

State feedback containment control of multi-agents system with lipschitz nonlinearity

Siti Nurfaridah Sheikh Hanis¹, Ahmad Sadhiqin Mohd Isira², Azdiana Md Yusop³,
Mohd Hendra Hairi⁴, Wong Chunyan⁵, Cherry D. Casuat⁶, Zaiton Abdul Mutalip⁷

^{1,2,3,4,7}Fakulti Kejuruteraan Elektronik dan Kejuruteraan Komputer, Universiti Teknikal Malaysia Melaka, Malaysia

⁴Fakulti Kejuruteraan Elektik, Universiti Teknikal Malaysia Melaka, Malaysia

⁵School of Aerospace Engineering, Beijing Institute of Technology (BIT), China

⁶Computer Engineering Department, Adamson University, Manila, Philippines

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ABSTRACT

This paper studies the containment control problem of the leader-follower configuration in a multi-agents system included with a type of nonlinearity such as Lipschitz concerning continuous-time and directed spanning forest communication network topology. A state feedback containment controller is designed and proposed with control theory and the Laplacian network structure, where it utilizes the relative information of each agent. The controller designed ensures that the followers are contained by the leaders that form the convex hull formation. For containment action, a minimum of one leader must have a direct communication trajectory to the followers. Lyapunov stability theory is used to provide stability conditions after analyzing the network structure. Finally, it has been shown from the simulation that the followers are contained successfully with the proposed controller.

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Corresponding Author:

Ahmad Sadhiqin Mohd Isira

Fakulti Kejuruteraan Elektronik dan Kejuruteraan Komputer

Universiti Teknikal Malaysia Melaka

Jalan Hang Tuah Jaya, 76100 Durian Tunggal, Melaka, Malaysia

Email: sadhiqin@utem.edu.my

1. INTRODUCTION

Consensus control originates from computer science [1] and has been widely accepted in the control system research community. In short, this type of control action is applied in a system with several agents or the multi-agents system, which forces each agent to come out with one outcome or consensus output. For the system to have a single outcome, it must have the Laplacian structure. With graph and control theory, this type of structure can be carefully understood, and the analysis of the system's stability can be provided. Through this knowledge, many research publications for consensus control have been reported for sensor network applications, automated highway systems, area surveillance, and other activities in the form of formation control, attitude alignment, swarming, flocking, task and role assignment, payload transport, air traffic control and cooperative search (See [2]-[7] and their references). In all publications, the consensus outcome relies heavily on the information transfer between agents, generally grouped into three types: mainly linear systems such as first-order systems [3], [4], [8], second-order systems, higher-order systems and general linear system nodes [9]-[14]. There are also works on the nonlinear systems published with Lipschitz and time delay [15]-[18].

Many works in consensus control employ leaderless configurations, which means there is no leader agent in the system. There is also another configuration known as leader-follower with a single leader that guarantees the consensus outcome. The extension to this configuration with the single leader, the containment control, is employed into the system, where multiple leaders will surround or contain the followers while in the movement. In other words, the leaders, either stationary or dynamic, form a convex hull to contain the followers in the multi-agents system. Works on containment control has been reported by looking at solving the containment problem (see [19], [20]).

The majority of the containment control publications are with linear multi-agents systems. As far as this author knows, the publications with systems with nonlinearity are small in number (see [21]-[26]). Therefore, motivated by Ding [15], Li *et al.* [21] and Wen *et al.* [22] for containment control of the multi-agents system, a containment controller with state feedback is proposed for a multi-agents system with Lipschitz nonlinearity. The agents are configured with the multiple leader-followers configuration. This configuration makes it possible to get a containment outcome even when there are no zero elements exist in the left eigenvector of the Laplacian matrix's eigenvalues.

Readers can refer to the papers mentioned for the basic structure of the controller. There are three critical features that can be considered as the main contributions for this paper:

- (i) The subsystem dynamics are influenced by nonlinearity in the form of Lipschitz.
- (ii) A containment controller with state feedback is proposed that depends on the relative information of follower subsystems.
- (iii) The type of containment controller proposed has never been looked at for a multi-agents system with (i) and (ii), as far as this author knows.

With the knowledge of the connection topology and control system tools, sufficient conditions are provided. A simulation examples are provided to validate the outcome.

