

Performance of similarity explicit group iteration for solving 2D unsteady convection-diffusion equation

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ABSTRACT

In this paper, a similarity finite difference (SFD) solution is addressed for the two-dimensional (2D) parabolic partial differential equation (PDE), specifically on the unsteady convection-diffusion problem. Structuring the similarity transformation using wave variables, we reduce the parabolic PDE into elliptic PDE. The numerical solution of the corresponding similarity equation is obtained using a second-order central SFD discretization scheme to get the second-order SFD approximation equation. We propose a four-point similarity explicit group (4-point SEG) iterative method as a numerical solution of the large-scale and sparse linear systems derived from SFD discretization of 2D unsteady convection-diffusion equation (CDE). To show the 4-point SEG iteration efficiency, two iterative methods, such as Jacobi and Gauss-Seidel (GS) iterations, are also considered. The numerical experiments are carried out using three different problems to illustrate our proposed iterative method's performance. Finally, the numerical results showed that our proposed iterative method is more efficient than the Jacobi and GS iterations in terms of iteration number and execution time.

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1. INTRODUCTION

Convection-diffusion equation (CDE) is one of the most challenging problems and frequently used in various branches of engineering and applied science, especially in radial transport in a porous medium [1], heat transfer in a nanofluid filled [2], heat transfer in a draining film [3], and water transport in soil [4]. Also, the applications of CDE can be found in [5]-[8]. Due to its application, these problems have received extensive attention, and many researchers attempted to solve these problems numerically to achieve the lowest computational complexity and highest performance. To achieve the low computational complexity, there are many studies on similarity solution techniques have been explored by many researchers and applied in partial differential equations (PDEs). For instance, Afify [9] presented similarity solutions in magnetohydrodynamic, which is obtained by using scaling transformations then solved numerically by using the shooting technique with fourth-order Runge-Kutta integration scheme. M. Siavashi *et al.* [10], the authors obtained a similarity solution of air and nanofluid impingement cooling of a cylindrical porous heat sink using similarity variables then solved numerically. Usman *et al.* [11], a similarity solution of the water/magnetite nanofluid modelled PDEs subject to thermal radiation and Lorentz force over stretchable rotating disks is obtained by supporting precise similarity transformation. The studies on the similarity

solution can also be found in [12]-[15]. Basically, the main idea of using a similarity solution is to reduce the number of independent variables of a system of PDEs at least one less than that of the original equation. Because of the low computational complexity and high computational performance of this method, its implementation is of interest.

Many researchers have solved the CDE using the finite difference method (FDM) in addition to the above numerical methods for solving the proposed CDE problem as noted in the first paragraph. As a result, a new finite difference (FD) discretization scheme has been proposed by combining several numerical discretization schemes, mostly from the FD scheme family. For instance, the nonstandard FD [16]-[18], the exponential FD [19]-[21] and perturbational FD [22] are implemented for solving CDE. No doubt that [16]-[22] can present outstanding performance, but it suffers from high computational complexity because the obtained approximation equation generated a sequence of linear systems for each time level, in which this phenomenon will increase computational complexity. The computational complexity of algorithms is determined by two norms: the number of nodes that have to be examined and the operational cost per node. The more linear systems, the more nodes will be used that will be affected by computational complexity. In conjunction with new FD discretization schemes in [16]-[22], we aim to introduce an entirely new FD scheme based on combination of the similarity and FD schemes, named as similarity finite difference (SFD) discretization schemes, to investigate its feasibility in solving CDE.

