Indonesian Journal of Electrical Engineering and Computer Science
Vol. 25, No. 1, January 2022, pp. 540~549
ISSN: 2502-4752, DOI: 10.11591/ijeecs.v25.i1.pp540-549

Fixed point theorem between cone metric space and quasi-cone metric space

Abdullah Al-Yaari[1,2], Hamzah Sakidin[1], Yousif Alyousifi[3], Qasem Al-Tashi[4,5]
[1]Department of Fundamental and Applied Sciences, Universiti Teknologi PETRONAS, Seri Iskandar, Malaysia
[2]Department of Mathematics, Faculty of Education, Thamar University, Dhamar, Yemen
[3]Department of Mathematics, Faculty of Applied Science, Thamar University, Dhamar, Yemen
[4]Department of Computer and Information Sciences, Universiti Teknologi PETRONAS, Seri Iskandar, Malaysia
[5]Faculty of Administrative and Computer Sciences, University of Albaydha, Albaydha, Yemen

Article Info

Article history:

Received Jul 28 2021
Revised Nov 1, 2021
Accepted Nov 26, 2021

Keywords:

Cone metric spaces second
Fixed point theorem
Quasi-cone metric spaces

ABSTRACT

This study involves new notions of continuity of mapping between quasi-cone metrics spaces (QCMSs), cone metric spaces (CMSs), and vice versa. The relation between all notions of continuity were thoroughly studied and supported with the help of examples. In addition, these new continuities were compared with various types of continuities of mapping between two QCMSs. The continuity types are $\boldsymbol{ff}$-continuous, $\boldsymbol{bb}$-continuous, $\boldsymbol{fb}$-continuous, and $\boldsymbol{bf}$-continuous. The results demonstrated that the new notions of continuity could be generalized to the continuity of mapping between two QCMSs. It also showed a fixed point for this continuity map between a complete Hausdorff CMS and QCMS. Overall, this study supports recent research results.

Corresponding Author:

Abdullah Al-Yaari
Department of Fundamental and Applied Sciences, Universiti Teknologi PETRONAS
32610 Seri Iskandar, Malaysia
E-mail: abdullah_20001447@utp.edu.my

1. INTRODUCTION

More than two decades ago, Huang and Zhang had proposed the notion of a cone metric space [1]. The Cauchy and convergent sequences and a few fixed-point theorems were used for the contractive form of mappings in cone metric spaces (CMSs). This notion is exciting because of the usual metric space where an ordered Banach space switches to the real numbers. Abbas and Jungck [2] have shown some noncommuting mapping causes CMSs. Furthermore, several researchers have proved different contractive-type maps in CMS and fixed points [3]-[8].

The function of a quasi-metric is to verify the triangle disparity. However, quasi-metric is considered an asymmetric metric. As compared to metric space, quasi-metric space (QMS) is more inclusive. It is also a topic of exhaustive research in computer science and the framework for topology. For instance, a quasi-cone metric space (QCMS) definition expands to the QMS given by Turkoglu and Abuloha [3]. Morales and Rojas have introduced the continuity of mapping between CMS (X, C), P a cone along with constant K and $T: X \longrightarrow X$ and it self [4]. Yaying *et al.* introduced the continuity of mapping between QCMS (X, q) and itself [5]. In addition, although the concept of normality in CMSs is monumental in developing fixed point theory in CMSs, Rezapour and Hamlbarani [6] have rejected it. Several researchers have simplified fixed points in CMSs in many directions. For instance, Jankovic *et al.* [7] surveyed the latest outcomes in CMSs. Abdeljawad and Karapinar [8] have verified fixed point theorems in QCMSs. They introduced many Cauchy sequences in QCMSs, which are studied as an extension of the Banach contraction mapping and other areas. Yaying *et al.* [9] have defined arithmetic fb- continuity and arithmetic ff-continuity based on the notions of forward and backward arithmetic convergence in asymmetric metric spaces. Also, they have proven some interesting results.

For functions between (MSs), (CMSs) and (QCMSs) and themselves, many continuity ideas have been established [4], [5]. The continuity function between (QCMSs) and (CMSs) and vice versa, however, has yet to be implemented. As a result, we have introduced these definitions and looked at how they relate to one another. Our ideas are based on the continuation of a function mapping between two (MSs), a function mapping between two (CMSs), and a function mapping between two (QCMSs). There have been numerous fixed-point theorems presented between (MSs) and themselves, (CMSs) and themselves, and (QCMSs) and themselves [1], [8]. We looked at prior results of a fixed-point to introduce a fixed point in a function mapping between (CMSs) and (QCMSs).

This paper introduces new notions of continuity of mapping, some theorems, and propositions. It compares the new notions to each other through specific examples. Additionally, the paper introduces the fixed-point theorem of the newly introduced notion. The paper also supports the results of other recent research with the help of examples. Finally, some general concepts, definitions, and outcomes are recalled and applied in this paper.

This study will make a significant contribution to the field of mapping continuity and fixed-point theorems. Fixed point theorems are used to solve problems in estimate theory, game theory, and mathematical economics in a variety of disciplines including mathematics, statistics, computer science, engineering, and economics. The following is how the rest of the paper is structured: The study's preliminary findings are presented in section 2, the results are clarified in section 3, and the paper is completed in section 4.

2. PRELIMINARIES

In this part, the elementary facts about the CMSs and QCMSs and their continuity, which can be observed in [1], [10]-[26], are given in a shortened form.

- Definition 2. 1 (see [17]). The cone is considered to be regular if each expanding sequence in P that is bounded from is convergent in E. If $\{u_n\}_{n\in\mathbb{N}}$ is a sequence such that,

$$u_1 \leq u_2 \leq \cdots \leq u_n \leq \cdots \leq v \tag{1}$$

  for a few $v \in E$, there is also some $u \in E$ that satisfies $\| u_n - u\| \longrightarrow 0_E$. As $(n \to \infty)$.
  In the following, we continually assume E is a space of Banach, P is a cone.
- Definition 2. 2 (see [18]). Assume X is a nonempty set. Presume the mapping $C : X \times X \longrightarrow E$ assures,
  (C1) $C(u,v) \geq 0_E$ for all $u, v \in X$ and $C(u,v) = 0_E$ if and only if $u = v$
  (C2) $C(u,v) = C(v,u)$ for all $u, v \in X$
  (C3) $C(u,v) \leq C(u,w) + C(v,w)$ for all $u, v, w \in X$
  then, C is a cone metric on X, and (X, C) is a CMS.
- Proposition 2. 3 (see [4]). Assume (X, C_1) and (Y, C_2) are a CMS, P a cone with constant K and $T: X \longrightarrow Y$. The following are equivalent,
  the map T is continuous.
  If $\lim_{n\to\infty} u_n = u$, implies that $\lim_{n\to\infty} Tu_n = Tu$ for every $\{u_n\}_{n\in\mathbb{N}}$ in X.
- Definition 2. 4 (see [24]). Assume X is a set. Presume that the mapping $q: X \times X \to E$ satisfies the subsequent,
  (Q1) $0_E \leq q(u,v)$ for all $u, v \in X$,
  (Q2) $q(u,v) = 0_E$ if and only if $u = v$,
  (Q3) $q(u,v) \leq q(u,w) + q(w,v)$, for all $u, v, w \in X$.
  Then, q is known as a QCM on X, and the pair (X, q) is called a QCMS.
- Example 2. 5 (see [25]). Assume $X = \mathbb{R}$, $E = \mathbb{R}^2$, $P = \{(u,v) \in E : u, v \geq 0\}$ and $q: \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}^2$ is defined by (2),

$$q(u,v) = \begin{cases} (0,0) \text{ if } u = v, \\ (1,0) \text{ if } u > v, \\ (0,1) \text{ if } u < v. \end{cases} \tag{2}$$

  Then, $(q, \mathbb{R})$ is a QCMS because it satisfies all conditions of QCMS, for all $u, v, w \in \mathbb{R}$.

