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Normal Vector Based Subdivision Scheme to Generate Fractal Curves

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Abstract

In this paper, we firstly devise a new and general p-ary subdivision scheme based on normal vectors with multi-parameters to generate fractals. Rich and colorful fractals including some known fractals and a lot of unknown ones can be generated directly and conveniently by using it uniformly. The method is easy to use and effective in generating fractals since the values of the parameters and the directions of normal vectors can be designed freely to control the shape of generated fractals. Secondly, we illustrate the technique with some design results of fractal generation and the corresponding fractal examples from the point of view of visualization, including the classical Lévy curves, Dragon curves, Sierpiński gasket, Koch curve, Koch-type curves and other fractals. Finally, some fractal properties of the limit of the presented subdivision scheme, including existence, self-similarity, non-rectifiability, and continuity but nowhere differentiability are described from the point of view of theoretical analysis.

Keywords: *p-ary subdivision scheme, normal vector, fractal, fractal property.*

1. Introduction

Subdivision scheme defines a curve from an initial control polygon or a surface from an initial control mesh by subdividing them according to some refining rules, recursively. Since the first subdivision schemes were proposed in 1970s [1, 2], a lot of reseachers have focused on subdivision scheme due to its efficiency and simplicity. Now subdivision is an important subject in its own right with many applications in the fields like Computer Graphics, Computer Aided Geometric Design, and Computer Animation, etc. Subdivision scheme can be divided into approximate and interpolatory ones. Interpolation is a requirement in some application, so interpolation by using subdivision is an attractive characteristic. Many interpolatory subdivision schemes have been proposed [3-8].

Fractal geometry [9], founded by Mandelbrot, deals with fractals, namely, geometric shape which is irregular, self-similar, and has fine or detailed structure at arbitrarily small scales. Since fractals were seen in many fields, there have been a lot of results in such varied branches of science as mathematics, computer science, chemistry, physics and biology, etc. Generation of fractal is an important issue in the fractal geometry. Many approaches to generate fractals have been proposed, such as iterated function systems method [10], L-system method [11], and subdivision method [12-14], etc [15, 16]. Using these subdivision methods fractal-like curves can be generated from the point of view of visualization, but few theoretical analysis about whether the limit curves are real fractals or not is presented. Furthermore, few general subdivision schemes which can be used to generate different types of fractals and can help us get desirable fractals. How to effectively generate known fractals or other fractals by using subdivision method is worth studying.

Recently it was shown that subdivision scheme method is not only an important tool for the fast generation of smooth geometric objects, but also an efficient tool for the fast generation of fractal by selecting the corresponding subdivision parameters within some special ranges [17-19]. But still very little attention has been paid to the fractal property analysis of subdivision schemes compared with the smooth properties analysis of those.

In this paper, we propose a new general multi-control *p-ary* $(p \geq 2, p \in Z)$ subdivision scheme based on normal vectors with multi-parameters to generate different fractals. We can

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freely design the values of the parameters and the directions of the normal vectors to control the shape of generated curves. We illustrate the technique with some design results based on the presented subdivision scheme. Some fractal examples, including a few known fractals and other fractals are given. Some fractal properties of the limit curve of the presented subdivision scheme, including fractal existence, self-similarity, non-rectifiability, and continuity but nowhere differentiability are analyzed theoretically, which shows that the generated curves can really be fractals.

2. Multi-control *p-ary* Subdivision Scheme

In this section we present a *p-ary* $\left(p \geq 2, p \in \mathrm{Z}\right)$ subdivision scheme based on normal vectors with multi-parameters which can be uniformly used to generate some known and new fractals.

