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New scaled algorithm for non-linear conjugate gradients in unconstrained optimization

Ghada M. Al-Naemi, Ahmed H. Sheekoo
Department of Energy Mathematics, Faculty of Computer Science and Mathematics, Mosul University, Mosul, Iraq

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ABSTRACT

A new scaled conjugate gradient (SCG) method is proposed throughout this paper, the SCG technique may be a special important generalization conjugate gradient (CG) method, and it is an efficient numerical method for solving nonlinear large scale unconstrained optimization. As a result, we proposed the new SCG method with a strong Wolfe condition (SWC) line search is proposed. The proposed technique's descent property, as well as its global convergence property, are satisfied without the use of any line searches under some suitable assumptions. The proposed technique's efficiency and feasibility are backed up by numerical experiments comparing them to traditional CG techniques.

Corresponding Author:

Ghada M. Al-Naemi
Department of Mathematics, Faculty of Computer Sciences and Mathematics
University of Mosul
Mosul, Iraq
Email: drghadaalnaemi@uomosul.edu.iq

1. INTRODUCTION

CG method is universal method for solving nonlinear large-scale unconstrained optimization problems, because it has simple iterations, low memory requirements and very fast convergence properties [1]. Therefore, in this work, we considered this general unconstrained optimization problem: indexing and abstracting services depend on the accuracy of the title, extracting from it keywords useful in cross-referencing and computer searching. An improperly titled paper may never reach the audience for which it was intended, so be specific.

$$Min\{f(x): x \in R^n\} \tag{1}$$

Where $f: R^n \to R$ is smooth and its gradient vector defined $g_n = \nabla f(x_n)$, and the initial point $x_0 \in R^n$ is usually solved iteratively according to the recursive formula,

$$x_{n+1} = x_n + \tau_n d_n, n \geq 0 \tag{2}$$

where x_n is current iteration, $\tau_n > 0$ is the step-size calculated by the SWC,

$$\begin{gathered} f(x_n+\tau_n d_n) \leq f(x_n)+\delta\tau_n g_n^T d_n \\ |g(x_n+\tau_n d_n)^T d_n| \leq -\sigma g_n^T d_n \end{gathered} \tag{3}$$

where $0 < \delta < \sigma < 1$ and d_n is a search direction. The classical search direction d_{n+1} are frequently defined by,

$$d_{n+1} = \begin{cases} -g_{n+1}, if\ n = 0 \\ -g_{n+1} + \beta_n d_n, if\ n \geq 1 \end{cases} \quad (4)$$

generally, the parameter β_n is selected so that if *f(x)* is a strictly convex quadratic function and if τ_n is calculated by the exact line search, then (2) and (4) can be simplified to the linear conjugate gradient technique [2]. Several formulas, such as hestenes and stiefel (HS), fletcher and reeves (FR), conjugate descent (CD), Polak-Ribiere (PRP), Liu and Storey (LS) and Dai-Yuan method (DY), have been proposed [3]-[9]. As demonstrated by the formula,

$$\beta_n^{HS} = \frac{g_n^T(g_n - g_{n-1})}{(g_n - g_{n-1})^T d_{n-1}}\ ;\ \beta_n^{FR} = \frac{g_n^T g_n}{g_{n-1}^T g_{n-1}}\ ;\ \beta_n^{PRP} = \frac{g_n(g_n - g_{n-1})}{||g_{n-1}||^2}$$

$$\beta_n^{CD} = -\frac{g_n^T g_n}{g_{n-1}^T d_{n-1}}\ ;\ \beta_n^{LS} = \frac{g_n^T y_{n-1}}{-g_{n-1}^T d_{n-1}}\ ;\ \beta_n^{DY} = \frac{g_n^T g_n}{y_{n-1}^T d_{n-1}}$$

the primary distinction between SCG and CG is the calculation of the search direction. SCG's typical search direction is as follows,

$$d_{n+1} = \begin{cases} -g_{n+1}, if\ n = 0 \\ -\vartheta_n g_{n+1} + \beta_n d_n, if\ n \geq 1 \end{cases} \quad (5)$$

where ϑ_n denotes a spectral parameter. Barzilai and Borwien [10] proposed the SCG method and developed their unconstrained optimization. Instead of global convergence, the idea is to use only teasing trends. Birgin and Martinez [11] proposed an unconstrained optimization method, but it lacked a sufficient descent condition. As a result, Andrai [12] proposed an accelerated CG technology that uses the Newton method to improve the CG method's performance. Following on from this thought, Farvaneh and Keyvan [13] proposed a new SCG [14]-[20] contain additional references in this field.