Based on the second paragraph, we can see that many researchers have proposed a new FD discretization scheme with getting high computational performance and lower computational complexity. Apart from that, we attempt to examine the feasibility of the SFD discretization scheme as a numerical method to solve 2D unsteady CDE. Based on the previous literature review on applying the similarity solutions in the first paragraph, the findings showed that the similarity solution technique has successfully reduced the independent variables of PDEs and then transform them into ordinary differential equation (ODE). It means that the computational cost of the similarity technique will decrease and make the computational complexity of the similarity approximation equation low compared to without using similarity solutions. However, most of them are solving PDEs using similarity solutions are concerned with PDE problems being reduced to ODE problems, but the studies concerning reducing parabolic PDEs into the corresponding elliptic PDEs have not been found yet. The main idea is to maintain the dimension of the model so that the characteristic of the object can be preserved. Inspired by this technique's low computational complexity and high computational performance, we propose newly SFD discretization schemes. With that, we applied the similarity solution specifically using wave variables transformation in order to reduce 2D parabolic PDEs in details 2D unsteady CDE problem into 2D elliptic PDEs.

The 2D unsteady CDE is first transformed into a 2D elliptic PDE via a similarity solution technique to solve the proposed problem. The SFD approximation equation is then formed by discretizing the corresponding elliptic PDE using the newly SFD discretization schemes. Since the SFD approximation equation produces a large-scale and sparse linear system with its matrix coefficient, an effective solver must solve the resulting large and sparse linear system. According to the results in [23]-[25], the iterative approach is most definitely the best linear solver for large and sparse linear systems. Several iterative methods for solving a linear system with a large-scale and sparse coefficient matrix are discussed in the literature. The implementations of the point iteration family, such as successive over-relaxation (SOR) [26], [27], accelerated over-relaxation (AOR) [28], [29], and kaudd successive over-relaxation (KSOR) [30], [31], can be used to solve this linear method. Evans [32] invented the explicit group (EG) iteration method for obtaining an approximate solution by grouping the linear system into a sequence of (4x4) linear systems based on the coefficient matrix's characteristics. Compared to the GS iteration, this block iterative method is faster. Although the convergence rate for EG iteration has been accelerated, several researchers have created new versions of the EG iteration family, including 4EGSOR via nine-point Laplacian (4EGSOR9L) [33], 4 point-explicit decoupled group (EDG) [34] and 9 Point-EDGSOR [35] in which all of these block iterations have significantly decreased their convergence rate. Ali *et al.* [36] and Bee *et al.* [37] used the EG method for solving the 2D CDE to solve the linear system of unsteady advection-diffusion problem iteratively, and their results indicated that this method has a fast convergence due to the large number of points that must be treated simultaneously. However, without using similarity solution techniques, the high computational complexity will occur to get the approximate solution of the 2D unsteady CDE. To achieve the low computational complexity, we propose a new variant of the EG iteration family to solve 2D unsteady CDE by solving the generated linear system.

As a result, the remainder of this paper focuses on evaluating the efficacy of the four-point similarity explicit group (4-point SEG) iterative method for solving the system of SFD approximation equations, which was inspired by the newly developed SFD discretization scheme. The combination of the SFD scheme and the EG iterative method can be used to formulate the 4 point-SEG iterative method. To illustrate the feasibility of 4-point SEG iteration, the following general equation for 2D unsteady CDE is

considered in this paper to investigate the feasibility of the SFD discretization schemes and performance of the 4 point-SEG iterative method:

$$\frac{\partial u}{\partial t} + \beta_1 \frac{\partial u}{\partial x} + \beta_2 \frac{\partial u}{\partial y} = \alpha_1 \frac{\partial^2 u}{\partial x^2} + \alpha_2 \frac{\partial^2 u}{\partial y^2} + R(x, y, t), \quad \Omega \times (0, T], \tag{1}$$

subject to the following conditions

$$\begin{aligned} u(x, y, t) &= g(x, y, t), \quad (x, y) \in \partial\Omega, \quad t \in (0, T], \\ u(x, y, 0) &= h(x, y), \quad (x, y) \in \Omega, \end{aligned} \tag{2}$$

where β_1 and β_2 are constant speeds of convection in the direction of x and y respectively; $\alpha_1 > 0$ and $\alpha_2 > 0$ are the coefficients of diffusivity in the x – and y – directions, respectively; Ω is a subset of \mathbb{R}^2 ; $(0, T]$ is the time interval; $g(x, y, t)$ and $h(x, y)$ denote the smooth functions.