- Definition 2. 6 (see [27]). A sequence $\{u_n\}_{n\in\mathbb{N}}$ forward converges to $u_0 \in X$ if for each $c \in E$ along with $0_E \ll c$ (i.e., $c \in \text{int}\, P$) there exists a number n_0 that satisfies $q(u_0, u_n) \ll c$ for all $n \geq n_0$. We indicate it through $u_n \xrightarrow{f} u_0$.
- Definition 2. 7 (see [3]). A sequence $\{u_n\}_{n\in\mathbb{N}}$ backward joins to $u_0 \in X$ if for each $c \in E$ along with $0_E \ll c$ (i.e., $c \in \text{int}\, P$) there is a number n_0 that satisfies $q(u_n, u_0) \ll c$ for all $n \geq n_0$. We show it by $u_n \xrightarrow{b} u_0$.
- Definition 2. 8 (see [5]). Assume (X, q_X) and (Y, q_Y) are two QCMSs. A map $T: X \longrightarrow Y$ is ff-continuous at $u \in X$ if whenever $u_n \xrightarrow{f} u$ in (X, q_X) we have $T(u_n) \xrightarrow{f} T(u)$ in (Y, q_Y).
- Definition 2. 9 (see [5]). Assume (X, q_X) and (Y, q_Y) are two QCMSs. A map $T: X \longrightarrow Y$ is fb-continuous at $u \in X$ if whenever $u_n \xrightarrow{f} u$ in (X, q_X) we have $T(u_n) \xrightarrow{b} T(u)$ in (Y, q_Y).
- Definition 2. 10 (see [5]). Assume (X, q_X) and (Y, q_Y) are two QCMSs. A map $T: X \longrightarrow Y$ is bf-continuous at $u \in X$ if whenever $u_n \xrightarrow{b} u$ in (X, q_X) we have $T(u_n) \xrightarrow{f} T(u)$ in (Y, q_Y).
- Definition 2. 11 (see [5]). Assume (X, q_X) and (Y, q_Y) are two QCMSs. A map $T: X \longrightarrow Y$ is bb-continuous at $u \in X$ if whenever $u_n \xrightarrow{b} x$ in (X, q_X) we have $T(u_n) \xrightarrow{b} T(u)$ in (Y, q_Y).
- Definition 2. 12 (see [5]). A set X is compact in the forward sequence if every sequence $\{u_n\}_{n\in\mathbb{N}}$ in X possesses a forward convergent subsequence $\{u_{n_k}\}_{n_k\in\mathbb{N}}$ to $u_0 \in X$ if for all $c \in E$ along with $0_E \ll c$ (i.e., $c \in \text{int}\, P$) there is number n_0 as a result $q(u_0, u_{n_k}) \ll c$ for all $n_k \geq n_0$.
- Definition 2. 13 (see [5]). A set X is compact in the backward sequence if every sequence $\{u_n\}_{n\in\mathbb{N}}$ in X possesses a backward convergent subsequence $\{u_{n_k}\}_{n_k\in\mathbb{N}}$ to $u_0 \in X$ if for all $c \in E$ along with $0_E \ll c$ (i.e., $c \in \text{int}\, P$) there exists a number n_0 as a result $q(u_{n_k}, u_0) \ll c$) for all $n_k \geq n_0$.
- Lemma 2. 14 (see [5]). Assume (X, q) is a QCMS. Then $u_n \xrightarrow{b} u$ if and only if each subsequence of it is backward convergent to u.

  Proof. Suppose $u_n \xrightarrow{b} u$. Then for $c \in E$ along with $0_E \ll c$ there exists an $n_0 \in \mathbb{N}$ such that $q(u_n, u) \ll c$, which means $c - q(u_n, u) \in \text{int}\, P$ for all $n \geq n_0$. Suppose $\{u_{n_k}\}_{k\in\mathbb{N}}$ is a random subsequence of $\{u_n\}_{n\in\mathbb{N}}$. If $n_k \geq n_0$, we got $c - q\left(u_{n_k}, u\right) \in \text{int}\, P$, i.e., $q\left(u_{n_k}, u\right) \ll c$. So $u_{n_k} \xrightarrow{b} u$.
- Lemma 2. 15 (see [5]). Assume (X, q) is a QCMS. Then $u_n \xrightarrow{f} u$ if and only if each subsequence of it is forward convergent to u.

  Proof. Suppose $u_n \xrightarrow{f} u$. Then, for $c \in E$ along sequence with $0_E \ll c$ there exists an $n_0 \in \mathbb{N}$ that satisfies $q(u, u_n) \ll c$, which means $c - q(u, u_n) \in \text{int}\, P$ for all $n \geq n_0$. Suppose $\{u_{n_k}\}_{k\in\mathbb{N}}$ is a random subsequence of $\{u_n\}_{n\in\mathbb{N}}$. If $n_k \geq n_0$, we got $c - q\left(u, u_{n_k}\right) \in \text{int}\, P$, i.e., $q(u, u_{n_k}) \ll c$. So $u_{n_k} \xrightarrow{f} u$.
  The converse' proof is evident. Thus, it is removed. As a result, the proof is complete.
- Lemma 2. 16 (see [5]). If $\{u_n\}_{n\in\mathbb{N}}$ forward converges to $u \in X$ and backward converges to $v \in X$, it means $u = v$.
- Theorem 2. 17 (see [5]). Assume $q: X \times X \to E$ is a QCMS. If (X, q) is a forward sequentially compact and $u_n \xrightarrow{b} u$. It means $u_n \xrightarrow{f} u$.

3. MAIN RESULTS

Because of the established concepts in CMSs and QCMSs [5] and [26], we introduce new notions of continuity of mapping between QCMS and CMS and vice versa. The relation between all notions of continuity is thoroughly studied and support with the help of examples. We also find a fixed point for this continuity map between a complete Hausdorff CMS and QCMS.