2. NEW ALGORITHM AND THE DESCENT PROPERTY

Obviously, for SCG, the method for selecting the spectral parameter ϑ_n and conjugate parameter β_n is critical. In this section, we explaine how our proposed SCG is dependent on the parameter β_n proposed by Wei *et al.* [21], which is defined as (6).

$$\beta_n^{WYL} = \frac{||g_n||^2 - \frac{||g_n||}{||g_{n+1}||} g_n^T g_{n+1}}{||g_{n+1}||^2} \quad (6)$$

The new spectral parameter ϑ_n is prposed by (7),

$$\vartheta_n = 1 + \frac{g_{n+1}^T d_n - \frac{(g_{n+1}^T g_n)(g_{n+1}^T d_n)}{||g_n||||g_{n+1}||}}{||g_n||^2} \quad (7)$$

note that, if an exact line search is used then $\vartheta_n = 1$, so (5) reduced to (4).

Algorithm SCG
Step1: Select a starting point $x_0 \in R$, given constand $0 < \delta < \sigma < 1$, stopping criteria $\varepsilon = 10^{-6} > 0$; Set $d_0 = -g_0$.
Step2: Compute $||g_n||$, if $||g_n|| \leq \varepsilon$, stop. Otherwise, continues.
Step3: Calculate $\beta_n^{WYL}, \vartheta_n$,by (6) and (7) respectively and compute step length τ_n by (3).
Step4: Update the new point by (2). Compute $g_{n+1} = g(x_{n+1})$; if $||g_{n+1}|| \leq \varepsilon$, stop; Otherwise, continues.
Step5: Compute search direction d_{n+1} by (5).
Step6: If the Powell restart criteria

$$|g_{n+1}^T g_n| \geq 0.2||g_{n+1}||^2 \quad (8)$$

is satisfied, set $d_{n+1} = -g_{n+1}$ and go back to Step3; otherwise continues.
Step7: Put $n = n + 1$ and go to step3.
We will discuss the sufficient descent property of the Algorithm SCG above without depending to any line search.

2.1. Theorm

It can be concluded that the SCG method with the line search direction (5), $\beta_n^{WYL}, \vartheta_n$ defined in (6) and (7) respectively, and then,

$$g_{n+1}^T d_{n+1} \leq -\xi \left|\left|g_{n+1}\right|\right|^2, \xi \geq 0 \tag{9}$$

holds for $\forall n \geq 0$.

Proof: To stimulate this confirmation, we use induction, if $n = 0$, then $g_o^T d_0 = -\left|\left|g_0\right|\right|^2$, as a result; condition (9) is established. Now, condition (9) is also true in order to notify that every $n \geq 0$ is true. Multiply both sides of (5) by g_{n+1}^T to obtain,

$$g_{n+1}^T d_{n+1} = -\left(1 + \frac{g_{n+1}^T d_n - \frac{(g_{n+1}^T g_n)(g_{n+1}^T d_n)}{||g_n|| ||g_{n+1}||}}{||g_n||^2}\right) \left|\left|g_{n+1}\right|\right|^2 + \frac{||g_{n+1}||^2 - \frac{||g_{n+1}||}{||g_n||} g_{n+1} g_n}{||g_n||^2} g_{n+1}^T d_n$$

$$= -\left|\left|g_{n+1}\right|\right|^2 - \frac{||g_{n+1}||^2 - \frac{||g_{n+1}||}{||g_n||} g_{n+1} g_n}{||g_n||^2} g_{n+1}^T d_n + \frac{||g_{n+1}||^2 - \frac{||g_{n+1}||}{||g_n||} g_{n+1} g_n}{||g_n||^2} g_{n+1}^T d_n$$

$$= -\left|\left|g_{n+1}\right|\right|^2 \tag{10}$$

therefore, the Algorithm SCG can satisfy the sufficient descent conditions without using any line searches.

3. THE GLOBAL CONVERGENCE ANALYSIS

The general situation of the objective function required for the overall global convergence of general CG in psychological analysis is as follows.

3.1. Assumption

- The function $f(x)$ is constrained from below to the level set $\Phi = \{x : x \in R^n / f(x) \leq f(x_0)\}$, where the point of departure is x_0 . i.e., there is a constant α >0, which means $\|x_n\| \leq \alpha \ \forall x \in \Phi$.
- In certain neighborhood N of the level set Φ, the function f(x) is continuously differentiable and its gradient $g(x)$ is Lipschitz continuous, i.e. ∃ a constant, $\underline{L} > 0$ s. t.

$$d_{n+1} = \begin{cases} -g_{n+1}, if\ n = 0 \\ -g_{n+1} + \beta_n d_n, if\ n \geq 1 \end{cases} \tag{11}$$

Assumption (I) clearly implies the existence of a constant $\omega > 0$, s. t.,

$$0 < \|g_{n+1}\| \leq \omega, \forall x \in \Phi \ \ [22] \tag{12}$$

the following Lemma, known as the Zountendijk condition Zountendijk [23], proposed it and is frequently used to demonstrate global convergence of CG techniques.