Given the set of initial control points $\mathrm{P}^0 = \left\{P_j^0\right\}_{j=0}^{n}, P_j^0 \in R^d$, let $\mathrm{P}^k = \left\{P_j^k\right\}_{j=0}^{p^k n}$ be the set of control points at level $k\left(k \geq 0, k \in \mathrm{Z}\right)$, and $\left\{P_j^{k+1}\right\}_{j=0}^{p^{k+1} n}$ satisfy the following rules recursively:

$$\begin{cases} P_{pi}^{k+1} = (1-\alpha_0)P_i^k + \alpha_0 P_{i+1}^k + \beta_0 d_i^k, 0 \leq i \leq p^k n, \\ P_{pi+1}^{k+1} = (1-\alpha_1)P_i^k + \alpha_1 P_{i+1}^k + \beta_1 d_i^k, 0 \leq i \leq p^k n - 1, \\ \qquad \vdots \\ P_{pi+p-1}^{k+1} = (1-\alpha_{p-1})P_i^k + \alpha_{p-1} P_{i+1}^k + \beta_{p-1} d_i^k, 0 \leq i \leq p^k n - 1. \end{cases} \tag{1}$$

Where $p\left(p \geq 2,\ p \in \mathrm{Z}\right),\ \alpha_j \in \mathrm{R},\ \beta_j \in \mathrm{R},\ 0 \leq j \leq p-1,$ are subdivision parameters, and $d_i^k = \left|\overrightarrow{P_i^k P_{i+1}^k}\right| \square \overrightarrow{n_i^k}$, where n_i^k is a unit normal vector perpendicular to vector $\overrightarrow{P_i^k P_{i+1}^k}$. n_i^k have two directions. In this paper, the direction of n_i^k which is obtained by rotating vector $\overrightarrow{P_i^k P_{i+1}^k}$ counterclockwise is called positive direction, and the direction of n_i^k which is obtained by rotating vector $\overrightarrow{P_i^k P_{i+1}^k}$ clockwise is called negative direction. The subdivision parameters and the directions of the normal vectors can help control the shape of the subdivision curves.

Remark 1. The geometric interpretation of the control effect of parameters $\alpha_j, \beta_j, 0 \leq j \leq p-1$ on locations of $P_{pi+j}^{k+1}, 0 \leq j \leq p-1$. For general $p\left(p \geq 2, p \in \mathrm{Z}\right)$ the subdivision scheme (1) contains *p* rules, and we can rewrite the subdivision scheme (1) simply as:

$$P_{pi+j}^{k+1} = (1-\alpha_j)P_i^k + \alpha_j P_{i+1}^k + \beta_j d_i^k = Q_j^k + \beta_j d_i^k, 0 \leq j \leq p-1,$$

Where $Q_j^k = (1-\alpha_j)P_i^k + \alpha_j P_{i+1}^k = P_i^k + \alpha_j (P_{i+1}^k - P_i^k)$ is the assistant point. Then we can find that parameters $\alpha_j, 0 \leq j \leq p-1$ determine the location of point Q_j^k , namely, the distance between P_i^k or P_{i+1}^k and the normal of segment $P_i^k P_{i+1}^k$ at point Q_j^k , while parameters $\beta_j, 0 \leq j \leq p-1$ determine the displacement of point Q_j^k along the direction of normal vector n_i^k , namely, the distance between P_{pi+j}^{k+1} and the segment $P_i^k P_{i+1}^k$.

Remark 2. The combined control effect of the parameters and the directions of the normal vectors. If we choose positive normal vectors, then we have nine different ranges that P_{pi+j}^{k+1} $(0 \leq j \leq p-1)$ possibly belong to (shown in Figure 1), namely,

Case 1. When $\alpha_j > 1, \beta_j > 0$, $P_{pi+j}^{k+1}, 0 \leq j \leq p-1$, are in the range □.

Case 2. When $0 \leq \alpha_j \leq 1, \beta_j > 0$, $P_{pi+j}^{k+1}, 0 \leq j \leq p-1$, are in the range □ including the two extended rays of normal of segment $P_i^k P_{i+1}^k$ passing through two points P_i^k and P_{i+1}^k going up.

Case 3. When $\alpha_j < 0, \beta_j > 0$, $P_{pi+j}^{k+1}, 0 \leq j \leq p-1$, are in the range □.

Case 4. When $\alpha_j > 1, \beta_j = 0$, $P_{pi+j}^{k+1}, 0 \le j \le p-1$, are in the range □, namely, in the extended ray of segment $P_i^k P_{i+1}^k$ to the right.