2. SFD APPROXIMATION EQUATION

Before we start the discretization process of problem (1), firstly, we transform problem (1) into the 2D elliptic PDE using similarity transformation specifically on wave variables as explained in the previous section. To start the transformation process, let us consider the wave variables as follows [27], [38], [39];

$$\begin{aligned} \xi &= x - ct, \\ \tau &= y - dt, \end{aligned} \tag{3}$$

and we use the transformation $u(x, y, t) = u(\xi, \tau)$. Using (3), in (1) reduces to an elliptic PDE as;

$$\zeta \left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \tau} \right) + \alpha \left(\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \tau^2} \right) = -R(\xi, \tau), \quad \Omega \in [\xi_a, \xi_b] \times [\tau_a, \tau_b], \tag{4}$$

where $\zeta = \frac{\eta}{2} - \beta$, $\beta = \beta_1 = \beta_2$, $\alpha = \alpha_1 = \alpha_2$. Let us build the distribution of uniformly node points as shown in Figure 1 to help us explore the derivation of the similarity approximation equation.

From the Figure 1, we must discretize the solution domain, (Ω) uniformly in both ξ and τ directions with a mesh size, h which is defined as $\Delta \xi = \frac{\xi_b - \xi_a}{m}$, $\Delta \tau = \frac{\tau_b - \tau_a}{m}$, $h = \Delta \xi = \Delta \tau$ and $m = n + 1$. Using the SFD scheme and the finite grid network in Figure 1, we discretize the 2D elliptic PDEs (4) as follows:

$$\begin{aligned} \left. \frac{du}{d\xi} \right|_{ij} &= \frac{U_{i+1,j} - U_{i-1,j}}{2h} \\ \left. \frac{d^2u}{d\xi^2} \right|_{ij} &= \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2} \\ \left. \frac{du}{d\tau} \right|_{ij} &= \frac{U_{i,j+1} - U_{i,j-1}}{2h} \\ \left. \frac{d^2u}{d\tau^2} \right|_{ij} &= \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{h^2} \end{aligned} \tag{5}$$

Then substitute (5) into (4), we have the following approximation equation;

$$\zeta \left(\frac{U_{i+1,j} - U_{i-1,j}}{2h} + \frac{U_{i,j+1} - U_{i,j-1}}{2h} \right) + \alpha \left(\frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2} + \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{h^2} \right) = -R_{i,j}, \tag{6}$$

By simplifying (6), we get;

$$AU_{i-1,j} + BU_{i+1,j} + AU_{i,j-1} + BU_{i,j+1} - 4U_{i,j} = r_{i,j} \tag{7}$$

where $A = \frac{\gamma_1 - \gamma_2}{\gamma_1}$, $B = \frac{\gamma_1 + \gamma_2}{\gamma_1}$, $\gamma_1 = \frac{\alpha}{h^2}$, $\gamma_2 = \frac{\zeta}{2h}$ and $r_{i,j} = -\frac{R_{i,j}}{\gamma_1}$.

Based on the SFD approximation (7), a linear system generates with the coefficient matrix is large-scale and sparse in matrix form as (8) by taking $i = 1, 2, 3, \dots, n$

$$F\underline{U} = \underline{r} \tag{8}$$

where;

$$F = \begin{bmatrix} G_2 & G_3 & 0 & 0 & \dots & 0 \\ G_1 & G_2 & G_3 & 0 & \dots & 0 \\ 0 & G_1 & G_2 & G_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & G_1 & G_2 & G_3 \\ 0 & 0 & 0 & 0 & G_1 & G_2 \end{bmatrix}, \underline{U} = \begin{bmatrix} \underline{U}_1 \\ \underline{U}_2 \\ \underline{U}_3 \\ \underline{U}_4 \\ \vdots \\ \underline{U}_n \end{bmatrix}, \underline{r} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ \vdots \\ r_n \end{bmatrix}$$