The paper is structured as follows. Section 2 describes the problem statement, and provides fundamental graph theory notations. The proposed state feedback containment controller is designed in section 3. Then, the main results in this paper are given in section 4. Simulations to verify the theoretical approach are included in section 5. Finally, the conclusion for the paper is in section 6.

2. PROBLEM STATEMENT

$N + 1$ nonlinear subsystems which are identical; described by:

$$\dot{x}_i = Ax_i + \phi(x_i) + Bu_i \tag{1}$$

$$y_i = Cx_i \tag{2}$$

where $x_i \in \mathbb{R}^n$ is the state vector of the subsystem for $i = 0, \dots, N$, the input of the i th subsystem is $u_i \in \mathbb{R}^p$, and the measured output vector is $y_i \in \mathbb{R}^q$. Matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times p}$, and $C \in \mathbb{R}^{q \times n}$ represents the appropriate matrices. The function $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ represents Lipschitz nonlinearity with γ as the Lipschitz constant. For any $x, y \in \mathbb{R}^n$, we have:

$$\|\phi(x) - \phi(y)\| \leq \gamma \|x - y\| \tag{3}$$

with x, y are two constant vectors.

The subsystems connections are specified by a directed graph \mathcal{G} . A set of vertices \mathcal{V} represents the subsystems and connections represented by the set of edges \mathcal{E} are included in this graph. The adjacency matrix \mathcal{A} is associated with graph \mathcal{G} . In \mathcal{G} , if subsystem j that is connected to i , we have $a_{ij} = 1$. Otherwise $a_{ij} = 0$. Next, we obtain the Laplacian matrix $\mathcal{L} = \{l_{ij}\}$ from the adjacency matrix with:

$$l_{ij} = -a_{ij}, \text{ if } j \neq i$$

$$l_{ii} = \sum_{j=1, j \neq i}^N a_{ij} \tag{4}$$

where the arrangement for Laplacian matrix \mathcal{L} is shown as:

$$\mathcal{L} = \begin{bmatrix} \mathcal{L}_1 & \mathcal{L}_2 \\ 0_{(N-M) \times M} & 0_{(N-M) \times (N-M)} \end{bmatrix} \quad (5)$$

with $\mathcal{L}_1 \in \mathbb{R}^{M \times M}$ and $\mathcal{L}_2 \in \mathbb{R}^{M \times (N-M)}$.

Before proceeding to the remaining of the paper, it is necessary to have the following assumptions, definitions, and lemmas.

Assumption 1 *A and B matrices are controllable.*

Assumption 2 *System dynamics (1) is stable.*

Assumption 3 *Every single leader subsystem is fixed.*

Assumption 4 *The communication network \mathcal{G} of the multi-agents system contains a directed spanning forest with any of the leaders has a path to the system.*

Definition 1 *It is assumed that the multiple leaders nonlinear subsystems are with M followers and $N < M$ and $N - M$ leaders. A subsystem that has a minimum of one neighbour subsystem is known as the follower and indexed by $1, \dots, M$, while a subsystem with no neighbours is the leader, indexed by $M + 1, \dots, N$ with zero control input. No information is transferred to the leader. The leader and the follower sets are denoted by $\mathcal{R} \triangleq \{M + 1, \dots, N\}$ and $\mathcal{F} \triangleq \{1, \dots, M\}$.*

Definition 2 [20] *Let \mathcal{C} be a set in a real vector space $V \subseteq \mathbb{R}^p$. The set \mathcal{C} is called convex if, for any x and y in \mathcal{C} , the point $(1 - z)x + zy$ in \mathcal{C} for any $z \in [0, 1]$. The convex hull for a set of points $X = \{x_1, \dots, x_q\}$ in V is minimal convex set containing all points in X , and defined as $\text{Co}\{x_j, j \in \mathcal{R}\}$.*

Lemma 1 [21] *From Assumption 1, matrices \mathcal{L}_1 contains all positive real parts, where $\mathcal{L}_1^{-1}\mathcal{L}_2$ entries are all non-negative, and the sum of each row for $\mathcal{L}_1^{-1}\mathcal{L}_2$ is equal to 1.*

Lemma 2 *Algebraic Riccati Equation (ARE) provides the solution for the stability matrix P [27]: To any ARE*

$$A^T P + PA + PRP + S = 0 \quad (6)$$

with the Hamiltonian matrix:

$$H = \begin{bmatrix} A & R \\ -S & -A^T \end{bmatrix} \quad (7)$$

If H eigenvalues are not on the imaginary axis, a solution of $P = P^T > 0$ is available if the pair (A, R) is stabilizable, and R is sign-definite (i.e semi-definite positive or semi-definite negative).

