower than the highest frequency signal of two times. However, acquisition digital images or video, in accordance with the SHANNON theorem bringing about the magnitude of sampling data, greatly increase the cost of storage and transmission. Compressed sensing uses a non-adaptive projection to keep the signal original structure, and can reconstruct the original signal through the numerical optimization problem accurately. This theory indicates that using a small number of observations can keep signal structure and related information if the original signal is sparse or in a compressed group. Based on this theory, demanded sampling quantity for accurate reconstruction signal can be far less than the dimension of observation, which greatly alleviate the pressure of the broadband signal processing.

In recent years there is a novel compressed sensing theory, it's another name is compressive sampling. There is no a unified Chinese vocabulary to the corresponding, someone call the compressed perception, others call the compression sensing (this paper uses concept of the compression sensing). In order to reduce cost of storage, processing and transmission, people often use compression way with less number of bits in the signal, but a large number of the important data are abandoned [2]. This high speed sampling and compression process will waste a large amount of sampling resources, using other transformation space description signal, establishing a new signal description and processing theoretical framework, which guarantee the information not loss, using far below the NYQUIST sampling theorem required rate sampling signal, at the same time, it can fully recover signal, greatly reduce the signal sampling frequency, data storage and transmission price, significantly reduce signal processing time and cost calculation [3]. Compression perception theory and traditional NYQUIST sampling theorem is different, it points out that, as long as the signal is compressible or sparse in a transform domain, it can transform the high dimensional signal projection to a low dimensional space by using a observation matrix not related to transformation base, then reconstruct the original signal at high probability by solving an optimization problem from these small amounts of projection [4]. Compressed sensing theory mainly includes signal sparse representation, sampling measurement and reconstruction algorithm and so on three aspects.

2.2. Mathematical Approaches in Signal Sparse Express

Considing R^N space is limited one-dimensional discrete time signals X, and assuming that $\{\psi_i \mid i = 1, 2, \cdots, N\}$ is a set of base vector of R^N , any signal X in R^N space can be expressed as linear:

$$\mathrm{x} = \sum_{i=1}^{N} s_i \psi_i \quad 或 \quad \mathrm{x} = \psi \mathrm{s} \tag{1}$$

Where $\psi = [\psi_1, \psi_2, \cdots, \psi_N\}$ is the series of base matrixs, S is transform vector of x in ψ domain. A hotspot in the field of sparse express mathematical approaches is signal sparse decomposition in the redundant dictionary. The choice of dictionary should as far as possible bring into correspondence with be approximation signal structure, its composition can be without any limitation [5]. Finding K atom of the best linear combination to represent a signal from the redundant dictionary, it is named signal sparse approximation or highly nonlinear approximation mathematic method [6]. A new signal representation theory in super complete library: the redundant dictionary is given a definition of super complete redundancy function library substituting for the basis it, the matching pursuit (MP) is one of the dictionary of the signal sparse representation approaches [7]. Some natural signals have compact and condense representation in over complete redundant dictionaries, which is a property called sparse.

Figure 1 shows the flowchart of MP method. In the frame-work, the algorithm of feature extraction based on the sparse representation is the key issue. If we look just at what happens to j during the algorithm above, it's sort of like a finite automaton. At each step j is set either to j+1 (in the inner loop, after a match) or to the overlap o (after a mismatch). At each step the value of o is just a function of j and doesn't depend on other information like the characters. So we can draw something like an automaton, with arrows connecting values of j and labeled with matches and mismatches.

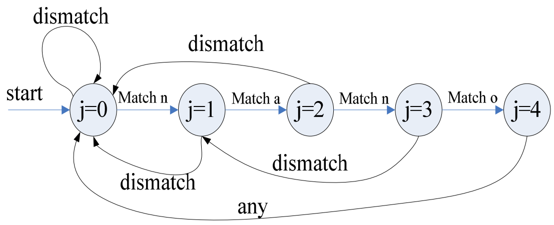

Figure 1. The MP Method

The mathematic method research on signal in the redundant dictionary of sparse representation focuses on two aspects: one is how to construct a suitable it, another is how to design the rapid and effective sparse decomposition mathematical method. These two problems have been the focus of research in this field, scholars have made some exploration [8], among them, a series of theoretical proof based on incoherent dictionary have been further improved .

