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Some Results of Bondage Number of (n,k)-Star Graphs

Yunchao Wei*, Hongxian Zhu, Junli Han
College of Information Technology, Shanghai Ocean University, Shanghai, 201306, P. R. China
***Corresponding author, e-mail: ycwei@shou.edu.cn**

Abstract

In the computer network, bondage number is one of the most important parameters to measure the control theory of the computer network, denoted by $b(G)$ for a network graph G. So computing $b(G)$ of some particular known gr-aphs is extremely valuable. In this paper, we determine $b(S_{n,2})$ and the precise lowerbound of $b(G)$ of (n,k)-star graphs, denoted by $S_{n,k}$, followed by some relative conclusions of n-star, denoted by S_n as the isomorphism of $S_{n,n-1}$.

Keywords: *(n,k)-star graph, bondage number, combinatories.*

1. Introduction

It is widely known that bondage number is one of the most important parameters to measure the resilience of graphs of computer network. For the particular known graphs, so far the results of this aspect are a few such as bondage number of de Bruijn and Kautz digraphs [9], bondage number in oriented graphs [10]. Especially, Huang and Xu [11] got a good lower-bound and a good upper-bound of bondage number of vertex-transitive graphs, but the precise lower-bound of bondage number of (n,k)-star graphs (vertex-transitive graphs) and $b(S_{n,2})$ can not be got by their results. Next, we see conception of bondage number:

Definition 1.1. Let G be a graph, and S be a nonempty subset of $V(G)$, then S is one dominating set of G if all nodes of G is either in S, or adjacent to a node of S. Moreover, we call that $|S|$ is dominating number of G if $|S|$ is minimum in all dominating sets of G, denoted by $\gamma(G)$.

Definition 1.2. Let G be a undirect graph, and B be a nonempty edge-subset of $E(G)$, then minimum $|B|$ is bondage number of G if $\gamma(G-B) > \gamma(G)$, denoted by $b(G)$.

In a network graph, predecessors have shown that computing $b(G)$ are extremely difficult. So computing $b(G)$ of some particular known graphs is very valuable. For example, the (n,k)-star graphs was first proposed in 1995 by W.K Chiang et al [1]. Because of good topological properties of $S_{n,k}$, its many properties have been researched such as diameter and connectivity [1, 8], pancyclicity [2], $\kappa_s^{(1)}(G)$ and $\kappa_s^{(2)}(G)$ [3-7], fault hamiltonicity and fault hamiltonicity connectivity [4, 12], independent number and dominating number [13] and so on. In this paper, we determine $b(G)$ of (n,k)-star graphs, so that can get $\gamma(S_n)$ and the good lower-bound of $b(S_n)$ of n-star, denoted by S_n as the isomorphism of $S_{n,n-1}$.

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2. Preliminaries

For given integers n and k, where $1 \le k \le n-1$, let $J_n = \{1, 2, ..., n\}$ and let $P(n, k)$ be the set of k-permutations on J_n for $1 \le k \le n-1$, that is, $P(n,k) = \{p_1 p_2 ... p_k : p_i \in J_n, p_i \ne p_j, 1 \le i \ne j \le k\}$.

Definition 2.1. The (n,k)-star graph, denoted by $S_{n,k}$, is an undirected graph with vertex-set $P(n,k)$. The adjacency is defined as follows: a vertex $p_1 p_2 ... p_i ... p_k$ is adjacent to a vertex

(1) $p_1 p_2 \cdots p_{i-1} p_1 p_{i+1} \cdots p_k$, where $2 \le i \le k$ (swap p_1 with p_i).

(2) $x p_2 ... p_k$, where $x \in J_n - \{p_i : 1 \le i \le k\}$ (replace p_1 by x).

Figure 1 shows a (4,2)-star graph $S_{4,2}$.

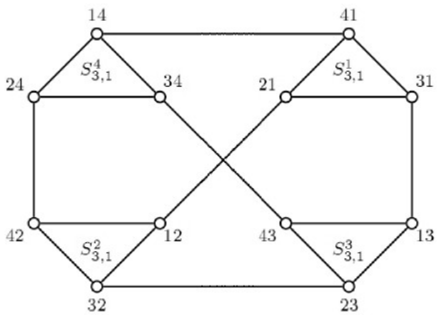

Figure 1. The Structure of a (4,2)-star Graph $S_{4,2}$

The edges of type (1) are referred to as i-edges ($2 \le i \le k$), and the edges of type (2) are referred to as 1-edge. The vertices of type (1) are referred to as swap-adjacent vertices, and the vertices of type (2) are referred to as unswap-adjacent vertices. We also call i-edge as swap-edge, and call 1-edges as unswap-edge. Clearly, every vertex in $S_{n,k}$ has $(k-1)$ swap-adjacent vertices and $(n-k)$ unswap-adjacent vertices. Usually, if $v = p_1 p_2 ... p_k$ is a vertex in $S_{n,k}$, we call that p_i is the i-th bit for each $i = 1, 2, ...k$.