Case 5. When $0 \le \alpha_j \le 1, \beta_j = 0$, $P_{pi+j}^{k+1}, 0 \le j \le p-1$, are in the range □, namely, in the segment $P_i^k P_{i+1}^k$.

Case 6. When $\alpha_j < 0, \beta_j = 0$, $P_{pi+j}^{k+1}, 0 \le j \le p-1$, are in the range □, namely, in the extended ray of segment $P_i^k P_{i+1}^k$ to the left.

Case 7. When $\alpha_j > 1, \beta_j < 0$, $P_{pi+j}^{k+1}, 0 \le j \le p-1$, are in the range □.

Case 8. When $0 \le \alpha_j \le 1, \beta_j < 0$, $P_{pi+j}^{k+1}, 0 \le j \le p-1$, are in the range □ including the two extended rays of normal of segment $P_i^k P_{i+1}^k$ passing through two points P_i^k and P_{i+1}^k going down.

Case 9. When $\alpha_j < 0, \beta_j < 0$, $P_{pi+j}^{k+1}, 0 \le j \le p-1$, are in the range □.

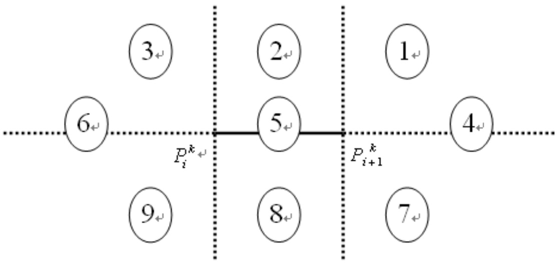


Figure 1. Nine Ranges P_{pi+j}^{k+1} Possibly belong to

3. Generation of Different Fractals

In the section we illustrate the technique by showing some of the special cases of subdivision scheme (1) by choosing the values of the parameters, and the correspondingly fractal curves generated by multi-control of the parameters and the directions of normal vectors from the point of view of visualization.

3.1. A binary interpolatory subdivision scheme to generate Lévy and dragon curves

Using subdivision scheme (1), let $p = 2, \alpha_0 = \beta_0 = 0, \alpha_1 = \beta_1 = \frac{1}{2}$, then we can get a binary interpolatory subdivision scheme as follows:

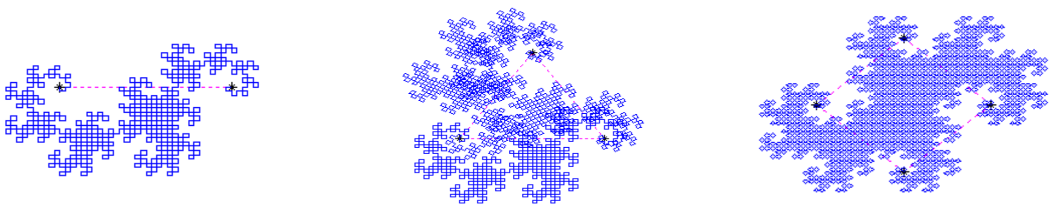
$$\begin{cases} P_{2i}^{k+1} = P_i^k, 0 \le i \le 2^k n, \\ P_{2i+1}^{k+1} = \frac{1}{2} P_i^k + \frac{1}{2} P_{i+1}^k + \frac{1}{2} d_i^k, 0 \le i \le 2^k n - 1. \end{cases} \tag{2}$$

Subdivision scheme (2) can be used to generate the classical Lévy curve [20]. Figure 2 shows the result after one subdivision step and the Lévy curve after eleven subdivision steps by using subdivision scheme (2) based on a unit segment as shown in Figure 2 (a). In Figure 2 and the following figures, the initial control polygons are shown in dashed lines, and the generated subdivision curves are shown in solid lines.