This work aims to solve the containment problem for a multi-agents system by ensuring the followers asymptotically converge to the convex hull formed by the multiple leaders by obeying definition 2. Hence this objective can be materialized by designing a containment controller with state feedback.

3. STATE-FEEDBACK CONTAINMENT CONTROLLER

The state-feedback containment controller is proposed as:

$$u_i = -K_c \sum_{j \in \mathcal{F} \cup \mathcal{R}} l_{ij}(x_i - x_j) \quad (8)$$

where the constant control gain matrix $K_c \in \mathbb{R}^{p \times n}$ is to be designed in the later section. The containment control problem is said to be solved if all followers always converge to the stationary convex hull $\text{Co}\{x_j, j \in \mathcal{R}\}$ as $t \rightarrow \infty$.

For the network dynamics, we have:

$$\dot{x} = (I_N \otimes A - \mathcal{L} \otimes BK_c)x + \Phi(x) \quad (9)$$

where \mathcal{L} is defined at (4) and (5), \otimes is the Kronecker product, $x = [x_f \ x_l]^T$ where $x_f = [x_1^T, \dots, x_M^T]^T$ and $x_l = [x_{M+1}^T, \dots, x_N^T]^T$ together with $\Phi(x) = [\Phi(x_f) \ \Phi(x_l)]^T$ where $\Phi(x_f) = [\phi(x_1)^T, \dots, \phi(x_M)^T]^T$ and $\Phi(x_l) = [\phi(x_{M+1})^T, \dots, \phi(x_N)^T]^T$. Hence, we obtain a value for x_f that satisfies the following dynamics:

$$\begin{aligned} \dot{x}_f &= (I_M \otimes A - \mathcal{L}_1 \otimes BK_c)x_f \\ &\quad - (\mathcal{L}_2 \otimes BK_c)x_l + \Phi(x_f) \end{aligned} \quad (10)$$

Let $\xi_i = \sum_{j \in \mathcal{F} \cup \mathcal{R}} l_{ij}(x_i - x_j)$, $i \in \mathcal{F}$ and we have:

$$\xi_f = (\mathcal{L}_1 \otimes I_m)x_f + (\mathcal{L}_2 \otimes I_m)x_l \quad (11)$$

Remark 1 Equation (11) is derived from:

$$\xi = (\mathcal{L} \otimes I_n)x \quad (12)$$

or

$$\begin{bmatrix} \xi_f \\ \xi_l \end{bmatrix} = \begin{bmatrix} \mathcal{L}_1 \otimes I_m & \mathcal{L}_2 \otimes I_m \\ 0_{(N-M) \times M} & 0_{(N-M) \times (N-M)} \end{bmatrix} \begin{bmatrix} x_f \\ x_l \end{bmatrix} \quad (13)$$

From (9) we can also have:

$$\dot{\xi} = (I_N \otimes A - \mathcal{L} \otimes BK_c)\xi + (\mathcal{L} \otimes I_n)\Phi(x) \quad (14)$$

Consider the structure of \mathcal{L} in (5), we can see from (9) and (10), with ξ_f that satisfies the following dynamics:

$$\begin{aligned} \dot{\xi}_f &= (\mathcal{L}_1 \otimes I_m)(I_M \otimes A - \mathcal{L}_1 \otimes BK_c)x_f \\ &\quad - (\mathcal{L}_1 \otimes I_m)(\mathcal{L}_2 \otimes BK_c)x_l + (\mathcal{L}_1 \otimes I_m)\Phi(x_f) \\ &\quad + (\mathcal{L}_2 \otimes I_m)(I_{N-M} \otimes A)x_l + (\mathcal{L}_2 \otimes I_m)\Phi(x_l) \\ &= (I_M \otimes A - \mathcal{L}_1^2 \otimes BK_c)x_f - (\mathcal{L}_1 \mathcal{L}_2 \otimes BK_c)x_l \\ &\quad + (\mathcal{L}_1 \otimes I_m)\Phi(x_f) + (\mathcal{L}_2 \otimes A)x_l + (\mathcal{L}_2 \otimes I_m)\Phi(x_l) \\ &= (I_M \otimes A - \mathcal{L}_1^2 \otimes BK_c) \times \\ &\quad \quad \quad [(\mathcal{L}_1^{-1} \otimes I_m)\xi_f - (\mathcal{L}_1^{-1} \mathcal{L}_2 \otimes I_m)x_l] \\ &\quad - (\mathcal{L}_1 \mathcal{L}_2 \otimes BK_c)x_l + (\mathcal{L}_1 \otimes I_m)\Phi(x_f) \\ &\quad + (\mathcal{L}_2 \otimes A)x_l + (\mathcal{L}_2 \otimes I_m)\Phi(x_l) \\ &= (I_M \otimes A - \mathcal{L}_1 \otimes BK_c)\xi_f + (\mathcal{L}_1 \otimes I_m)\Phi(x_f) \\ &\quad + (\mathcal{L}_2 \otimes I_m)\Phi(x_l) \end{aligned}$$

Hence, we have:

$$\begin{aligned} \dot{\xi}_f &= (I_M \otimes A - \mathcal{L}_1 \otimes BK_c)\xi_f + (\mathcal{L}_1 \otimes I_m)\Phi(x_f) \\ &\quad + (\mathcal{L}_2 \otimes I_m)\Phi(x_l) \end{aligned} \quad (15)$$

Remark 2 Based on Theorem 3.1 in [20], without the nonlinearity term $\Phi(x_f)$ in (10), and if the topology is directed, the final positions of the followers are given by $-(\mathcal{L}_1^{-1} \mathcal{L}_2 \otimes I_p)x_l$. This is related to the property of $\mathcal{L}_1^{-1} \mathcal{L}_2$, where each row of $\mathcal{L}_1^{-1} \mathcal{L}_2$ has a sum equal to 1, which is stated in Lemma 1. Hence, incorporating nonlinear terms in (15), we can use the Lyapunov method to analyse its stability, and to find a suitable matrix K for the control design. This can only be achieved by transforming (15) into a diagonally-dominant matrix.

Hence, we introduce transformations:

$$\eta = (T^{-1} \otimes I_n)\xi \tag{22}$$

and

$$\Psi(x) = (T^{-1} \otimes I_n)(\mathcal{L} \otimes I_n)\Phi(x) \tag{23}$$

where $\eta = [\eta_f \ \eta_l]^T$ and $\Psi(x) = [\Psi(x_f) \ \Psi(x_l)]^T$. We then obtain:

$$\begin{aligned} \begin{bmatrix} \dot{\eta}_f \\ \dot{\eta}_l \end{bmatrix} &= \begin{bmatrix} I_M \otimes A & 0 \\ 0 & I_{N-M} \otimes A \end{bmatrix} \begin{bmatrix} \eta_f \\ \eta_l \end{bmatrix} \\ &- \begin{bmatrix} J_f \otimes BK & 0 \\ 0 & 0_{(N-M) \times (N-M)} \otimes 0_{M \times M} \end{bmatrix} \begin{bmatrix} \eta_f \\ \eta_l \end{bmatrix} \\ &+ \begin{bmatrix} \Psi(x_f) \\ \Psi(x_l) \end{bmatrix} \end{aligned} \tag{24}$$

where:

$$\begin{aligned} \begin{bmatrix} \Psi(x_f) \\ \Psi(x_l) \end{bmatrix} &= (T^{-1} \otimes I_n) \times \\ &\begin{bmatrix} \mathcal{L}_1 \otimes I_m & \mathcal{L}_2 \otimes I_m \\ 0_{(N-M) \times M} \otimes I_m & 0_{(N-M) \times (N-M)} \otimes I_m \end{bmatrix} \\ &\times \begin{bmatrix} \Phi(x_f) \\ \Phi(x_l) \end{bmatrix} \end{aligned} \tag{25}$$

and

$$(T^{-1} \otimes I_n) = \begin{bmatrix} U^{-1} \otimes I_m & 0 \\ 0 & 0_{(N-M) \times (N-M)} \otimes I_m \end{bmatrix} \tag{26}$$

From (24) we obtain:

$$\dot{\eta}_f = (I_M \otimes A - J_f \otimes BK_c)\eta_f + \Psi(x_f) \tag{27}$$

and

$$\dot{\eta}_l = (I_{N-M} \otimes A)\eta_l \tag{28}$$

Remark 4 The dynamics (10) becomes (27), which is utilized in the stability analysis.