By Definition 2.1, we know $S_{n,n-1} \cong S_n$ and $S_{n,1} \cong K_n$ where S_n is n-star graph and K_n is complete graph with order n. So $S_{n,k}$ is a generalization of S_n. It has been shown by Chiang and Chen [1] that S_n is an $(n-1)$-regular, $(n-1)$-connected vertex-transitive graph with $n!/(n-k)!$ vertices.

The following content, we mainly determine the dominating number of $S_{n,k}$ for obtaining main results of Section 3. Since $S_{n,1} \cong K_n$, we only consider the case $k \ge 2$ in the following discussion.

Lemma 2.2. $\gamma(S_{n,k}) \ge \dfrac{(n-1)!}{(n-k)!}$ for $2 \le k \le n-1$.

Proof. Let $S(\subset V(S_{n,k}))$ be a minimum dominating set of $S_{n,k}$, then $|S| = \gamma(S_{n,k})$ by Definition 1.1. By Definition 2.1, we have known that $S_{n,k}$ is a $(n-1)$-regular graph, so each vertex of S can at most dominate $(n-1)$ vertices in $S_{n,k}-S$. If $\gamma(S_{n,k}) \le \frac{(n-1)!}{(n-k)!}-1$ then S can at most dominate $\left[\frac{(n-1)!}{(n-k)!}-1\right](n-1)$ vertices in $S_{n,k}-S$. Thus, we can get:

$$|S|+|V(S_{n,k}-S)| \le \left[\frac{(n-1)!}{(n-k)!}-1\right](n-1)+\frac{(n-1)!}{(n-k)!}-1$$
$$=\frac{n!}{(n-k)!}-n<\frac{n!}{(n-k)!}=|V(S_{n,k})|$$

It is contrary to the definition of dominating number. □

Theorem 2.3. $\gamma(S_{n,k})=\frac{(n-1)!}{(n-k)!}$ for $2 \le k \le n-1$.

Proof. By Lemma 2.2, we have shown $|S|=\gamma(S_{n,k}) \ge \frac{(n-1)!}{(n-k)!}$. Thus, by Definition 1.1, Theorem 2.3 can be proved if we can construct a dominating set S, so that $|S|=\frac{(n-1)!}{(n-k)!}$.

We now split $V(S_{n,k})$ into three vertex-subsets: $V_n=\{\mathrm{n}\alpha \,|\, \alpha \in P(n-1,k-1)\}$, $V_n'=\{\alpha \,|\, \alpha \in P(n-1,k)\}$ and $V_n''=\{p_1p_2\dots p_{a-1}np_{a+1}\dots p_k \,|\, p_i \in J_{n-1}, a \ge 2\}$. It is easy to verify that V_{n}, V_n' and V_n'' have no intersection, and $|V_{\mathrm{n}}|+|V_{\mathrm{n}}'|+|V_n''|=|V(S_{n,k})|=\frac{n!}{(n-k)!}$ since $|V_{\mathrm{n}}|=\frac{(\mathrm{n}-1)!}{(\mathrm{n}-\mathrm{k})!}, |V_n'|=\frac{(n-1)!}{(n-k-1)!}$ and $|V_{\mathrm{n}}''|=\frac{(n-1)!}{(n-k)!}(k-1)$.

Let $\mathrm{p}_1p_2\dots p_k$ be any one vertex of V_{n}, then all neighboring-edges of $\mathrm{p}_1p_2\dots p_k$ must have one unswap-edge connected to $np_2\dots p_k$ of V_{n}.

Let $p_1p_2\dots p_{a-1}np_{a+1}\dots p_k$ be any one vertex of V_n'', then all neighboring-edges of $p_1p_2\dots p_{a-1}n\ \ p_{a+1}\dots p_k$ must have one swap-edge connected to $np_2p_{a-1}p_1p_{a+1}\dots p_k$ of V_{n}. Thus, we can let $V_{\mathrm{n}}=S$, and $|V_{\mathrm{n}}|=\frac{(n-1)!}{(n-k)!}$. □

Corollary 2.4. In n-star graph S_{n}, $\gamma(S_n)=(n-1)!$.