- Definition 3.1. Suppose (X, q) is a QCMS and (Y, C) a CMS. A map $T: X \longrightarrow Y$ is b-continuous at u if for any sequence $\{u_n\}_{n\in\mathbb{N}}$ in X backward converges to an element u in X $\left(u_n \xrightarrow{b} u \text{ as } n \to \infty\right)$, then the sequence $\{T(u_n)\}_{n\in\mathbb{N}}$ converges to $T(u)$ in Y, $(C(T(u_n), T(u)) \to 0_E \text{ as } n \to \infty)$. T is considered b-continuous if it is b-continuous at each $u \in X$.
- Example 3.2. Suppose (X, q) is a QCMS where $X = [0,1]$, $E = \mathbb{R}^2$, $\alpha \in [0,1)$, $P = \{(u, v) \in \mathbb{R}^2 : u, v \geq 0\}$ and $q: [0,1] \times [0,1] \longrightarrow \mathbb{R}^2$ is defined by (3),

$$q(u, v) = \begin{cases} \big(u - v, \alpha(u - v)\big), \text{if } u \geq v, \\ (\alpha, 1), \text{if } u < v. \end{cases} \tag{3}$$

assume (X,C) is a CMS where $X=[0,1]$, $E=\mathbb{R}^2$, $\alpha\in[0,1)$, $P=\{(u,v)\in\mathbb{R}^2: u,v\geq 0\}$ and $C:[0,1]\times[0,1]\longrightarrow\mathbb{R}^2$ is defined by $C(u,v)=(|u-v|,\alpha|u-v|)$. A map $T:[0,1]\longrightarrow[0,1]$ defined by $T(u)=\frac{u}{2}$ is b-continuous at $\{0\}$.

- Proposition 3.3. Assume $T:(X,q)\longrightarrow(Y,C)$ is b-continuous at u. Then, a map T is bb-continuous at u and bf-continuous at u.
  Proof. Since a map $T:(X,q)\longrightarrow(Y,C)$ is b-continuous at u, then for every sequence $\{u_n\}_{n\in\mathbb{N}}$ in (X,q) that backward converges to an element u in (X,q), i.e., $q(u_n,u)\longrightarrow 0_E$ as $n\to\infty$, we have a sequence $\{T(u_n)\}_{n\in\mathbb{N}}$ that converges to $T(u)$ in (Y,C), i.e., $C(T(u_n),T(u))\longrightarrow 0_E$ as $n\to\infty$, where every CMS is QCMS and $C(T(u_n),T(u))=C(T(u),T(u_n))\longrightarrow 0_E$ as $n\to\infty$, thus, a map T is bb-continuous at u and bf-continuous at u.
- Remark 3.4. Every bb-continuous (or bf-continuous) is not always the case b-continuous as every QCMS is not always CMS.
- Definition 3.5. Assume (X,q) is a QCMS and (Y,C) a CMS. A map $T:X\longrightarrow Y$ is f-continuous at u if for any sequence $\{u_n\}_{n\in\mathbb{N}}$ in X forward converges to an element u in $\left(u_n\xrightarrow{f}u \text{ as } n\to\infty\right)$. Then, the sequence $\{T(u_n)\}_{n\in\mathbb{N}}$ approaches to $T(u)$ in $(C(T(u_n),T(u))\to 0_E \text{ as } n\to\infty)$. T is considered f-continuous if it is f-continuous at all $u\in X$.
- Example 3.6. Assume (X,q) is a QCMS, where $X=[0,1]$, $E=\mathbb{R}^2$, $\alpha\in[0,1)$, $P=\{(u,v)\in\mathbb{R}^2: u,v\geq 0\}$ and $q:[0,1]\times[0,1]\longrightarrow\mathbb{R}^2$ is defined by (4),

$$q(u,v)=\begin{cases}(v-u,\alpha(v-u)), \text{if } v\geq u,\\ \quad (\alpha,1), \text{if } v<u.\end{cases} \tag{4}$$

  suppose (X,C) is a CMS where $X=[0,1]$, $E=\mathbb{R}^2$, $\alpha\in[0,1)$, $P=\{(u,v)\in\mathbb{R}^2: u,v\geq 0\}$ and $C:[0,1]\times[0,1]\longrightarrow\mathbb{R}^2$ is defined by $C(u,v)=(|u-v|,\alpha|u-v|)$. A map $T:[0,1]\longrightarrow[0,1]$ defined by $T(u)=\frac{u}{3}$ is f-continuous at $\{0\}$.
- Proposition 3.7. Assume $T:(X,q)\longrightarrow(Y,C)$ is f-continuous at x. Then, a map T is ff-continuous at x and fb-continuous at x. Proof. Since a map $T:(X,q)\longrightarrow(Y,C)$ is f-continuous at u, then for all sequence $\{u_n\}_{n\in\mathbb{N}}$ in (X,q) forward approaches to an element u in (X,q), i.e., $q(u,u_n)\longrightarrow 0_E$ as $n\to\infty$, we have a sequence $\{T(u_n)\}_{n\in\mathbb{N}}$ that converges to $T(u)$ in (Y,C), i.e., $C(T(u_n),T(u))\longrightarrow 0_E$ as $n\to\infty$, where every CMS is QCMS and $C(T(u_n),T(u))=C(T(u),T(u_n))\longrightarrow 0_E$ as $n\to\infty$, thus, a map T is ff-continuous at u and fb-continuous at u.
- Remark 3.8. Every ff-continuous (or fb-continuous) is not always the case f-continuous as every QCMS is not CMS.
- Theorem 3.9. Assume $T:(X,q)\longrightarrow(Y,C)$ is b-continuous at u. If (X,q) is forward sequentially compact, then, T is f-continuous at u.
  Proof. Since $T:X\longrightarrow Y$ is b-continuous at u, any sequence $\{u_n\}_{n\in\mathbb{N}}$ in X backward converges to $u\in X$. Then, the sequence $\{T(u_n)\}_{n\in\mathbb{N}}$ approaches to $T(u)\in Y$.
  In other words, $u_n\xrightarrow{b}u\Longrightarrow T(u_n)\to T(u)$ as $n\to\infty$, where (X,q) is forward sequentially compact, the sequence $\{u_n\}_{n\in\mathbb{N}}$ possesses a forward convergent subsequence in X say $u_{n_k}\xrightarrow{f}v$ by Lemma 2.15, so $u_n\xrightarrow{f}v\in X$. Since $u_n\xrightarrow{b}u\in X$ and $u_n\xrightarrow{f}v\in X$ by Lemma 2.16, subsequently $u=v$. Thus, $u_n\xrightarrow{f}u$. Since $T(u_n)\to T(u)$ whenever $u_n\xrightarrow{b}u$ as $n\to\infty$, so $T(u_n)\to T(u)$ whenever $u_n\xrightarrow{f}u$ as $n\to\infty$. Then, T is f-continuous at u.
- Example 3.10. Assume (X,q) is a QCMS where $X=[0,1]$, $E=\mathbb{R}^2$, $\alpha\in[0,1)$, $P=\{(u,v)\in E: u,v\geq 0\}$
  and $q:[0,1]\times[0,1]\longrightarrow\mathbb{R}^2$ is defined by (5),

$$q(u,v)=\begin{cases}(0,0) \text{ if } v\geq u,\\ (u,\alpha u) \text{ if } v<u.\end{cases} \tag{5}$$