3.2. Lemma

Suppose Assumption (I) holds. Suppose a general iterative method (2) and the direction (4) is descent direction. So, we have got,

$$\sum_{n=0}^{\infty} \frac{(g_n^T d_n)^2}{||d_n||^2} < \infty \tag{13}$$

according to Assumptions (3.1), Theorem (2.1) and Lemma (3.1), the following results can be proved.

3.3. Theorem

Suppose that Assumption (I) holds. Any CG method of the form (2) and (5) with d_n is a descending search direction and τ_n satisfies SWC. Then,

$$\liminf_{n\to\infty} ||g_n|| = 0 \tag{14}$$

or,

$$\sum_{n\geq 1} \frac{||g_n||^4}{||d_n||^2} < +\infty \tag{15}$$

Proof: Assume, for the sake of argument that the conclusion is not true. Then there exists a positive constant $\varpi > 0$ s.t. $\|g_{n+1}\| \geq \varpi, \forall n$. We can deduce from (5) that $d_{n+1} + \vartheta_n g_{n+1} = \beta_n^{WYL} d_n$. When we square both sides of this equation, we get,

$$(d_{n+1} + \vartheta_n g_{n+1})(d_{n+1} + \vartheta_n g_{n+1}) = (\beta_n^{WYL})^2 ||d_n||^2$$

$$||d_{n+1}||^2 = -(\vartheta_n)^2\, ||g_{n+1}||^2 - 2\vartheta_n g_{n+1}^T d_{n+1} + (\beta_n^{WYL})^2 ||d_n||^2$$

dividing both sides of the above equation by $||g_{n+1}||^4$, and use (10) we get,

$$\frac{||d_{n+1}||^2}{(g_{n+1}^T d_{n+1})^2} = \frac{||d_{n+1}||^2}{||g_{n+1}||^4} = -\frac{(\vartheta_n)^2}{||g_{n+1}||^2} - \frac{2\vartheta_n}{||g_{n+1}||^2} + (\beta_n^{WYL})^2 \frac{||d_n||^2}{||g_{n+1}||^4}$$

$$= -\frac{((\vartheta_n)^2 + 2\vartheta_n)}{||g_{n+1}||^2} + (\beta_n^{WYL})^2 \frac{||d_n||^2}{||g_{n+1}||^4}$$

$$= -\frac{((\vartheta_n)^2 + 2\vartheta_n + 1 - 1)}{||g_{n+1}||^2} + (\beta_n^{WYL})^2 \frac{||d_n||^2}{||g_{n+1}||^4}$$

$$= \frac{1}{||g_{n+1}||^2} + (\beta_n^{WYL})^2 \frac{||d_n||^2}{||g_{n+1}||^4} - \left(\frac{1}{||g_{n+1}||^2} + \frac{(\vartheta_n + 1)^2}{||g_{n+1}||^2}\right)$$

$$\leq \frac{1}{||g_{n+1}||^2} + (\beta_n^{WYL})^2 \frac{||d_n||^2}{||g_{n+1}||^4}$$

in [16] they proved $0 \leq \beta_n^{WYL} \leq \frac{2||g_{n+1}||^2}{||g_n||^2}$ $\forall n \geq 0$

$$\frac{||d_{n+1}||^2}{||g_{n+1}||^4} \leq \frac{1}{||g_{n+1}||^2} + \left(\frac{2||g_{n+1}||^2}{||g_n||^2}\right)^2 \frac{||d_n||^2}{||g_{n+1}||^4}$$

$$= \frac{1}{||g_{n+1}||^2} + \frac{4||d_n||^2}{||g_n||^4}$$

$$\frac{||d_{n+1}||^2}{||g_{n+1}||^4} \leq \frac{4||d_n||^2}{||g_n||^4} + \frac{1}{||g_{n+1}||^2}$$

In terms of $\frac{||d_1||^2}{(g_1^T d_1)^2} = \frac{1}{||g_1||^2}$, together with the above relations and $||g_n||^2 \geq \underline{\omega}$, we have,

$$\frac{||d_{n+1}||^2}{||g_{n+1}||^4} \leq \frac{4||d_n||^2}{||g_n||^4} + \frac{1}{||g_{n+1}||^2} + \frac{1}{||g_n||^2}$$
$$\leq \cdots \leq \sum_{i=1}^{n} \frac{1}{||g_i||^2} \leq \frac{n}{\tilde{\omega}^2}$$

that is, $\frac{||g_{n+1}||^4}{||d_{n+1}||} \geq \frac{\tilde{\omega}^2}{n}$. Hence $\sum_{n\geq 1}\frac{||d_{n+1}||^2}{||g_{n+1}||^4} \geq +\infty$, this is contradicts lemma (3.1). Therefore, the proof is complete.