From the nonlinear approximation perspective, signal sparse approximation mathematics method contains two aspects: On the one hand, the good or best base should been selected based on the objective function from a given base library; on the other hand, the best K item combination should been selected from the good base. From the redundant dictionary structure perspective, the literature [9] put forward to depict the voice signal local frequency domain characteristics making use of local Cosine base; to depict the image edge geometry by Bandlet base. Also, it can been put the basis function of other different shapes into a dictionary, such as the Gabor base for suitable texture characterization, the Curvelet base for suitable contour characterization, etc.

From mathematical method of sparse decomposition method, in audio and video signal processing, MP method based on greed iterative thought show great superiority, but not is the global optimal solution. Donoho puts forward the base tracking (Basis Pursuit, BP) mathematical method, which has the advantage of global optimal, but high computational complexity, such as for length for 9248 signal, using wavelet decomposition, dictionary equivalent to solving a length for 9248*212992 linear programming. Although the MP method convergence speed is faster than BP, but not the global optimal, moreover, the computation complexity is very big. Then there was a series of the same thought based on greed iteration mathematical approaches, such as orthogonal matching pursuit mathematical approaches (OMMM), tree matching mathematical approaches (TMMM), block matching pursuit (STOMMM) mathematics method, etc.

2.3. Mathematical Approaches in Signal Sampling Measurement

Seting original signal X_n, the parameterized mathematical model can been said in the following description:

$$X_n = \sum_{p=1}^{P} \pi_p \partial(a_n, b_p), \qquad n = 1, 2, \cdots, N \tag{2}$$

Where P is model order number, $\partial(a,b)$ is complex domain waveform based on the parameters b, π_p and b_p Respectively show No.P component amplitude of signal and parameter values. in the compressed sensing, through the transformation mathematical method, the signal sparse coefficient vector is defined as:

$$\gamma = \psi\chi + \varepsilon \tag{3}$$

Where, $\psi \in \mathbf{R}^{M\times N}$ is a Measurement matrix under the condition of $M \prec N$, $\varepsilon \in \mathbf{R}^{\mathrm{M}}$ is the measurement noise, $\gamma \in \mathbf{R}^{\mathrm{M}}$ is X compression measurement. It is also need to design compression sampling system observation part around the observation matrix. Observer design purpose is how to get M a sampling observation, and ensure that it can reconstruct the signal X whose length is N or the equivalent sparse coefficient vector based on ψ [10]. Obviously, if the signal X information is destroyed in observation process, reconstruction is impossible. The actual observation process is to use M row vector of M*N observation matrix to project for sparse coefficient vector, etc, calculating the inner product of each observation vector, M observation value is got, which is marked $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_m,)$. Sampling process is adaptive, that is to say, it is not necessary to change according to the signal X, observation result is no longer the signal point sample but signal more general K linear function [11].

Getting X form a given Y is a linear program mathematics method problem, but as a result of M is far less than the unknown number of N equation, generally, no sure solution will been got because this is a under determined problem. However, if there are K sparse characteristic, it is expected to find out determined solution. At this time, as long as trying to identify the right position of K non-zero coefficient, the observation vector Y is linear combination of the K columns of the non-zero coefficient corresponding, which can form a system of linear equations to solve these nonzero item specific value. Limited Isometric Property (RIP) is given to determine the sufficient and necessary conditions for the solution. D. L. Donoho and X. Huo had proposed that the sparse signal in the observation matrix must be kept consistent under the action of the geometric properties [12]. In other word, to make the perfect reconstruction signal, we must ensure that observation matrix won't put two different K sparse sign mapping to the same sample set, matrix is singular which consist of M column vector from the observation matrix extraction. Obviously, the key problem is how to determine the position of the nonzero coefficient to construct a solvable K linear equations system.

The observation matrix mathematical approaches research is an important aspect in compressed perception theory. In this theory, the observation matrix constraints are more relaxed, Weng, Q., Lu, D. and Schubring, J. had Put forward three prerequisite conditions of observation matrix in the literature [13], and had point out that the most of uniform distribution random matrix can act as the observation matrix because that they own these three conditions, such as: some Fourier set, some Wavelet set, uniform distribution of random projection set, it bring into correspondence with RIP, however, after observation in the above various observation matrix, it can't guarantee to reconstruct signal at one hundred percent accurate ,but to restore signal at a high probability. It is still a worth studying problem for any stable reconstruction algorithm matched a real deterministic observation matrix. The literature [14] had described the relationship between the information theory and CS from the angle of information theory. It pointed out that the observation noise is an important factor of influence the observation number in simulation system, to illustrate this which, the author studied the sparse signal rate distortion function, given the observation noise on the effects of signal reconstruction from the perspective of the information theory.