(a) Result after one subdivision step (b) Lévy curve after eleven subdivision steps

Figure 2. Results by using Subdivision Scheme (2)

Figure 3 shows the subdivision parameter $\beta_1 = (-1)^i \frac{1}{2}$ and the other parameters in subdivision scheme (2) unchanged, with the positive directions of the normal vectors based on a unit segment, an equilateral triangle and a quadrilateral, respectively. We can generate the dragon curves [21] after ten subdivision steps.



(a) Dragon curve based on a unit segment (b) Dragon curve based on an equilateral triangle (c) Dragon curve based on a quadrilateral

Figure 3. Dragon curves by using subdivision scheme (2) with subdivision parameter $\beta_1 = (-1)^i \frac{1}{2}$ and the other parameters in subdivision scheme (2) unchanged after ten subdivision steps

If we set $p = 2, \beta_0 = \beta_1 = 0, \alpha_0 = 1-\alpha, \alpha_1 = 1-\beta$ in subdivision scheme (1), we can get the generalized Chaikin corner-cutting subdivision scheme [17]. Apparently using the presented multi-control *p*-ary subdivision scheme (1) we can generate more vivid fractals than using it.

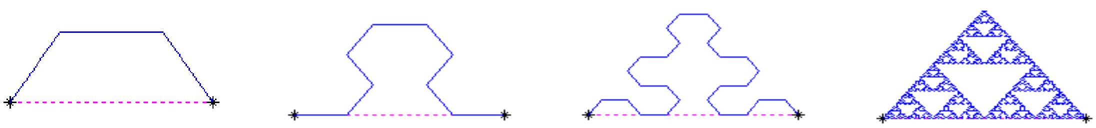
3.2. A Ternary Interpolatory Subdivision Scheme to Generate Classical Sierpiński Gasket

Using subdivision scheme (1), let $p = 3, \alpha_0 = \beta_0 = 0, \alpha_1 = \frac{1}{4}, \alpha_2 = \frac{3}{4}, \beta_1 = \beta_2 = (-1)^{k+i} \frac{\sqrt{3}}{4}$, then we can get a non-stationary and non-uniform ternary interpolatory subdivision scheme as follows:

$$\begin{cases} P_{3i}^{k+1} = P_i^k, 0 \le i \le 3^k n, \\ P_{3i+1}^{k+1} = \frac{3}{4} P_i^k + \frac{1}{4} P_{i+1}^k + (-1)^{k+i} \frac{\sqrt{3}}{4} d_i^k, 0 \le i \le 3^k n - 1, \\ P_{3i+2}^{k+1} = \frac{1}{4} P_i^k + \frac{3}{4} P_{i+1}^k + (-1)^{k+i} \frac{\sqrt{3}}{4} d_i^k, 0 \le i \le 3^k n - 1. \end{cases} \quad (3)$$

Subdivision scheme (3) can be used to generate the classical Sierpiński gasket, also called Sierpiński triangle [15], [22]. Figure 4 shows the results after one subdivision step, two subdivision steps, three subdivision steps, and the classical Sierpiński gasket after seven subdivision steps by using subdivision scheme (3) based on the same unit segment as shown in Figure 3(a).

Remark 3. The method to generate Sierpiński gasket presented here is quite different from the classical one, which can be thought as a procedure of repeatedly replacing an equilateral triangle by three triangles of half the height [15].



(a)Result after one subdivision step (b)Result after two subdivision steps (c)Result after three subdivision steps (d)Sierpiński gasket after seven subdivision steps

Figure 4. Results by using Subdivision Scheme (3)

Figure 5 shows two more Sierpiński gasket-like curves generated by applying subdivision scheme (3) with the positive directions of the normal vectors after seven subdivision steps based on the same initial control polygons as shown in Figure 3(b) and Figure 3(c).