From Assumption 4 and Lemma 1, we observe that all the eigenvalues of \mathcal{L}_1 have positive real parts. Let $U \in \mathbb{R}^{M \times M}$ be a nonsingular matrix such that $U^{-1}\mathcal{L}_1U = J_f$, with $\lambda_i = 1, \dots, M$ as its diagonal entries. Hence, in order to further manipulate the special characteristic of \mathcal{L}_1 , (22) and (23) with respect to η_f and $\Psi(x_f)$ are derived as:

$$\eta_f = (U^{-1} \otimes I_m)\xi_f \tag{29}$$

and

$$\Psi(x_f) = (U^{-1} \otimes I_m)(\mathcal{L}_1 \otimes I_m)\Phi(x_f) \tag{30}$$

(29) and (30) are used for the design procedure.

4. STABILITY ANALYSIS

For stability analysis, we introduce the following lemma to provide a bound that is needed for the transformed function η_f .

Lemma 3 A bound in terms of the state η_f can be established for nonlinear element $\psi_i(x_f)$ from the nonlinear term $\Psi(x_f)$ in the closed loop network dynamics (27) transformed, as shown by:

$$\|\psi_i(x_f)\| \leq \frac{\varrho_0}{\sqrt{N}}\eta_f \tag{31}$$

with

$$\varrho_0 = \gamma\lambda_\sigma(U^{-1})\lambda_\sigma(\mathcal{L}_1)\lambda_\sigma(\mathcal{L}_1^{-1})\lambda_\sigma(U)\sqrt{N} \tag{32}$$

where the matrix's maximum singular value is represented by $\lambda_\sigma(\cdot)$.

Proof 1 From (30) we have:

$$\begin{aligned}\|\psi_i(x_f)\| &\leq \|o_i \otimes I_n\| \|l_{ij} \otimes I_n\| \|\Phi(x_f)\| \\ &\leq \lambda_\sigma(U^{-1}) \lambda_\sigma(\mathcal{L}_1) \gamma \|x_f\| \\ &\leq \lambda_\sigma(U^{-1}) \lambda_\sigma(\mathcal{L}_1) \gamma \|\mathcal{L}_1^{-1} \otimes I_m\| \|\xi_f\| \\ &\leq \lambda_\sigma(U^{-1}) \lambda_\sigma(\mathcal{L}_1) \gamma \lambda_\sigma(\mathcal{L}_1^{-1}) \|\xi_f\|\end{aligned}$$

where o_i denotes the i^{th} row of U^{-1} and l_{ij} denotes the i^{th} row of \mathcal{L}_1 . From (29) we have:

$$\|\xi_f\| \leq \|\wedge_i \otimes I_n\| \|\eta_f\| \leq \lambda_\sigma(U) \|\eta_f\|$$

where \wedge_i is the i^{th} row of U . Then,

$$\|\psi_i(x_f)\| \leq \frac{\gamma \lambda_\sigma(U^{-1}) \lambda_\sigma(\mathcal{L}_1) \lambda_\sigma(\mathcal{L}_1^{-1}) \lambda_\sigma(U) \sqrt{N}}{\sqrt{N}} \|\eta_f\| \leq \frac{\varrho_0}{\sqrt{N}} \|\eta_f\|$$

Note that in *Proof 1* and *Lemma 3*, $\|\cdot\|$ denotes the Euclidean norm for vectors $x \in \mathbb{R}^n$, defined by $\|x\| = \sqrt{x^T x}$, and the induced norm corresponding to the vector Euclidean norm for matrices $A \in \mathbb{R}^{m \times n}$, defined by $\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}$. With the induced norm, the inequality $\|\psi_i(x, x_0)\| \leq \|t_i \otimes I_n\| \|\Phi(x) - \Phi(x_0)\|$ holds. Then we arrive at the following theorem that utilizes the bound in *Lemma 3*.