and

$$G_1 = \begin{bmatrix} A & 0 & 0 & \dots & 0 \\ 0 & A & 0 & \dots & 0 \\ 0 & 0 & A & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & A \end{bmatrix}, G_2 = \begin{bmatrix} -4 & B & 0 & \dots & 0 \\ A & -4 & B & \dots & 0 \\ 0 & A & -4 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & A & -4 \end{bmatrix}, G_3 = \begin{bmatrix} B & 0 & 0 & \dots & 0 \\ 0 & B & 0 & \dots & 0 \\ 0 & 0 & B & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & B \end{bmatrix},$$

$$\underline{U}_i = [U_{1,i} \quad U_{2,i} \quad U_{3,i} \quad U_{4,i} \quad \dots \quad U_{n,i}]^T,$$

for $i = 1, 2, 3, \dots, n$.

3. DERIVATION OF 4-POINT SEG ITERATION

Since the coefficient matrix for the linear system (8) has large-scale and sparse characteristics, this study proposed a faster numerical solver by employing a 4-point SEG iteration. Now, in this section, the formulation of 4 point-SEG attempts to be established. To obtain the formulation of 4-point SEG, we consider the grid network in Figure 1 and a group of block node points concept in Figure 2. The finite grid network of the SFD approximation equation is depicted in Figure 2, where the block iteration approach has been made until iteration convergence is achieved.

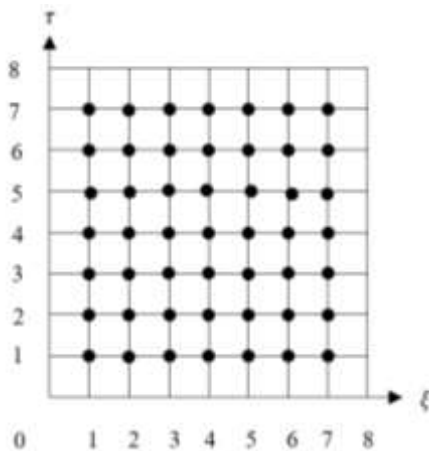


Figure 1. Finite grid networks at $m=8$

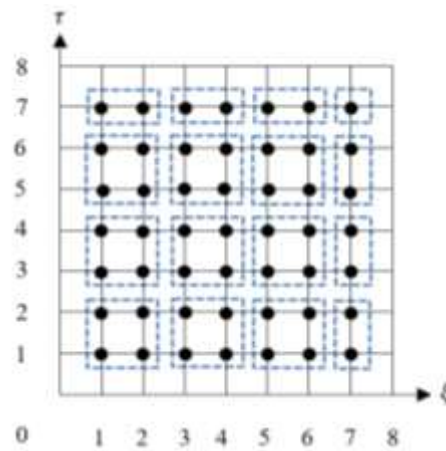


Figure 2. Implementation of the 4-point SEG iteration at solution domain, Ω

Apply (7) on any group of four node points in the solution domain, the 4-point SEG iteration scheme can be formulated as;

$$\begin{bmatrix} 4 & -B & 0 & -B \\ -A & 4 & -B & 0 \\ 0 & -A & 4 & -A \\ -A & 0 & -B & 4 \end{bmatrix} \begin{bmatrix} u_{i,j} \\ u_{i+1,j} \\ u_{i+1,j+1} \\ u_{i,j+1} \end{bmatrix} = \begin{bmatrix} s1 \\ s2 \\ s3 \\ s4 \end{bmatrix} \tag{9}$$

where,

$$\begin{aligned} s1 &= AU_{i-1,j} + AU_{i,j-1} - r_{i,j}, & s2 &= BU_{i+2,j} + AU_{i+1,j-1} - r_{i+1,j}, \\ s3 &= BU_{i+2,j+1} + BU_{i+1,j+2} - r_{i+1,j+1}, & s4 &= AU_{i-1,j+1} + BU_{i,j+2} - r_{i,j+1}. \end{aligned}$$