Corollary 2.5 If let $V_x=\{xp_1p_2\dots p_{k-1} \,|\, p_j \in J_n \setminus x\}(x \in J_n)$, then each V_x is a minimum dominating set of $S_{n,k}$ for $x=1,2,\dots,n$.

Lemma 2.6. If S is a minimum dominating set of $S_{n,k}$, then any two vertices of S aren't adjacent in $S_{n,k}$, and any two neighboring-vertices of S aren't common.

Proof. Let v_1 and v_2 be any two vertices of S, if v_1 and v_2 are adjacent in $S_{n,k}$, then v_1 and v_2 can at most dominate $(2n-4)$ vertices of $S_{n,k}-S$ since either v_1 or v_2 only dominate $(n-2)$ vertices of $S_{n,k}-S$. Thus, we can get that S can at most dominate $[(|S|-2)(n-1)+2n-4]$ vertices of $S_{n,k}-S$, and can get $|S|+(|S|-2)(n-1)+2n-4=n|S|-2=|V(S_{n,k})|-2<|V(S_{n,k})|$, a contradiction.

If there exist two neighboring-vertices of S who are common, then S can at most dominate $|S|(n-1)-1$ vertices of $S_{n,k}-S$ Therefore, we have $|S|+|S|(n-1)-1=|V(S_{n,k})|-1<|V(S_{n,k})|$, a contradiction. □

3. The Important Results of Bondage Number of $S_{n,k}$

In this section, we mainly consider the bondage number of $S_{n,k}$ ..

Lemma 3.1. If let $V_x=\{xp_1p_2\dots p_{k-1}|p_j\in J_n\setminus x\}(x\in J_n)$, then:

(a) Any two vertices of $V_x\,(x\in J_n)$ aren't adjacent.

(b) Any two set V_x and $V_y\,(x,y\in J_n,x\neq y)$ have no intersection, and $V(S_{n,k})=V_1\bigcup V_2\bigcup\cdots\bigcup V_n$.

(c) Any one vertex of $V_x\,(x\in J_n)$ has exactly a neighbor respectively in $V_y(y\in J_n\setminus x)$.

Proof. In fact, the conclusion (a) is the same as Lemma 2.6. By Definition 2.1, it is easy to verify that (b) is correct. Next, we prove conclusion (c). Let $v_x=xp_2p_3\dots p_{k-1}$ be any one vertex of V_x. If element y isn't in v_x, then v_x is only adjacent to $v_y=yp_1p_2\dots p_k$ of V_y by a swap-edge v_xv_y. If element y is in v_x, i.e the t-th bit $p_t=y$ for each $t\in J_k\setminus\{1\}$, then v_x is only adjacent to $v_y=yp_1p_2\cdots p_{k-1}$ of V_y by an swap-edge v_xv_y, and the t-th bit $p_t=x$ of v_y. □

Corollary 3.2. The induced subgraph $S_{n,k}[V_x\bigcup V_y]$ of any two set V_x and $V_y(x,y\in J_n,x\neq y)$ is a bipartite graph, denoted by $[V_x,E_{xy},V_y]$. Moreover, E_{xy} is a unique complete matching of $S_{n,k}[V_x\bigcup V_y]$.

Lemma 3.3. $S_{n,2}$ has exactly n minimum dominating sets, which are $V_x=\{xp|p\in J_n\setminus x\}(x\in J_n)$.

Proof. By Corollary 2.5, we only prove no other dominating sets except $V_1,V_2,...,V_n$. Let X be a minimum dominating set and different from $V_X\,(x\in J_n)$, and let nonempty $X_{i_m}=X\bigcap V_{i_m}(m\in J_{b,},2\le b\le n,i_m\in J_n)$ and $X_{i_m}\bigcap X_{i_t}=\varnothing$ for $m\neq t$ by the conclusion (b) of Lemma 3.1.

By the conclusion (a) of Lemma 3.1 and Lemma 2.6, each vertex of $V_{i_1}-X_{i_1}$ has exactly one neighbor in $X-X_{i_1}$ since $V-X_{i_1}$ must be dominated by $X-X_{i_1}$. By Lemma 2.6, we know that all neighboring-vertices of $V_{i_1}-X_{i_1}$ aren't common, so we have $|X|=|X_{i_1}|+|V_{i_1}-X_{i_1}|=\dfrac{(n-1)!}{(n-2)!}=\gamma(S_{n,2})$. Now, let U_{i_1} be a subset of $N_{S_{n,2}}(V_{i_1}-X_{i_1})$, and denote that neighbors of each vertex of $V_{i_1}-X_{i_1}$ only have one in U_{i_1}, then $X=X_{i_1}\bigcup U_{i_1}$, clearly, $|U_{i_1}|=|V_{i_1}-X_{i_1}|$.