Theorem 1 If there exist a solution of $P = P^T > 0$ for the nonlinear system (1), the distributed controller (8) with $K_c = B^T P$ solves the containment control problem with the communication topology \mathcal{G} , when *Lemma 1* and *Assumption 1* to *Assumption 4* are satisfied, specified by either these two cases:

1. For matrix \mathcal{L}_1 with distinct eigenvalues, i.e., $n_k = 1$ for $k = 1, \dots, q$, if the matrix P satisfies:

$$A^T P + PA - 2\alpha P B B^T P + \kappa P P + \frac{\varrho_0^2}{\kappa} I_n < 0 \quad (33)$$

where $\kappa > 0$ and $\alpha = \min\{\lambda_1, \dots, \lambda_p, \alpha_{p+1}, \dots, \alpha_q\}$.

2. For matrix \mathcal{L}_1 with multiple eigenvalues, i.e. $n_k > 1$ for any $k \in \{1, \dots, q\}$, if the matrix P satisfies:

$$A^T P + PA - 2(\alpha - 1) P B B^T P + \kappa P P + \frac{\varrho_0^2}{\kappa} I_n \quad (34)$$

with $\kappa > 0$.

Proof 2 Notice that each Jordan block J_f takes the form of (17), where within each real Jordan block J_k , for $k \leq p$, we have $i = N_{k-1} + 1, \dots, N_k - 1$,

$$\dot{\eta}_{f_i} = (A - \lambda_i B K_c) \eta_{f_i} - B K_c \eta_{f_{i+1}} + \psi_i(x_f) \quad (35)$$

and

$$\dot{\eta}_{f_i} = (A - \lambda_i B K_c) \eta_{f_i} + \psi_i(x_f) \quad (36)$$

for $i = N_k$.

The dynamics of the state variables that is related to the Jordan blocks J_k for $k > p$ are considered in pairs for complex eigenvalues. Let:

$$\begin{aligned}i_1(j) &= N_{k-1} + 2j - 1 \\ i_2(j) &= N_{k-1} + 2j\end{aligned}$$

for $j = 1, \dots, n_k/2$. The dynamics of $\eta_{f_{i_1}}$ and $\eta_{f_{i_2}}$ for $j = 1, \dots, n_k/2 - 1$ are represented as:

$$\begin{aligned}\dot{\eta}_{f_{i_1}} &= (A - \alpha_k B K_c) \eta_{f_{i_1}} - \beta_k B K_c \eta_{f_{i_2}} - B K_c \eta_{f_{i_1+2}} + \psi_{i_1}(x_f) \\ \dot{\eta}_{f_{i_2}} &= (A - \alpha_k B K_c) \eta_{f_{i_2}} + \beta_k B K_c \eta_{f_{i_1}} - B K_c \eta_{f_{i_2+2}} + \psi_{i_2}(x_f)\end{aligned}$$

and

$$\begin{aligned} \dot{\eta}_{f_{i_1}} &= (A - \alpha_k BK_c)\eta_{f_{i_1}} - \beta_k BK_c \eta_{f_{i_2}} + \psi_{i_1}(x_f) \\ \dot{\eta}_{f_{i_2}} &= (A - \alpha_k BK_c)\eta_{f_{i_2}} + \beta_k BK_c \eta_{f_{i_1}} + \psi_{i_2}(x_f) \end{aligned}$$

for $j = n_k/2$.

Let $W_i = \eta_{f_i}^T P \eta_{f_i}$. Choose $V_k = \sum_{j=1}^{n_k} \sigma^{2(j-1)} W_{j+N_{k-1}}$ for $k = 1, \dots, p$ and $V_k = \sum_{j=1}^{n_k/2} \sigma^{2(j-1)} (W_{i_1(j)} + W_{i_2(j)})$ for $k = p + 1, \dots, q$, where $\sigma > 0$.

Next, by having the Lyapunov function $V = \sum_{i=1}^q V_k$ and the controller gain $K_c = B^T P$, we can obtain the following:

Case 1. We can obtain the following for the distinct eigenvalues,

$$\dot{V} \leq \sum_{i=1}^M \eta_{f_i}^T \left(A^T P + PA - 2\alpha P B B^T P + \kappa P P + \frac{\rho_0^2}{\kappa} I_n \right) \eta_{f_i} \tag{37}$$

with $\dot{V} < 0$ is guaranteed from condition (33).