2.4. Mathematical Approaches in Signal Reconstruction

In the compression perception theory, for the observation number M is far less than the signal length N, we have to face to solve underdetermined equations mathematical approaches. On the surface, it seems to be hopeless in solving underdetermined equations, however, the literature [11] and the literature [12] pointed out that the problem could be solved for the signal X was sparse or compressible, it provided the theoretical guarantee in restoring signal from M observations value accurately for the observation matrix own RIP nature also. How to reconstruct the X signal from the observed signal Y, it can adopt the following optimization problem to solve.

$$\min\|x\|_0, \quad s.t.\ y = \psi x \tag{4}$$

Practice, it is allowed to exist some degree error, therefore, the original optimization problem can be turn into a simpler form of approximate solution, among them, δ is very small constant：

$$\min\|x\|_0, \quad s.t.\ \|y - \psi x\|_2^2 \leq= \delta \tag{5}$$

The above L_0 norm minimum problem is a NP problem, it is difficult to solve directly. A kind of sampling method had appeared which use sparse matrix as the observation matrix. Y. Eldar had put forward that the position and value of sparse signal non-zero coefficient was estimated through selection grouping test and random subset, which need hit M=O(K$\log_2$N), signal reconstruction algorithm complexity was O(K$\log_2$N), it reconstruct signal even faster than others. Gilbert had put forward chaining Pursuit method to restore compressible signal in April 2006. Using O(K$\log_2$N) sampling observation to reconstruct signal, its time complexity was O (K$\log_2$N$\log_2$K), this method for special sparse signal recovery calculated performance was higher than others, but when the signal sparse degree reduction, necessary sampling points would increase quickly, even more than the length of the signal itself, it would lost the meaning of compressed sampling.

It can be classified as three types in the reconstruction of the mathematics approaches. The first, greedy Tracking Mathematical Approaches: the method is through when each iteration to choose a local optimal solution to gradually approaching the original signal. These mathematics approaches are abundant, such as the MP, OMP, Piecewise OMP (POMP) and Regularization OMP (ROMP) [15]. ROMP approache has a second choice by using the atomic quadratic regularization process, the atomic correlation coefficien can be divided into severa groups on the index value corresponding:

$$|\tau(p)| \leq 2|\tau(q)|, \quad p, q \in H \tag{6}$$

Where τ is the absolute value of the inner product of each atom in the matrix between allowance and measurement,H is a Hou selection.

Convex Relaxation Method Mathematical Approaches: this method find signal approximation through transforming into convex problem solving, such as BP algorithm, interior point method, gradient projection method and iterative threshold method [16]. Combinatorial Mathematics Approaches: the method for signal sampling support through the grouping test fast reconstruction, such as Fourier sampling, chain tracking and HHS (Heave Hitters on Steroids) tracking, etc.

It can be seen that each algorithm has its inherent shortcomings. Convex relaxation method for reconstruction of signal observation times is least than others, but often computation burden is heavy. Greed tracking algorithm is located in intermediate of the other two group algorithms in operation time and sampling efficiency [17].

From the above analysis, we can conclude that reconstruction algorithm and the number of observed closely related. At present, signal reconstruction method of mathematics research in compressed sensing theory mainly focus on how to reconstruct accurately to recover the original signal by constructing stable and less computational complexity and less observed quantity requirement.

3. Conclusion

It has some achievements in compressed sensing processing mathematical approaches, but there are still a lot of problems deserving studied. Summarized as the following several aspects: The first, It is very important whether an optimal deterministic observation matrix for the reconstruction of the stable mathematical approaches. Second, how to construct stable and computational complexity is low, the observation times fewer restrictions for the reconstruction of the mathematics method to accurately recover compressible signal. Third, how to find a kind of effective and rapid sparse decomposition mathematical method is redundant dictionary of compression perception theory difficulty. Fourth, how to design effective software and hardware to apply compressed sensing theory to solve a lot of practical problems, this aspect research is still far from enough. Fifth, it is also not enough study for p-norm optimization mathematical method to solve. Sixth, it is difficulty in noise signal reconstruction mathematics method with noise signal or sampling process, and the results are not ideal. In addition, it is also not far enough in the field of compressed sensing theory and the signal procession, such as signal detection, feature extraction, etc. CS theory and machine learning and other areas of inner link research work had been started. Compressed sensing theory is the new born, although there are many problems to study, but it is a very good supplement and perfect for the traditional signal processing, it is a strong vitality theory and great influence on the fields in the study of mathematics method of signal processing results.

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