  Assume (X,C) is a CMS where $X=[0,1]$, $E=\mathbb{R}^2$, $\alpha\in[0,1)$, $P=\{(u,v)\in E: u,v\geq 0\}$ and $C:[0,1]\times[0,1]\longrightarrow\mathbb{R}^2$ is defined by $C(u,v)=(|u-v|,\alpha|u-v|)$. A map $T:[0,1]\longrightarrow[0,1]$ defined by $T(u)=\frac{u}{4}$ is b-continuous at $\{0\}$.
- Lemma 3.11. Assume $q:X\times X\to E$ is a QCM. If (X,q) is backward sequentially compact and $u_n\xrightarrow{f}u$. Then $u_n\xrightarrow{b}u$.

Proof. Assume $\{u_n\}_{n\in\mathbb{N}}$ is a sequence so $u_n \xrightarrow{f} u$ for a few $u \in X$. Via the backward sequentially compactness, the sequence $\{u_n\}_{n\in\mathbb{N}}$ has a backward convergent subsequence, as $u_{n_k} \xrightarrow{b} v$ by lemma 2.14, so, $u_n \xrightarrow{b} v$. Since $u_n \xrightarrow{f} u \in X$ and $u_n \xrightarrow{b} v \in X$, by Lemma 2.16, subsequently $u = v$. Thus, $u_n \xrightarrow{b} u$.

- Theorem 3.12. Assume $T: (X, q) \longrightarrow (Y, C)$ is f-continuous at x. If (X, q) is backward sequentially compact. Then, T is b-continuous at x.
  Proof. Since $T: X \longrightarrow Y$ is f-continuous at u that means any sequence $\{u_n\}_{n\in\mathbb{N}}$ in X forward converges to $u \in X$. Then, a sequence $\{T(u_n)\}_{n\in\mathbb{N}}$ approaches to $T(u) \in Y$. In other words, $u_n \xrightarrow{f} u \implies T(u_n) \to T(u)$ as $n \to \infty$, by Lemma 3.11, so $u_n \xrightarrow{b} u$. Since $T(u_n) \to T(u)$ whenever $u_n \xrightarrow{f} u$ as $n \to \infty$, so $T(u_n) \to T(u)$ whenever $u_n \xrightarrow{b} u$ as $n \to \infty$. Then, T is b-continuous at u.
- Example 3.13. Assume (X, q) is a QCMS where $X = [0,1]$, $E = \mathbb{R}^2$, $\alpha \in [0, 1)$, $P = \{(u, v) \in E: u, v \geq 0\}$ and $q: [0, 1] \times [0, 1] \longrightarrow \mathbb{R}^2$ is defined by (6),

$$q(u, v) = \begin{cases} (0,0) \text{ if } u \geq v, \\ (v, \alpha v) \text{ if } u < v. \end{cases} \tag{6}$$

  assume (X, C) is a CMS where $X = [0,1]$, $E = \mathbb{R}^2$, $\alpha \in [0, 1)$, $P = \{(u, v) \in E: u, v \geq 0\}$ and $C: [0, 1] \times [0, 1] \longrightarrow \mathbb{R}^2$ is defined by $C(u, v) = (|u - v|, \alpha|u - v|)$. A map $T: [0, 1] \longrightarrow [0, 1]$ defined by $T(u) = \frac{u}{3}$ is f-continuous at $\{0\}$.
- Remarks 3.14. Assume (X, q) is a QCMS and $\{u_n\}_{n\in\mathbb{N}}$ a backward convergent sequence in X, the sequence $\{u_n\}_{n\in\mathbb{N}}$ is not always the case a forward convergent sequence in X. Assume (X, q) is a QCMS and $\{u_n\}_{n\in\mathbb{N}}$ a forward convergent sequence in X, the sequence $\{u_n\}_{n\in\mathbb{N}}$ is not always the case a backward convergent sequence in X.
- Corollary 3.15. Assume $T: (X, q) \longrightarrow (Y, C)$ is b-continuous at x. If (X, q) is forward sequentially compact, then T is bb-continuous, bf-continuous, fb-continuous and ff-continuous at u.
  Proof. Since $T: X \longrightarrow Y$ is b-continuous at u and (X, q) is forward sequentially compact by –
  Proposition 3.3, a map T is bb-continuous and bf-continuous at u, and by Theorem 3.9, a map T is f-continuous then, by Proposition 3.7, a map T is ff-continuous and fb-continuous at u.
- Corollary 3.16. Assume $T: (X, q) \longrightarrow (Y, C)$ is f-continuous at u. If (X, q) is backward sequentially compact, then T is bb-continuous, bf-continuous, fb-continuous and ff-continuous at u.
  Proof. Since $T: X \longrightarrow Y$ is b-continuous at u and (X, q) is backward sequentially compact by Proposition 3.7, a map T is ff-continuous and fb-continuous at u, and by Theorem 3.12, a map T is b-continuous then, by Proposition 3.3, a map T is bb-continuous and bf-continuous at u.
- Definition 3.17. Assume (X, C) is a CMS and (Y, q) a QCMS. A map $T: X \longrightarrow Y$ is b^*-continuous at u if for any sequence $\{u_n\}_{n\in\mathbb{N}}$ in (X, C) converges to an element u in (X, C), $(C(u_n, x) \to 0_E \text{ as } n \to \infty)$. Then, a sequence $\{T(u_n)\}_{n\in\mathbb{N}}$ in (Y, q) backward converges to $T(u)$ in (Y, q) $\left(T(u_n) \xrightarrow{b} T(u) \text{ as } n \to \infty\right)$. Provided T is b^*-continuous at each $u \in X$, then it is called b^*-continuous.
- Example 3.18. Assume (X, C) is a CMS where $X = [0,1]$, $E = \mathbb{R}^2$, $\alpha \in [0, 1)$, $P = \{(u, v) \in \mathbb{R}^2: u, v \geq 0\}$ and $C: [0, 1] \times [0, 1] \longrightarrow \mathbb{R}^2$ is defined by $C(u, v) = (|u - v|, \alpha|u - v|)$. Suppose$(X, q)$ is a QCMS where $X = [0,1]$, $E = \mathbb{R}^2$, $\alpha \in [0, 1)$, $= \{(u, v) \in \mathbb{R}^2: u, v \geq 0\}$ and $q: [0, 1] \times [0, 1] \longrightarrow \mathbb{R}^2$ is defined by (7),

$$q(u, v) = \begin{cases} \big(u - v, \alpha(u - v)\big), \text{if } u \geq v, \\ (\alpha, 1), \text{if } u < v. \end{cases} \tag{7}$$