By establishing the inverse matrix of the coefficient system (9) and manipulating it, the general scheme of the 4-point SEG iteration can be written as;

$$\begin{bmatrix} u_{i,j} \\ u_{i+1,j} \\ u_{i+1,j+1} \\ u_{i,j+1} \end{bmatrix}^{(k+1)} = \frac{1}{det4} \begin{bmatrix} c1s1 + c3s2 + c6s3 + c3s4 \\ c2s1 + c1s2 + c3s3 + c4s4 \\ c5s1 + c2s2 + c1s3 + c2s4 \\ c2s1 + c4s2 + c3s3 + c1s4 \end{bmatrix} \tag{10}$$

where $det4 = 8(4 - AB)$, $c1 = 8 - AB$, $c2 = 2A$, $c3 = 2B$, $c4 = AB$, $c5 = A^2$ and $c6 = B^2$. The 4-point SEG iteration which has been used to solve the proposed problem (1), is summarised in Algorithm 1.

Algorithm 1: 4-point SEG iteration

- i. Initialize $U \leftarrow 0$ and $\varepsilon \leftarrow 10^{-10}$
- ii. Calculate F and \underline{r}
- iii. For $i = 1, 2, 3, \dots, n$, calculate the equation (10).
- iv. Perform the convergence test, $|U_i^{(k+1)} - U_i^{(k)}| < \varepsilon = 10^{-10}$. If yes, go to step (v).
Otherwise, repeat step (iii).
- v. Display the numerical outputs.

4. NUMERICAL EXPERIMENT

Three selected numerical experiments were performed in this section to illustrate the feasibility of 4-point SEG iteration in solving the 2D unsteady CDE (1) as compared with the Jacobi and GS iterations. For the sake of comparison, we considered three criteria include iteration number (*Iter.*), execution time in second (*Time*) and maximum absolute error (*Err.*). All the numerical experiments were run with different mesh sizes (*m*) such as 64, 128, 256, 512, and 1024.

Problem 1 [40]

Consider (1) in the unit square domain $[0,1] \times [0,1]$ with diffusion coefficients $\alpha = 0.05$, convection coefficients $\beta = 0.8$ and an analytic solution is;

$$u(x, y, t) = \frac{1}{1+4t} \exp\left(-\frac{(x-\beta_1 t-0.5)^2}{\alpha_1(1+4t)} - \frac{(y-\beta_2 t-0.5)^2}{\alpha_2(1+4t)}\right). \tag{11}$$

Problem 2 [41]

To demonstrate the benefits of the 4-point SEG iteration, we created a new test problem with an analytic solution as follow;

$$u(x, y, t) = \exp(-2t) \cos(x) \sin(2y). \tag{12}$$

and the source term is;

$$R(x, y, t) = \exp(-2t) \left[\cos(x + 2y) + \cos(x) \left(\frac{1}{2} \sin(2y) + \cos(2y) \right) \right]. \tag{13}$$

Consider (1) in $[0,1] \times [0,1]$ with diffusion coefficients $\alpha = 0.5$, and convection coefficients $\beta = 1.0$.

Problem 3 [42]

In this problem, we consider (1) in the unit square domain $[0,1] \times [0,1]$ with diffusion coefficients $a = b = \alpha = 1.0$, convection coefficients $\beta = 64.0$ and the analytic solution is

$$u(x, y, t) = e^{-(a^3+b^3)t} \sin(ax + by) \tag{14}$$

and the source term is;

$$R(x, y, t) = (a\beta + b\beta)e^{-(a^3+b^3)t} \cos(ax + by) \tag{15}$$

All the numerical results for 4-point SEG iteration together with Jacobi and GS iterations in solving the above three problems were collected and tabulated in Tables 1, 2 and 3, respectively.

Based on Tables 1 to 3, obviously, it shows that our proposed iterative method, namely 4-point SEG iteration, gives tremendously improve in term of iteration number and execution time, which significantly has approximately reduced iteration number and execution time by 58.32-68.34% and 50.86-66.89% for

Problem 1, 69.83-72.17% and 64.55-71.89% for Problem 2 and 74.43-79.33% and 69.44-73.53% for Problem 3 respectively. Also, for the maximum absolute error, all iterative methods show excellent agreement and close to the exact solution.