(a)Sierpiński gasket based on an equilateral triangle (b)Sierpiński gasket based on a quadrilateral

Figure 5. Sierpiński Gasket-like curves by using subdivision scheme (3) after seven subdivision steps

3.3. A Quaternary Subdivision Scheme to Generate Koch-type Curves and other Curves

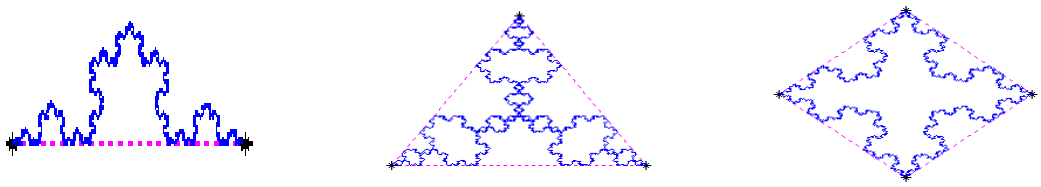
Setting $p=4$ in subdivision scheme (1), we can get a quaternary interpolatory subdivision scheme.

$$\begin{cases} P_{4i}^{k+1} = (1-\alpha_0)P_i^k + \alpha_0 P_{i+1}^k + \beta_0 d_i^k, 0 \le i \le 4^k n, \\ P_{4i+1}^{k+1} = (1-\alpha_1)P_i^k + \alpha_1 P_{i+1}^k + \beta_1 d_i^k, 0 \le i \le 4^k n-1, \\ P_{4i+2}^{k+1} = (1-\alpha_2)P_i^k + \alpha_2 P_{i+1}^k + \beta_2 d_i^k, 0 \le i \le 4^k n-1, \\ P_{4i+3}^{k+1} = (1-\alpha_3)P_i^k + \alpha_3 P_{i+1}^k + \beta_3 d_i^k, 0 \le i \le 4^k n-1. \end{cases} \quad (4)$$

By choosing values of the eight parameters and designating the directions of the normal vectors we can generate many fractal curves and control the shape of the generated curves freely. If we let $\alpha_1 = \frac{1}{3}, \alpha_2 = \frac{1}{2}, \alpha_3 = \frac{2}{3}, \beta_2 = \frac{\sqrt{3}}{6}$, and the other parameters being 0 in subdivision scheme (4), we have a quaternary interpolatory subdivision scheme.

$$\begin{cases} P_{4i}^{k+1} = P_i^k, 0 \le i \le 4^k n, \\ P_{4i+1}^{k+1} = \frac{2}{3}P_i^k + \frac{1}{3}P_{i+1}^k, 0 \le i \le 4^k n-1, \\ P_{4i+2}^{k+1} = \frac{1}{2}P_i^k + \frac{1}{2}P_{i+1}^k + \frac{\sqrt{3}}{6}d_i^k, 0 \le i \le 4^k n-1, \\ P_{4i+3}^{k+1} = \frac{1}{3}P_i^k + \frac{2}{3}P_{i+1}^k, 0 \le i \le 4^k n-1. \end{cases} \quad (5)$$

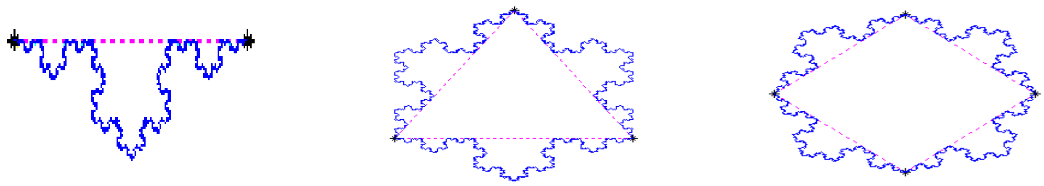
Subdivision scheme (5) can be used to generate the classical Koch curve [15], [23]. Figure 6 shows three curves generated by applying subdivision scheme (5) with the positive directions of the normal vectors after five subdivision steps based on the same initial control polygons as shown in Figure 3.



(a) The classical Koch curve based on a unit segment (b) Koch-type curve based on an equilateral triangle (c) Koch-type curve based on a quadrilateral

Figure 6. Koch-type curves by using subdivision scheme (5) with the positive directions of the normal vectors after five subdivision steps

Figure 7 shows three curves generated by applying subdivision scheme (5) with the negative directions of the normal vectors after five subdivision steps based on the same initial control polygons as shown in Figure 3. The curve in Figure 7(a) is the same as in Figure 6(a) but pointing downward. The curve in Figure 7(b) is called Koch snowflake, which can be thought to be formed by fitting three classical Koch curves together.