  a map $T: [0, 1] \longrightarrow [0, 1]$ defined by $T(u) = \frac{u}{2}$ is b^*-continuous at $\{0\}$.
- Definition 3.19. Assume (X, C) is a CMS and (Y, q) a QCMS. A map $T: X \longrightarrow Y$ is f^*-continuous at u if for any sequence $\{u_n\}_{n\in\mathbb{N}}$ in (X, C) converges to an element u in (X, C) $(C(u_n, u) \to 0_E \text{ as } n \to \infty)$. Then, a sequence $\{T(u_n)\}_{n\in\mathbb{N}}$ in (Y, q) forward converges to $T(u)$ in (Y, q) $\left(T(u_n) \xrightarrow{f} T(u) \text{ as } n \to \infty\right)$. Provided T is f^*-continuous at each $u \in X$, then it is called f^*-continuous.
- Example 3.20. Assume (X, C) is a CMS where $X = [0,1]$, $E = \mathbb{R}^2$, $\alpha \in [0, 1)$, $P = \{(u, v) \in \mathbb{R}^2: u, v \geq 0\}$ and $C: [0, 1] \times [0, 1] \longrightarrow \mathbb{R}^2$ is defined by $C(u, v) = (|u - v|, \alpha|u - v|)$. Assume (X, q) is a QCMS

where $X=[0,1]$, $E=\mathbb{R}^2$, $\alpha\in[0,1)$, $=\{(u,v)\in\mathbb{R}^2: u,v\geq 0\}$ and $q:[0,1]\times[0,1]\longrightarrow\mathbb{R}^2$ is defined by (8),

$$q(u,v)=\begin{cases}\big(v-u,\alpha(v-u)\big), \text{if } v\ \geq\ u,\\ \quad(\alpha,1), \text{if } v<u.\end{cases} \tag{8}$$

a map $T:[0,1]\longrightarrow\ [0,1]$ defined by $T(u)=\frac{u}{3}$ is f^*-continuous at $\{0\}$.

- Proposition 3.21. Assume $T:(Y,C)\longrightarrow(X,q)$ is b^*-continuous at u. Then, a map T is bb-continuous at u and fb-continuous at u.
  Proof. Since a map $T:(Y,C)\longrightarrow(X,q)$ is b^*-continuous at u, then for every sequence $\{u_n\}_{n\in\mathbb{N}}$ in (Y,C) approaches to an element u in (Y,C), i.e., $C(u_n,u)\longrightarrow 0_E$ as $n\to\infty$, we have a sequence $\{T(u_n)\}_{n\in\mathbb{N}}$ that backward approaches to $T(u)$ in (X,q), i.e., $q(T(u_n),T(u))\longrightarrow 0_E$ as $n\to\infty$, where every CMS is QCMS and $C(u_n,u)=C(u,u_n)\longrightarrow 0_E$ as $n\to\infty$. Thus, a map T is bb-continuous at u and fb-continuous at u.
- Remark 3.22. Assume $T:(X,q)\longrightarrow(Y,q)$ is bb-continuous at u, or fb-continuous at u; then, a map T is not b^*-continuous at u because not every QCMS is necessarily CMS.
- Proposition 3.23. Assume $T:(Y,C)\longrightarrow(X,q)$ is f^*-continuous at u. Then a map T is ff-continuous at u and bf-continuous at u.
  Proof. Since a map $T:(Y,C)\longrightarrow(X,q)$ is f^*-continuous at u, then for any sequence $\{u_n\}_{n\in\mathbb{N}}$ in (Y,C) approaches to an element u in (Y,C), i.e., $C(u_n,u)\longrightarrow 0_E$ as $n\to\infty$, we have a sequence $\{T(u_n)\}_{n\in\mathbb{N}}$ that forward approaches to $T(u)$ in (X,q), i.e., $q(T(u_n),T(u))\longrightarrow 0_E$ as $n\to\infty$, where every CMS is a QCMS and $C(u_n,u)=C(u,u_n)\longrightarrow 0_E$ as $n\to\infty$. Thus, a map T is ff-continuous at u and bf-continuous at u.
- Theorem 3.24. Assume $T:(Y,C)\longrightarrow(X,q)$ is b^*-continuous at u. If (X,q) is forward sequentially compact, then T is f^*-continuous at u.
  Proof. Since $T:Y\longrightarrow X$ is b^*-continuous at u, every sequence $\{u_n\}_{n\in\mathbb{N}}$ in Y converges to $u\in Y$, then the sequence $\{T(u_n)\}_{n\in\mathbb{N}}$ backward approaches to $T(u)\in X$.In other words, $u_n\to u\Longrightarrow T(u_n)\xrightarrow{b}T(u)$ as $n\to\infty$, where (X,q) is forward sequentially compact, the sequence $\{T(u_n)\}_{n\in\mathbb{N}}$ has a forward convergent subsequence in X say $T(u_{n_k})\xrightarrow{f}T(v)$ by Lemma 2.15, so $T(u_n)\xrightarrow{f}T(v)\in X$. Since $T(u_n)\xrightarrow{b}T(u)\in X$ and $T(u_n)\xrightarrow{f}T(v)\in X$ by Lemma 2.16, subsequently $T(u)=\ T(v)$. Thus, $T(u_n)$ $\xrightarrow{f}T(u)$. Since $T(u_n)\xrightarrow{b}T(u)$ whenever $u_n\to u$ as $n\to\infty$, $T(u_n)\xrightarrow{f}T(u)$ whenever $u_n\to u$ as $n\to\infty$. Then, T is f^*-continuous at u.
- Theorem 3.25. Assume $T:(Y,C)\longrightarrow(X,q)$ is f^*-continuous at u. If (X,q) is backward sequentially compact, then T is b^*-continuous at u.
  Proof. Since $T:Y\longrightarrow X$ is f^*-continuous at u, then any sequence $\{u_n\}_{n\in\mathbb{N}}$ in Y converges to $u\in Y$, then the sequence $\{T(u_n)\}_{n\in\mathbb{N}}$ forward approaches to $T(u)\in X$.
  In other words, $u_n\to u\Longrightarrow T(u_n)\xrightarrow{f}T(u)$ as $n\to\infty$, where (X,q) is backward sequentially compact, the sequence $\{T(u_n)\}_{n\in\mathbb{N}}$ has a backward convergent subsequence in X say $T(u_{n_k})\xrightarrow{b}T(v)$ by Lemma 2.14, so $T(u_n)\xrightarrow{b}T(v)\in X$. Since $T(u_n)\xrightarrow{b}T(u)\in X$ and $T(u_n)\xrightarrow{f}T(v)\in X$ by Lemma 2.16, subsequently $T(u)=T(v)$. Thus, $T(u_n)\xrightarrow{b}T(u)$. Since $T(u_n)\xrightarrow{f}T(u)$ whenever $u_n\to u$ as $n\to\infty$, so $T(u_n)\xrightarrow{b}T(u)$ whenever $u_n\to u$ as $n\to\infty$. Then T is b^*-continuous at u. Proof. Since $T:X\longrightarrow Y$ is f^*-continuous at u and (X,q) is backward sequentially compact by Proposition 3.23, a map T is f-continuous and bf-continuous at u, and by Theorem 3.25, a map T is b^*-continuous then, by Proposition 3.21, a map T is bb-continuous and fb-continuous at u.
- Corollary 3.26. Assume $T:(X,C)\longrightarrow(Y,q)$ is b^*-continuous at u, if (Y,q) is forward sequentially compact. Then a map T is bf-continuous, ff-continuous, bb-continuous and fb-continuous at u.
  Proof. Since $T:X\longrightarrow Y$ is b^*-continuous at u and (X,q) is forward sequentially compact by Proposition 3.21, a map T is bb-continuous and fb-continuous at u, and by Theorem 3.24, a map T is f^*-continuous then, by Proposition 3.23, a map T is ff-continuous and bf-continuous at u.
- Corollary 3.27. Assume $T:(X,C)\longrightarrow(Y,q)$ is f^*-continuous at u, if (Y,q) is backward sequentially compact, then a map T is bf-continuous, ff-continuous, bb-continuous and fb-continuous at u.
- Lemma 3.28. Assume (X,C) is a CMS and $\{u_n\}_{n\geq 1}$ a sequence in X. Assume there is an order of non-negative natural numbers $\{\lambda_n\}_{n\geq 1}$ such that $\sum_{n=1}^{\infty}\gamma_n<\infty$, in which $C(u_n,u_{n+1})\leq\gamma_n M$, for some $M\in P$, and for all $n\in\mathbb{N}$. Then, the sequence $\{u_n\}_{n\geq 1}$ is Cauchy order in (X,C).
  Proof. For $n>m$, we get,