Case 2. We can obtain the following for multiple eigenvalues,

$$\begin{aligned} \dot{V} &\leq \sum_{i=1}^M \eta_{f_i}^T \left[A^T P + PA - 2 \left(\alpha - \frac{1}{\sigma} \right) P B B^T P + \kappa P P + \frac{\rho_0^2}{\kappa} I_n \right] \eta_{f_i} \\ &\leq \sum_{i=1}^M \eta_{f_i}^T \left[A^T P + PA - 2(\alpha - 1) P B B^T P + \kappa P P + \frac{\rho_0^2}{\kappa} I_n \right] \eta_{f_i} \end{aligned} \tag{38}$$

with $\sigma = 1$. By having Lemma 1 and Assumption 4, the condition (34) guarantees $\dot{V} < 0$ with $\eta_{f_i} \rightarrow 0$ as $t \rightarrow \infty, \forall i = 1, \dots, N$ as $t \rightarrow \infty$. Thus containment is achieved and the proof is completed.

5. SIMULATION

In order to verify the theoretical approach of this paper, a simulation example is given with a system that contains three leader and six follower subsystems, described as:

$$\dot{x}_i = Ax_i + \phi(x_i) + Bu_i \tag{39}$$

with:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [1 \quad 0] \tag{40}$$

where the nonlinear term is:

$$\phi(x_i) = \begin{bmatrix} 0.05 \cos(Cx_i) \\ 0 \end{bmatrix} \tag{41}$$

Without the nonlinearity, the result is similar to [28]. In this paper, Lipschitz nonlinearity is included in the system dynamics as shown in (39). The connection between leaders and the followers is shown in Figure 1.

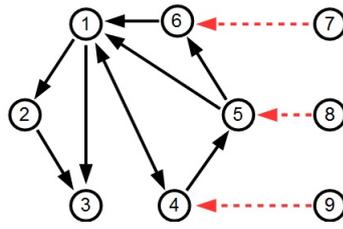


Figure 1. The containment configuration where the connections between the leaders and the followers are represented by the dotted lines

Clearly we have satisfied *Assumption 4* by looking at Figure 1. Hence, the Laplacian matrix is:

$$\mathcal{L} = \begin{bmatrix} 3 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where:

$$\mathcal{L}_1 = \begin{bmatrix} 3 & 0 & 0 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

Please note that (A, B) is controllable and verified. We can easily obtain the eigenvalues of \mathcal{L}_1 as $\{0.8213, 1, 2, 2.3329 \pm 0.6708j, 3.5129\}$. By observation, the eigenvalues obtained are distinct and satisfy *Lemma 1*.

From the transformation (16), the Jordan matrix J is:

$$J = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.8213 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2.3329 & 0.6708 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.6708 & 2.3329 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3.5129 \end{bmatrix}$$

where:

$$J_f = \begin{bmatrix} 0.8213 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.3329 & 0.6708 & 0 \\ 0 & 0 & 0 & -0.6708 & 2.3329 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3.5129 \end{bmatrix}$$

Note that $0.05 \sin(Cx_i)$, is nonlinear function chosen for the system and it is globally Lipschitz. The followers are initiated with the values of:

$$x_1 = [0.15; -0.15], x_2 = [0.25; -0.25], x_3 = [0.35; -0.35], x_4 = [0.45; -0.45],$$

$$x_5 = [0.55; -0.55], x_6 = [0.65; -0.65]$$

The leader values are set as:

$$x_7 = [0.7; -0.7], x_8 = [0.8; -0.8], x_9 = [0.9; -0.9] x_9 = [0.9; -0.9]$$

From \mathcal{L}_1 , $\alpha = 0.8213$ is obtained. Then, based on (41),

$$\phi(x_i) = \begin{bmatrix} \gamma \sin(Cx_i) \\ 0_{N-1} \end{bmatrix}$$

where $\gamma = 0.05$ is set, and the bound in Lemma 3 can be obtained as $\varrho_0 = 7.5545 \times 10^{-7}$ with κ set at 10. Next, Algebraic Riccati Equation (ARE) is utilized to get the solution for P , with:

$$P = \begin{bmatrix} 0.0999 & 0.0054 \\ 0.0054 & 0.1201 \end{bmatrix}$$

Thus, we can easily obtain $K_c = B^T P$, which is the controller gain as:

$$K_c = [0.0054 \quad 0.1201]$$

Previously, we have shown that matrices (A, B) are controllable. In addition, the solution of P has satisfied Lemma 2. The Hamiltonian matrix is then obtained as:

$$H = \begin{bmatrix} 0 & 1.0000 & -10.0000 & 0 \\ 0 & 0 & 0 & -8.3574 \\ -1.0000 & 0 & 0 & 0 \\ 0 & -1.0000 & -1.0000 & 0 \end{bmatrix}$$

with $-3.0632 + 0.4524i$, $-3.0632 - 0.4524i$, $3.0632 + 0.4524i$, and $3.0632 - 0.4524i$ as the eigenvalues of H . Clearly, these eigenvalues are not located on the imaginary axis and satisfies Lemma 2.

The plots for the multiple leaders-followers system is shown in Figure 2, and Figure 3 with $\gamma = 0.05$ with the application of controller (8). Without the nonlinearity element $\phi(x_i)$, the controller is proved to be stable, and we obtain:

$$eig(A - \lambda_i B K_c) = \begin{bmatrix} -0.2122 + 0.4472i \\ -0.2122 - 0.4472i \end{bmatrix} \tag{42}$$

where $\lambda_i = 0.8231$ is the minimum eigenvalue of Laplacian matrix \mathcal{L} , with the location on the plane for all its eigenvalues is LHS and Hurwitz-stable.

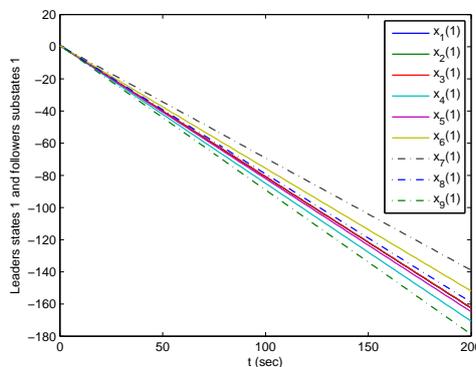


Figure 2. The containment plot the system that contains multiple leaders (dotted lines) and followers system (normal lines) with $\gamma = 0.05$ which represents the Lipschitz nonlinearity for substates 1

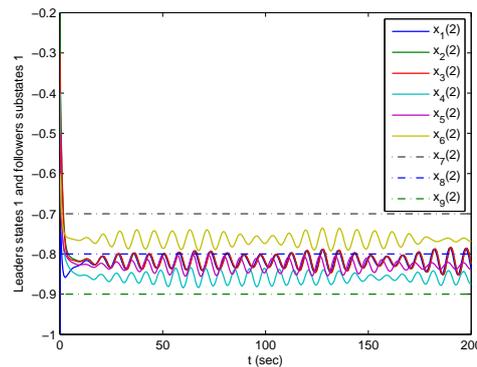


Figure 3. The containment plot the system that contains multiple leaders (dotted lines) and followers system (normal lines) with $\gamma = 0.05$ which represents the Lipschitz nonlinearity for substates 2

Remark 5 The constant γ range implemented in the experiment is between 0.01 and 1.05. This value range is when the followers are able to be contained by the containment controller designed.

The control gain can provide the containment action with conditions (33) and (34) when the nonlinearities exist in the system. Bear in mind, the conditions provided are conservative when we look at the Lipschitz nonlinear function is included in the steps when designing of the controller. Similar to when $\gamma = 0.05$, when the nonlinearity is increased to $\gamma = 0.07$, the containment controller could still achieve containment for the first substates of the followers. However, containment was not achieved for the second substates of the followers where the signal went slightly above the bound set by the leaders. For $\gamma = 0.2$, containment has not been realized for substates 2 of each subsystem, but it remained stable and oscillated within the bound of the leaders as $t \rightarrow \infty$.

6. CONCLUSION

The containment controller with state feedback proposed has successfully enabled the leaders' subsystems to contain the follower subsystems in the system with Lipschitz nonlinearity and directed spanning forest topology network. A specific measure of nonlinearity was included, deemed conservative, but still enable the containment outcome to be obtained. With careful evaluation of the Laplacian structure and Lyapunov stability analysis, stability conditions for the system have been provided. The conditions are verified with simulations which shows the successful containment action. Since not all states are measurable for real systems, an observer may be considered for the unmeasurable states in the multi-agent systems for future work.

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