4. THE NUMERICAL RESULTS

In this section, we will present the outcomes of various test functions. To evaluate the new method, some test functions were chosen. These functions are taken into account by CUTE test function [24], [25]. Using SWC line search, the new SCG method, the classic [21] (WYL) method, the FR method, and the LS method are compared in terms of the number of iterations (NI) and the number of function evaluations (NF). All symbols are written in FORTRAN 77 double precision and collected as Visual FORTRAN (F6.6). The new SCG method is implemented using the SWC line search (3), and with $\delta = 0.001$, $\sigma = 0.9$, we tested 15 well-known test functions, the dimensions of which are given below (1000, 5000, 10000, 50000, and 100000). This algorithm's stopping criterion is $||g_{n+1}|| \leq 10^{-6}$ and we enter 600 if the (NI) equal to or more than 600. The results obtained by the newly proposed method outperform those obtained by the other methods mentioned in the Table 1.

Table 1. The comparison between the proposed method and the other classical methods

No	Test Function	Dimension	SCG method		WYL method		FR method	
			ni	nf	ni	nf	ni	nf
1	ROSEN	1000	26	67	30	78	30	78
		10000	26	67	30	78	30	78
		100000	27	70	32	83	31	81
2	WOLFE	1000	110	225	116	230	111	229
		10000	120	244	123	248	135	280
		100000	121	247	130	255	134	276
	EX-BLOCK DIAGONAL	1000	23	49	21	45	22	46
	BD1	10000	24	51	23	49	26	48
		100000	26	53	25	53	27	52
4	SHALLOW	1000	10	25	10	25	10	25
		10000	10	25	10	25	10	25
		100000	11	27	11	27	11	27
5	WOOD	1000	28	64	29	66	29	66
		10000	28	64	29	66	29	66
		100000	30	68	29	66	30	68
6	BEAL	1000	11	27	11	27	11	27
		10000	11	27	11	27	600	490
		100000	11	27	600	523	600	490
7	POWELL	1000	38	122	56	162	36	110
		10000	38	122	56	162	36	110
		100000	41	138	56	162	39	131
8	CUBIC	1000	16	45	16	45	16	45
		10000	16	45	16	45	16	45
		100000	16	45	16	45	16	45
9	HIMMELBAU	1000	24	251	26	266	26	276
		10000	8	391	10	405	10	401
		100000	12	485	6	523	12	490
10	DQDRTIC	1000	5	11	6	13	5	11
		10000	5	11	6	13	5	11
		100000	5	11	13	13	5	11
11	DIXMAANB	1000	5	13	6	15	6	15
		10000	6	16	7	16	7	16
		100000	6	16	7	16	7	16
12	STRAIT	1000	6	14	8	21	7	18
		10000	6	14	8	21	6	15
		100000	6	14	8	21	6	15
13	BEAL U63	1000	10	27	13	32	12	29
		10000	10	27	13	32	12	29
		100000	11	29	15	39	13	32
14	HILICAL	1000	30	78	29	75	29	73
		10000	33	82	30	78	31	82
		100000	36	92	36	85	36	88
15	DENSCHNB	1000	9	21	9	21	9	21
			9	21	9	21	9	21
			9	21	9	21	9	21

Total 1069 3589 1760 4339 2297 4629

Table 2 compares the performance percentages of the FR, WYL, and proposed SCG technologies. When compared to the FR-method, the WYL technique saves (NI 23.38%), (NF 6.26%) and the SCG technique saves (NI 53.46%). (NF 22.47%). Under the strong Wolfe line search, the proposed method outperformed the existing methods in terms of number of iterations and number of function evaluations.

Table 2. The percentage performance of the proposed methods

Measures	FR method	WYL method	SCG method
NI	100%	76.62%	46.54%
NF	100%	93.74%	77.53%

5. CONCLUSION

In this paper, a new scaled conjugate gradient algorithm for unconstrained optimization problems is proposed. This method, independent of the line search, satisfies the sufficient descent condition. The proposed method has the advantage of being applicable to large-scale problems. The strong Wolfe line search is used to perform numerical computations on some standard benchmark problems. Preliminary findings indicate that the proposed method is both efficient and promising. As a result, it can be used as a different approach for large-scale unconstrained optimization problems. Furthermore, future research can focus on demonstrating the convergence of this method under different line search methods.

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