Table 1. Comparison of *Iter.*, *Time*, and *Err.* for Jacobi, GS and 4-point SEG iterations of Problem 1

	Method	Mesh Size				
		64x64	128x128	256x256	512x512	1024x1024
<i>Iter.</i>	Jacobi	2236	6732	19017	49868	119729
	GS	1268	3919	11434	31146	78287
	4-point SEG	708	2242	6744	19038	49908
<i>Time</i>	Jacobi	1.48	4.79	38.20	391.40	3785.64
	GS	0.82	2.77	25.30	261.91	2628.67
	4-point SEG	0.49	1.76	16.91	179.43	1860.27
<i>Err.</i>	Jacobi	1.505817E-05	1.515919E-05	1.537614E-05	1.620934E-05	1.927445E-05
	GS	1.504651E-05	1.511897E-05	1.522844E-05	1.565958E-05	1.725004E-05
	4-point SEG	1.504112E-05	1.509937E-05	1.515330E-05	1.537699E-05	1.620543E-05

Table 2. Comparison of *Iter.*, *Time*, and *Err.* for Jacobi, GS and 4-point SEG iterations of Problem 2

	Method	Mesh Size				
		64x64	128x128	256x256	512x512	1024x1024
<i>Iter.</i>	Jacobi	9906	35147	122640	418746	1387700
	GS	5231	18688	65793	227295	765604
	4-point SEG	2757	9907	35143	122627	418714
<i>Time</i>	Jacobi	5.94	22.20	247.69	3522.95	44301.54
	GS	3.14	12.52	145.05	1923.93	26991.47
	4-point SEG	1.67	7.45	86.06	1167.63	15703.74
<i>Err.</i>	Jacobi	3.268668E-03	3.269326E-03	3.270326E-03	3.273919E-03	3.288190E-03
	GS	3.268631E-03	3.269176E-03	3.269730E-03	3.271549E-03	3.278696E-03
	4-point SEG	3.268612E-03	3.269102E-03	3.269434E-03	3.270365E-03	3.273953E-03

Table 3. Comparison of *Iter.*, *Time*, and *Err.* for Jacobi, GS and 4-point SEG iterations of Problem 3

	Method	Mesh Size				
		64x64	128x128	256x256	512x512	1024x1024
<i>Iter.</i>	Jacobi	421	1697	6611	25519	98073
	GS	158	750	3138	12552	49152
	4-point SEG	87	387	1604	6403	25082
<i>Time</i>	Jacobi	0.34	1.23	14.90	200.16	3082.46
	GS	0.13	0.62	7.18	107.98	1657.81
	4-point SEG	0.09	0.33	4.01	61.16	932.81
<i>Err.</i>	Jacobi	1.147132E-03	1.138314E-03	1.136067E-03	1.135537E-03	1.135553E-03
	GS	1.147132E-03	1.138313E-03	1.136061E-03	1.135513E-03	1.135449E-03
	4-point SEG	1.147132E-03	1.138312E-03	1.136058E-03	1.135501E-03	1.135397E-03

5. CONCLUSION

In this paper, we have been successfully reduced 2D parabolic PDEs, particularly on the convection-diffusion problem, into 2D elliptic PDEs using the similarity solutions technique via wave variables in which we manage to get a low computational complexity as desired in this study. The similarity approximation equation has been discretizing by using the SFD discretization scheme to get the SFD approximation equation. This approximation equation generated a large-scale and sparse linear system then solved using 4-point SEG, GS and Jacobi iterations. The 4-point SEG iteration has achieved the highest performance based on the implementation of these three iterations since the iteration number and execution time was smaller than Jacobi and GS iterations. Thus, we can conclude that our proposed method is more efficient than GS and Jacobi iterations. This research will be further expanded into the use of SFD discretization scheme via half-sweep and quarter-sweep iteration families for solving the 2D unsteady CDE.

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