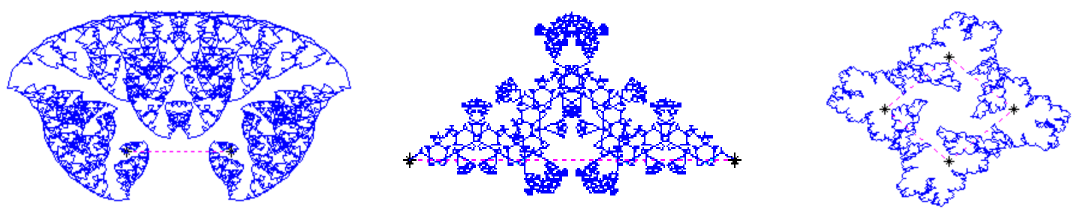


(a) Koch curve based on a unit segment (b) Koch snowflake based on an equilateral triangle (c) Koch-type curve based on a quadrilateral

Figure 7. Koch-type curves by using subdivision scheme (5) with the negative directions of the normal vectors after five subdivision steps

From Figure 6 to Figure 7 we know that given the parameters, the directions of the normal vectors can help control the shape of the generated Koch-type curves.

In fact, subdivision scheme (4) can be used to generate more interesting curves including approximating curves. Figure 8(a) and Figure 8(b) shows two interpolatory curves by applying subdivision scheme (4) with the positive directions of the normal vectors based on the same unit segment as shown in Figure 3(a). Figure 8(a) shows a tigerhead-like curve with subdivision parameters $\alpha_1 = -\frac{1}{4}, \alpha_2 = \frac{1}{2}, \alpha_3 = \frac{5}{4}, \beta_1 = \frac{1}{3}, \beta_2 = \frac{1}{2}, \beta_3 = \frac{1}{3}$, and the other parameters being 0 after six subdivision steps. Figure 8(b) shows a human-like curve with subdivision parameters $\alpha_1 = \frac{5}{9}, \alpha_2 = \frac{1}{2}, \alpha_3 = \frac{4}{9}, \beta_1 = \beta_3 = (-1)^{(k+1)} \frac{1}{3}, \beta_2 = (-1)^{(k+1)} \frac{1}{2}$, and the other parameters being 0 after seven subdivision steps. Figure 8(c) shows an approximating curve with subdivision parameters $\alpha_0 = -\frac{1}{2}, \alpha_1 = -\frac{1}{4}, \alpha_2 = \frac{1}{4}, \alpha_3 = \frac{1}{2}, \beta_1 = \beta_2 = \frac{1}{5}$, and the other parameters being 0 with the negative directions of the normal vectors based on the same initial control polygon as shown in Figure 3(c) after five subdivision steps.



(a) Tigerhead-like curve (b) Human-like curve (c) Approximating curve

Figure 8. More Interesting Curves

Due to the limitation of space, in the flowing part of this section we just show some other special cases of the presented *p*-ary subdivision scheme (1) and the correspondingly generated fractal curves by using them with the positive directions of the normal vectors based on the same simple unit segment as in Figure 3(a).

3.4. *P-ary* Interpolatory Subdivision Scheme to Generate other Special Curves

Using subdivision scheme (1), let $p = 6, \alpha_0 = \beta_0 = 0, \alpha_1 = \alpha_2 = \frac{1}{3}, \alpha_3 = \frac{1}{2}, \alpha_4 = \alpha_5 = \frac{2}{3}$, $\beta_1 = \beta_5 = -\frac{\sqrt{3}}{9}, \beta_2 = \beta_4 = \frac{\sqrt{3}}{9}, \beta_3 = -\frac{\sqrt{3}}{18}$, then we can get a corresponding 6-ary subdivision scheme and use it to generate a crown-like fractal curve shown in Figure 9(a) after four subdivision steps.