$$C(u_m,u_n) \leq C(u_m,u_{m+1}) + C(u_{m+1},u_{m+2}) + \cdots + C(u_{n-1},u_n) \leq M \sum_{m=1}^{\infty} \gamma_i$$

suppose $c \in \operatorname{int} P$ and choose $\delta > 0$ such that $c + N_\delta(0) \subset P$ where,

$$N_\delta(0) = \{v \in E : \|v\| < \delta\}$$

since $\sum_{n=1}^{\infty} \gamma_n < \infty$, there exists an actual number n_0 to the extent that for all $m \geq n_0$ $M\sum_{m=1}^{\infty} \gamma_i \in N_\delta(0)$, also $-M\sum_{m=i}^{\infty} \gamma_i \in N_\delta(0)$ since $c + N_\delta(0)$ is open. Thus, $c + N_\delta(0) \in \operatorname{int} P$; that is $c - M\sum_{m=i}^{\infty} \gamma_i \in \operatorname{int} P$. Thus, $M\sum_{m=i}^{\infty} \gamma_i \ll c$ for $m \geq n_0$ and so, $C(u_m,u_n) \ll$ c for $n > m \geq n_0$. Thus, $\{u_n\}_{n\geq 1}$ is a Cauchy order.

- Theorem 3.29. Assume (X, C) is a complete Hausdorff CMS, (X, q) a QCMS and suppose $T: (X, C) \longrightarrow (X, q)$ is a b^*-continuous mapping. Suppose that there are functions $\eta, \lambda, \zeta, \mu, \xi: X \longrightarrow [0,1)$ which satisfy the following for $u, v \in X$,

(1) $\eta(T(u)) \leq \eta(u), \lambda(T(u)) \leq \lambda(u), \zeta(T(u)) \leq \zeta(u), \mu(T(u)) \leq \mu(u)$ and $\xi(T(u)) \leq \xi(u)$
(2) $\eta(u) + \lambda(u) + \zeta(u) + \mu(u) + \xi(u) < 1$
(3) $q(T(u),T(v)) \leq \eta(u)q(u,v) + \lambda(u)q(u,T(u)) + \zeta(u)q(v,T(v)) + \mu(u)q(T(u),v) + \xi(u)q(v,T(v))$
(4) $C(T(u),T(v)) \leq \eta(u)C(u,v) + \lambda(u)C(u,T(u)) + \zeta(u)C(v,T(v)) + \mu(u)C(T(u),v) + \xi(u)C(u,T(v))$

T then has a distinctive fixed point.

Proof. Let $u_0 \in X$ be arbitrary and fixed, and we consider the sequence of orbit of u_0, $u_n = T(u_{n-1})$ for all $n \in \mathbb{N}$. If we take $u = u_{n-1}$ and $v = u_n$ in (4) we have,

$$C(u_n,u_{n+1}) = C(T(u_{n-1}),T(u_n))$$

$$\leq \eta(u_{n-1})C(u_{n-1},u_n) + \lambda(u_{n-1})C(u_{n-1},T(u_{n-1})) +$$

$$\zeta(u_{n-1})C(u_n,T(u_n)) + \mu(u_{n-1})C(T(u_{n-1}),u_n) + \xi(u_{n-1})C(u_{n-1},T(u_n))$$

$$= \eta(T(u_{n-2}))C(u_{n-1},u_n) + \lambda(T(u_{n-2}))C(u_{n-1},u_n) +$$

$$\zeta(T(u_{n-2}))C(u_n,u_{n+1}) + \mu(T(u_{n-2}))T(u_n,u_n) +$$

$$\xi(T(u_{n-2}))C(u_{n-1},u_{n+1}))$$

$$\leq \eta(u_{n-2})C(u_{n-1},u_n) + \lambda(u_{n-2})C(u_{n-1},u_n) + \zeta(u_{n-2})C(u_n,u_{n+1}) +$$

$$\xi(u_{n-2})(C(u_{n-1},u_n) + C(u_n,u_{n+1}))$$

$$\vdots$$

$$\leq \eta(u_0)C(u_{n-1},u_n) + \lambda(u_0)C(u_{n-1},u_n) + \zeta(u_0)C(u_n,u_{n+1}) + \xi(u_0)(C(u_{n-1},u_n) + C(u_n,u_{n+1}))$$

$$= \eta(u_0)C(u_{n-1},u_n) + \lambda(u_0)C(u_{n-1},u_n) + \zeta(u_0)C(u_n,u_{n+1}) + \xi(u_0)C(u_{n-1},u_n) + \xi(u_0)C(u_n,u_{n+1})$$

$$= (\eta(u_0) + \lambda(u_0) + \xi(u_0))C(u_{n-1},u_n) + \zeta(u_0)C(u_n,u_{n+1}) + \xi(u_0)C(u_n,u_{n+1})$$

$$= (\eta(u_0) + \lambda(u_0) + \xi(u_0))C(u_{n-1},u_n) + (\zeta(u_0) + \xi(u_0))C(u_n,u_{n+1}).$$

so,

$$C(u_n,u_{n+1}) - (\zeta(u_0) + \xi(u_0))C(u_n,u_{n+1}) \leq (\eta(u_0) + \lambda(u_0) + \xi(u_0))C(u_{n-1},u_n)$$