Next, by the proof of Theorem 2.3, we let $V_{i_1}'=\{\alpha \mid \alpha\in P(n,2), i_1\notin\alpha\}$, $V_{i_1}''=\{pi_1 \mid p\in J_n\setminus i_1\}$ and $V_{i_1}-X_{i_1}=X_{i_1}'+X_{i_1}''$, where $U_{i_1}=U_{X_{i_1}'}+U_{X_{i_1}''}$, and let $U_{X_{i_1}'}$ be in V_{i_1}', $U_{X_{i_1}''}$ be in V_{i_1}'', clearly, $\left|U_{X_{i_1}'}\right|=\left|X_{i_1}'\right|$ and $\left|U_{X_{i_1}''}\right|=\left|X_{i_1}''\right|$.

If $U_{X_{i_1}'}=\varnothing$, then neighboring-vertices of X_{i_1}'' in V_{i_1}' can't be dominated since it is easy to verify $S_{n,2}\left[V_{i_1}'\right]\cap S_{n,2}\left[V_{i_1}''\right]=\varnothing$, a contradiction.

If $U_{X_{i_1}''}=\varnothing$, then neighboring-vertices of X_{i_1}'' in V_{i_1}' can't be dominated, a contradiction.

If $U_{X_{i_1}''}\neq\varnothing$ and $U_{X_{i_1}''}\neq\varnothing$, then $U_{X_{i_1}'}$ can exactly dominate neighboring-vertices of X_{i_1}' and X_{i_1}'' in V_{i_1}' except X_{i_1}'. Therefore, we have:

$$\left|U_{X_{i_1}'}\right|(n-2)=\left|X_{i_1}'\right|(n-2)=\left|X_{i_1}'\right|(n-3)+\left|X_{i_1}''\right|(n-2)\Rightarrow\left|X_{i_1}''\right|(n-2)=\left|X_{i_1}''\right| \quad (1)$$

In addition, we have known:

$$\left|X_{i_1}''\right|+\left|X_{i_1}'\right|+\left|X_{i_1}\right|=\left|V_{i_1}\right|=\frac{(n-1)!}{(n-2)!}=n-1 \quad (2)$$

By (3.1) (3.2), we can get $\left|X_{i_1}''\right|(n-1)+\left|X_{i_1}\right|=n-1$, a contradiction for $\left|X_{i_1}''\right|\neq 0$ and $\left|X_{i_1}\right|\neq 0$. Thus, X does not exist. □

Theorem 3.4. $b(S_{n,k})\geq\left\lceil\frac{n}{2}\right\rceil$, and $b(S_{n,2})=\left\lceil\frac{n}{2}\right\rceil$ for $n\geq 3$.

Proof. Let B be a minimum bondage set of $S_{n,k}$. If $b(S_{n,k})<\left\lceil\frac{n}{2}\right\rceil$, then there at least exists a V_x, all neighboring-edges of which aren't in B such that V_x is still a minimum dominating set of $S_{n,k}$ by Corollary 3.2 since each edge of B can exactly connect two elements of $\{V_x : x\in J_n\}$, that is, $2|B|<n$, a contradiction. So we have $b(S_{n,k})\geq\left\lceil\frac{n}{2}\right\rceil$.

Next, we construct a set B such that $|B|=b(S_{n,2})=\left\lceil\frac{n}{2}\right\rceil$. Let e_{xy} be any one edge of $S_{n,2}[V_x\cup V_y]$, then $e_{xy}\in E_{xy}$ by Corollary 3.2. Now, we let $B=\{e_{12}, e_{34},\ldots,e_{(n-1)n}\}$ for even n, or $B=\{e_{12}, e_{34},\ldots,e_{(n-2)(n-1,)},e_{(n-1)n}\}$ for odd n. It is easy to verify that $\gamma(S_{n,2}-B)=\gamma(S_{n,2})+1$ by Corollary 3.2 and Lemma 3.3. □

4. Conclusion

In fact, we conjecture $b(S_{n,k})=\left\lceil\frac{n}{2}\right\rceil$, but need to find a suitable method for proving the conjecture.

In any case, in Graph Theory, it is rather difficult to compute bondage number of the graphs. Up to now, the conclusions in this respect are confined only to a few specific graphs such as cube, de Bruijn and Kautz digraphs and so on. Thus, the paper is very valuable since it solves bondage number of $(n,2)$-star graphs and the precise lower- bound of bondage number of (n,k)-star graphs and n-star graphs.

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