Using subdivision scheme (1), let $p=7,\alpha_0=0,\alpha_1=\alpha_2=\frac{1}{4},\alpha_3=\alpha_4=\frac{1}{2},\alpha_5=\alpha_6=\frac{3}{4}$, $\beta_0=\beta_1=\beta_6=0,\beta_2=\beta_3=\frac{1}{4},\beta_4=\beta_5=-\frac{1}{4}$, then we can get a corresponding 7-ary subdivision scheme and use it to generate Minkowski curve [16] shown in Figure 9(b) after five subdivision steps.

(a) Crown-like fractal curve (b) Minkowski curve

Figure 9. Special Curves

From the above examples we know that many fractals including some known fractals and a lot of unknown ones can be generated directly and quickly by using subdivision scheme (1) based on the same simple unit segment as in Figure 3(a) only. So we can conclude that rich and colorful fractals can be generated directly and quickly by using subdivision scheme (1), and the presented method in this paper is easy to use and effective in generating various fractals.

4. Analysis of Fractal Properties

In this section we show fractal properties of the presented subdivision scheme (1) from the point of view of theoretical analysis. For simplicity, we suppose the directions of normal vectors are all positive and we take one of the special cases of presented subdivision scheme (1), the subdivision scheme (5) as an example. The analysis of subdivision scheme (1) is similar but complicated. We will prove that the limit curve of subdivision scheme (5) exists, which is a Koch-type curve, and it is non-rectifiable, and continuous but nowhere differentiable.

4.1. The Property of Convergence

Theorem 1. The subdivision scheme (5) is convergent.

Proof. Suppose $F^{(k)}$ is a piecewise linear interpolatory function which satisfies $F^{(k)}\left(4^{-k}i\right)=P_i^k,\ \ 0\le i\le 4^k n$. Then $F^{(k)}\in C[0,n],\ \forall k\ge 0,\ k\in Z$. And we have

$$\left\|F^{(k+1)}(t)-F^{(k)}(t)\right\|_\infty=\max_i\left\|P_{4i+2}^{k+1}-\tfrac{P_i^k+P_{i+1}^k}{2}\right\|_\infty=\tfrac{\sqrt{3}}{6}\max_i\left|P_i^kP_{i+1}^k\right|=\tfrac{\sqrt{3}}{6}\max_i\left|P_{i+1}^k-P_i^k\right|.$$

According to subdivision scheme (5), we have:

$$\begin{aligned}
P_{4i+1}^{k+1}-P_{4i}^{k+1}&=\tfrac{1}{3}(P_{i+1}^k-P_i^k),\\
P_{4i+2}^{k+1}-P_{4i+1}^{k+1}&=\tfrac{1}{6}(P_{i+1}^k-P_i^k)+\tfrac{\sqrt{3}}{6}\,|\,P_i^kP_{i+1}^k\,|\,n_i^k,\\
P_{4i+3}^{k+1}-P_{4i+2}^{k+1}&=\tfrac{1}{6}(P_{i+1}^k-P_i^k)-\tfrac{\sqrt{3}}{6}\,|\,P_i^kP_{i+1}^k\,|\,n_i^k,\\
P_{4i+4}^{k+1}-P_{4i+3}^{k+1}&=\tfrac{1}{3}(P_{i+1}^k-P_i^k).
\end{aligned}$$

So, $\max_i\left|P_{i+1}^{k+1}-P_i^{k+1}\right|\le\max\left\{\tfrac{1}{3},\tfrac{1}{6}+\tfrac{\sqrt{3}}{6}\right\}\max_i\left|P_{i+1}^k-P_i^k\right|=\tfrac{\sqrt{3}+1}{6}\max_i\left|P_{i+1}^k-P_i^k\right|$.