so,

$$C(u_n,u_{n+1})(1 - (\zeta(u_0) + \xi(u_0))) \leq (\eta(u_0) + \lambda(u_0) + \xi(u_0))C(u_{n-1},u_n)$$

so,

$$C(u_n, u_{n+1}) \leq \left(\frac{\eta(u_0)+\lambda(u_0)+\xi(u_0)}{1-\zeta(u_0)-\xi(u_0)}\right) C(u_{n-1}, u_n)$$

$$= h\, C(u_{n-1}, u_n)$$
$$\leq h^2\, C(u_{n-2}, u_{n-1})$$

$$\vdots$$

$$\leq h^n\, C(u_0, u_1)$$

thus, by Lemma 3.28, $\{u_n\}_{n\geq 1}$ is Cauchy in X. Because of completeness of X and T is a b^*-continuous mapping, there exists $w \in X$ such that $u_n \to w$ and $u_{n+1} = T(u_n) \overset{b}{\to} T(w)$.
For uniqueness, let w_1 be another fixed point of T, then,

$$q(w, w_1) = q(T(w), T(w_1))$$

$$\leq \eta(w)q(w, w_1) + \lambda(w)q\big(w, T(w)\big) + \zeta(w)q\big(w_1, T(w_1)\big) + \mu(w)q(T(w), w_1) + \xi(w)q(w, T(w_1))$$

$$= \eta(w)q(w, w_1) + \lambda(w)q(w, w) + \zeta(w)q(w_1, w_1) + \mu(w)q(w, w_1) + \xi(w)q(w, w_1)$$

$$= \eta(w)q(w, w_1) + \mu(w)q(w, w_1) + \xi(w)q(w, w_1)$$

$$= (\eta(w) + \mu(w) + \xi(w))q(w, w_1)$$

Therefore; $\eta(w) + \mu(w) + \xi(w) \geq 1$, contradicts to (2). Thus, $q(w, w_1) = 0 \implies w = w_1$.

- Corollary 3.30. Assume (X, C) is a complete Hausdorff CMS, (X, q) a QCMS and suppose $T: (X, C) \longrightarrow (X, q)$ is a b^*-continuous mapping. Suppose that there are functions $\eta, \lambda, \mu: X \to [0, 1)$, which gratify the following for $u, v \in X$,

$\eta(T(u)) \leq \eta(u)$, $\lambda(T(u)) \leq \lambda(u)$ and $\mu(T(u)) \leq \mu(u)$,

(1) $\eta(u) + 2\lambda(u) + 2\mu(u) < 1$

(2) $q\big(T(u), T(v)\big) \leq \eta(u)q(u, v) + \lambda(u)(q(u, T(v)) + q(v, T(v))) + \mu(u)(q(T(u), v) + q(u, T(v)))$

(3) $C\big(T(u), T(v)\big) \leq \eta(u)C(u, v) + \lambda(u)(C(u, T(u)) + C(v, T(v))) + \mu(u)(C(T(u), v) + C(u, T(v)))$

then, T has a unique fixed point.

4. CONCLUSION

This study was mainly concerned with introducing four new notions of continuity of mapping, some theorems and propositions comparing these new notions to each other, illustrating them with some examples. Furthermore, in this paper, the fixed-point theorem of this newly introduced notion has been introduced. Future research can investigate a unique fixed point between CMS and QCMS to prove a fixed point between QCMS and CMS.

ACKNOWLEDGEMENTS

Authors would like to thank Department of fundamental and applied science, Universiti Teknologi PETRONAS UTP for their support. The funder of this work is Yayasan YUTP under grant cost center 015LC0-272.

REFERENCES

[1] L. G. Huang and X. Zhang, "Cone metric spaces and fixed point theorems of contractive mappings," *J. Math. Anal. Appl.*, vol. 332, no. 2, pp. 1468-1476, 2007, doi: 10.1016/j.jmaa.2005.03.087.
[2] M. Abbas and G. Jungck, "Common fixed point results for noncommuting mappings without continuity in cone metric spaces," *J. Math. Anal. Appl.*, vol. 341, no. 1, pp. 416-420, 2008, doi: 10.1016/j.jmaa.2007.09.070.
[3] D. Turkoglu and M. Abuloha, "Cone metric spaces and fixed point theorems in diametrically contractive mappings," *Acta Math. Sin. Engl. Ser.*, vol. 26, no. 3, pp. 489-496, 2010, doi: 10.1007/s10114-010-8019-5.