Hence, $\left\|F^{(k+1)}(t)-F^{(k)}(t)\right\|_\infty=\tfrac{\sqrt{3}}{6}\max_i\left|P_{i+1}^k-P_i^k\right|$

$$\le\tfrac{\sqrt{3}}{6}\tfrac{\sqrt{3}+1}{6}\max_i\left|P_{i+1}^{k-1}-P_i^{k-1}\right|\le\cdots\le\tfrac{\sqrt{3}}{6}\left(\tfrac{\sqrt{3}+1}{6}\right)^k\max_i\left|P_{i+1}^0-P_i^0\right|.$$

Since $\frac{\sqrt{3}+1}{6}<1$, so when k tends to infinity, the maximum error between functions $F^{(k)}$ and $F^{(k+1)}$ tends to 0, which means the function sequence $\{F^{(k)}\}$ is a Cauchy sequence. Since every function $F^{(k)}$ is a continuous function, so the function sequence $\{F^{(k)}\}$ is convergent to a continuous function as its limit. The limit curve is exactly a Koch-type curve. Theorem 1 holds.

4.2. The Property of Non-rectifiability

Theorem 2. The Koch-type curve generated by subdivision scheme (5) is non-rectifiable.

Proof. Suppose P_i and P_j are arbitrarily fixed two control points on the subdivided control polygon after an arbitrary number of subdivision steps. We need to analyze the sum of the length of all the small edges between these two points after another k subdivision steps. For simplicity, it is sufficient to discuss two adjacent fixed points: P_0^0 and P_1^0. After arbitrary k subdivision steps, there are 4^k small edge segments between P_0^0 and P_1^0. The sum of the length of all the small edges is $L_k=[\frac{4}{3}]^k$. We have $L_k=[\frac{4}{3}]^k\to\infty\left(k\to\infty\right)$, which means the limit curve of subdivision scheme (5) is non-rectifiable. Theorem 2 holds.

4.3. The Property of Continuity but Nowhere Differentiability

Theorem 3. The Koch-type curve generated by subdivision scheme (5) is continuous but nowhere differentiable.

Proof. From Theorem 1 we know that the Koch-type curve is continuous. To show it is nowhere differentiable it is sufficient to show that there exists no tangent at any points on the Koch-type curve generated by subdivision scheme (5).

On the Koch-type curve, which is the result corresponding to subdivision step $k=\infty$, there are only four types of points, namely, $P_{4i}^\infty, P_{4i+1}^\infty, P_{4i+2}^\infty,$ and P_{4i+3}^∞, due to the infinite iteration property of subdivision. From the construction of the Koch-type curve, we know that they are connected orderly. The unit edge vectors $\frac{\overrightarrow{P_{4i+j}^\infty P_{4i+j+1}^\infty}}{\left|\overrightarrow{P_{4i+j}^\infty P_{4i+j+1}^\infty}\right|}$ and $\frac{\overrightarrow{P_{4i+j}^\infty P_{4i+j-1}^\infty}}{\left|\overrightarrow{P_{4i+j}^\infty P_{4i+j-1}^\infty}\right|}$, $j=0,1,2,3,$ located at the both sides of point $P_{4i+j,}^\infty\, j=0,1,2,3,$ are not collinear. Hence there exists no tangent at point $P_{4i+j,}^\infty\, j=0,1,2,3.$ So the Koch-type curve is continuous but nowhere differentiable. Theorem 3 holds.

Based on the similar theoretical analysis process and results shown in this section, we can deduce that the limit curve of subdivision scheme (1) exists in a convergent subdivision range, which depends on the subdivision parameters. And it can exhibit some fractal features in a fractal subdivision range, which depends on the subdivision parameters too, including the self-similarity, which is easy to be seen from the natural construction of subdivision scheme, non-rectifiability, and continuity but nowhere differentiability. So we can conclude that the generated curves by using subdivision scheme (1) can be real fractals [15] in the fractal subdivision range. Hence, the presented method in this paper offers an effective means to generate fractals.

5. Conclusion

In this paper, a general multi-control *p-ary* subdivision scheme based on normal vectors acting as a uniform fractal generation method is proposed. Combining choosing of the values of the parameters with designating of the directions of normal vectors we can have a lot of flexibility to control the shape of generated fractals. Many fractal curves generated by using the presented subdivision scheme show that the method is easy to use and effective in generating various fractals.

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