[4] Z. E. D. D. Olia, M. E. Gordji, and D. E. Bagha, "Banach Fixed Point Theorem on Orthogonal Cone Metric Spaces," *Facta Univ. Ser. Math. Informatics*, vol. 35, p. 1239, 2021, doi: 10.22190/fumi2005239e.
[5] T. Yaying, B. Hazarika, and H. Cakalli, "New results in quasi cone metric spaces," *J. Math. Comput. Sci.*, vol. 16, no. 03, pp. 435-444, 2017, doi: 10.22436/jmcs.016.03.13.
[6] S. Rezapour and R. Hamlbarani, "Some notes on the paper 'Cone metric spaces and fixed point theorems of contractive mappings," *J. Math. Anal. Appl.*, vol. 345, no. 2, pp. 719-724, 2008, doi:10.1016/j.jmaa.2008.04.049.
[7] S. Janković, Z. Kadelburg, and S. Radenović, "On cone metric spaces: a survey," *Nonlinear Anal. Theory, Methods & Appl.*, vol. 74, no. 7, pp. 2591-2601, 2011, doi:10.1016/j.na.2010.12.014.
[8] T. Abdeljawad and E. Karapinar, "Quasicone metric spaces and generalizations of caristi kirk's theorem," *Fixed Point Theory Appl.*, vol. 2009, no. 1, pp. 9-18, 2009, doi:10.1155/2009/574387.
[9] T. Yaying, B. Hazarika, and S. A. Mohiuddine, "Some New Types of Continuity in Asymmetric Metric Spaces," *Facta Univ. Ser. Math. Informatics*, p. 485, 2020, doi: 10.22190/FUMI2002485Y.
[10] M. Abbas, B. E. Rhoades, and T. Nazir, "Common fixed points for four maps in cone metric spaces," *Appl. Math. Comput.*, vol. 216, no. 1, pp. 80-86, 2010, doi: 10.1016/j.amc.2010.01.003.
[11] Q. Yan, J. Yin, and T. Wang, "Fixed point and common fixed point theorems on ordered cone metric spaces over Banach algebras," *J. Nonlinear Sci. Appl.*, vol. 9, no. 4, pp. 1581-1589, 2016, doi: 10.22436/jnsa.009.04.15.
[12] Altun and G. Durmaz, "Some fixed point theorems on ordered cone metric spaces," *Rend. del Circ. Mat. di Palermo*, vol. 58, no. 2, pp. 319-325, 2009, doi: 10.1007/s12215-009-0026-y.
[13] D. Ilic and V. Rakocevic, "Common fixed points for maps on cone metric space," *J. Math. Anal. Appl.*, vol. 341, no. 2, pp. 876-882, 2008, doi:10.1016/j.jmaa.2007.10.065.
[14] D. Ilić and V. Rakočević, "Quasi-contraction on a cone metric space," *Appl. Math. Lett.*, vol. 22, no. 5, pp. 728-731, 2009, doi: 10.1016/j.aml.2008.08.011.
[15] S. Radenović and B. E. Rhoades, "Fixed point theorem for two non-self mappings in cone metric spaces," *Comput. Math. with Appl.*, vol. 57, no. 10, pp. 1701-1707, 2009, doi: 10.1016/j.camwa.2009.03.058.
[16] N. Sadigh and S. Ghods, "Coupled coincidence point in ordered cone metric spaces with examples in game theory," *Int. J. Nonlinear Anal. Appl.*, vol. 7, no. 1, pp. 183-194, 2016, doi: 10.22075/IJNAA.2015.305.
[17] X. Ge and S. Yang, "Some fixed point results on generalized metric spaces [J]," *AIMS Math.*, vol. 6, no. 2, pp. 1769-1780, 2021, doi: 10.3934/Math.2021106.
[18] R. Mustafa, S. Omran, and Q. N. Nguyen, "Fixed Point Theory Using $ψ$ Contractive Mapping in C∗-Algebra Valued B-Metric Space," *Mathematics*, vol. 9, no. 1, p. 92, 2021, doi: 10.3390/math9010092.
[19] J. Fernandez, N. Malviya, Z. D. Mitrović, A. Hussain, and V. Parvaneh, "Some fixed point results on Nb -cone metric spaces over Banach algebra," *Adv. Differ. Equations*, vol. 2020, no. 1, pp. 1-15, 2020, doi: 10.1186/s13662-020-02991-5.
[20] N. Sadigh and S. Ghods, "Coupled coincidence point in ordered cone metric spaces with examples in game theory," *Int. J. Nonlinear Anal. Appl.*, vol. 7, no. 1, pp. 183-194, 2016, doi: 10.22075/ijnaa.2015.305.
[21] L.-G. Huang and X. Zhang, "Cone metric spaces and fixed point theorems of contractive mappings," *J. Math. Anal. Appl.*, vol. 332, no. 2, pp. 1468-1476, 2007, doi: 10.1016/j.jmaa.2006.03.087.
[22] Z. M. Fadail and S. M. Abusalim, "T-Reich contraction and fixed point results in cone metric spaces with c-distance," *Int. J. Math. Anal.*, vol. 11, no. 8, pp. 397-405, 2017, doi: 10.12988/ijma.2017.7338.
[23] S. U. Rehman and H.-X. Li, "Fixed point theorems in fuzzy cone metric spaces," *J. Nonlinear Sci. Appl. JNSA*, vol. 10, no. 11, pp. 5763-5769, 2017, doi:10.22436/jnsa.010.11.14.
[24] M. Abbas, A. R. Khan, and T. Nazir, "Common fixed point of multivalued mappings in ordered generalized metric spaces," *Filomat*, vol. 26, no. 5, pp. 1045-1053, 2012, doi: 10.2298/FIL1205045A.
[25] R. George, H. A. Nabwey, J. Vujaković, R. Rajagopalan, and S. Vinayagam, "Dislocated quasi cone b-metric space over Banach algebra and contraction principles with application to functional equations," *Open Math.*, vol. 17, no. 1, pp. 1065-1081, 2019, doi: 10.1515/math-2019-0086.
[26] T. Abdeljawad and E. Karapinar, "Quasicone metric spaces and generalizations of Caristi Kirk's theorem," *Fixed Point Theory Appl.*, vol. 2009, pp. 1-9, 2009, doi: 10.1155/2009/574387.
[27] K. Abodayeh, T. Qawasmeh, W. Shatanawi, and A. Tallafha, "Eϕ-Contraction and Some Fixed Point Results Via Modified Ω-Distance Mappings in the Frame of Complete Quasi Metric Spaces and Applications," *International Journal of Electrical and Computer Engineering (IJECE)*, vol. 10, no. 4, pp. 3839-3853, 2020, doi: 10.11591/ijece.v10i4.pp3839-3853.

BIOGRAPHIES OF AUTHORS

Abdullah Al-Yaari received a B.Sc. degree in Mathematics (Honors) from Thamar University, in 2007, as well as M.Sc. in Mathmatics from University Kebangsaan Malaysia. He is currently a doctoral stusent at Universiti Teknologi PETRONAS (UTP) and an Academic Staff at Thamar University, Yemen. His research interests lie in fixed point theorem and nanofluid flow in porous media. He can be contacted Phone: +60-1-8310-3931, Fax: +60-3-8925-4519 and at email: abdullah_20001447@utp.edu.my.

Hamzah Bin Sakidin is a Senior Lecturer at Universiti Teknologi PETRONAS (UTP) Seri Iskandar Perak since September 2013. His research interest is in the field of Applied Mathematics and mathematical modelling. He has received several research grants, acting as the head researcher or co-researcher for some research grants. He is also the author of several mathematics book for tertiary level of education. The author also has more than 20 years of experience teaching in secondary and tertiary level of education. He can be contacted at email: hamzah.sakidin@utp.edu.my.

Yousif Alyousifi received a B.Sc. degree in Mathematics (Honors) from Thamar University, in 2007, as well as M.Sc. and PhD degrees in Statistics from University Kebangsaan Malaysia. He is currently an Academic Staff at Thamar University, Yemen. His research interests lie in statistical modeling and advanced time series analysis and stochastic process under classic and Bayesian approaches. He can be contacted at email: yalyousify@tu.edu.ye.

Qasem Al-Tashi is currently a Postdoctoral Fellow at The University of Texas MD Anderson Cancer Center, Houston, Texas, United States. He was a Research Scientist at Universiti Teknologi PETRONAS. He received the B.Sc. degree in software engineering from Universiti Teknologi Malaysia, in 2012, and the M.Sc. degree in software engineering from Universiti Kebangsaan Malaysia, in 2017. He obtained his PhD in Information in early 2021 from Universiti Teknologi PETRONAS. He is also an Academic Staff with Albaydha University, Yemen. His research interests include artificial neural networks, multi-objective optimization, feature selection, swarm intelligence evolutionary algorithms, classification and data analytics. He is a section editor for the Journal of Applied Artificial Intelligence (JAAI) and Journal of Information Technology and Computing (JITC) and a reviewer for several high impact factor journals such as Artificial Intelligence Review, the IEEE ACCESS, Knowledge-Based Systems, Soft Computing, Journal of Ambient Intelligence and Humanized Computing, Applied Soft Computing, Neurocomputing, Applied Artificial Intelligence, and Plos One. He can be contacted at email: qaal@mdanderson.org, qasemacc22@gmail.com.