Extended modified three-term conjugate gradient method for large-scale nonlinear equations

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ABSTRACT

In this research paper, we introduce a novel gradient-free modified three-term conjugate gradient method designed to solve nonlinear equations subject to convex constraints. Our approach incorporates the projection scheme, which enhances the effectiveness of the proposed method. Building upon the modified three-term conjugate gradient method for solving $\mathcal{M}$-tensor systems and ℓ_1-norm-based nonsmooth optimization problems, our method can be regarded as an extension of their technique. By making mild assumptions, we establish the theoretical convergence properties of our iterative method. Through extensive numerical experiments, we demonstrate that our proposed approach is not only highly efficient but also outperforms other existing methods in terms of performance and accuracy.

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1. INTRODUCTION

Nonlinear systems of equations frequently arise in various real-life scenarios. Solving such problems necessitates the utilization of various methods, each with its own set of strengths and weaknesses. This research paper focuses on a specific type of problem: convex constraint nonlinear equations. Our objective is to identify a vector $e \in \mathcal{D}$ that satisfies the given constraints satisfying the property.

$$J(e) = 0. \tag{1}$$

The mapping $J : \mathcal{D} \to \mathbb{R}^n$ is a continuous nonlinear mapping, and $\mathcal{D} \subseteq \mathbb{R}^n$ is a closed convex set. The problem of solving nonlinear equations subject to constraints is encountered in various practical applications, such as heat transfer problems [1], physics phenomena [2], fuzzy problems [3], and economics problems [4], among others. To address this problem, researchers have proposed several methods, including the Newtonian, Quasi-Newton, and Levenberg methods, for solving the nonlinear (1) (see [5]-[8]). These methods exhibit appealing characteristics, such as fast convergence and straightforward implementation. However, they are not well-suited for solving large-scale nonlinear equations with convex constraints, as they require the computation and storage of the Jacobian matrix or its approximation at each iteration.

The conjugate gradient (CG) method is a well-known iterative technique commonly employed for solving large-scale unconstrained optimization problems. Over the years, numerous variations of the conjugate gradient method have been developed and extended to address the nonlinear (1). For instance, Ibrahim *et al.* [9] expanded the hybrid Liu–Storey (LS)-Fetcher–Reeves (FR) conjugate gradient approach proposed by Djordjevi'c [10] to tackle (1) using the Solodov and Svaiter [11] projection technique. Notably, their method does not involve the storage of matrices at each iteration. Yamashita and Fukushima [12] introduced a novel three-term conjugate gradient method specifically designed for nonlinear monotone equations with convex constraints. They established the global convergence and convergence rate of their method under mild assumptions and demonstrated its superior numerical performance compared to other approaches. For comprehensive references on methods for solving nonlinear equations with convex constraints, please refer to [13]-[16]. Additionally, Liu and Du [17] recently proposed a modified three-term conjugate gradient method that proves effective in solving M-tensor systems and nonsmooth optimization problems incorporating the ℓ_1-norm. They also provided theoretical analysis to support the global convergence of their method. For further convergence results concerning the CG method, please consult [18]-[23].

Building upon the work of Liu and Du [17], we present a novel gradient-free method for solving the nonlinear (1). Our proposed method can be considered as an extension of the approach developed by Liu and Du [17]. The global convergence of our method is established, assuming Lipschitz continuity of the underlying mapping and a weaker monotonicity condition.

The remainder of this paper is structured as follows: in section 2, we introduce the gradient-free modified three-term conjugate gradient method for solving the constrained nonlinear (1). Section 3 provides the theoretical analysis, establishing the global convergence of the method under mild assumptions. In section 4, we present preliminary numerical results to demonstrate the efficiency of our proposed method. Finally, we conclude the paper. Throughout this manuscript, the Euclidean norm is denoted by $|\cdot|$.

2. METHOD

Expanding upon the conjugate gradient method developed by Liu and Du [17] for solving m-tensor systems and ℓ_1-norm problems, we present a gradient-free projection method for addressing (1). Our method involves the generation of a trial point k_t using the following relation:

$$k_t = e_t + \alpha_t p_t \tag{2}$$

and the search direction p_t is computed by:

$$p_t := \begin{cases} -J(e_t) & \text{if } t = 0, \\ -J(e_t) + \beta_t^{EMTT} p_{t-1} - \vartheta_t y_{t-1} & \text{if } t > 0, \end{cases} \tag{3}$$

where β_t^{EMTT} and ϑ_t are defined as:

$$\beta_t^{EMTT} := \frac{J(e_t) y_{t-1}}{\|p_{t-1}\|^2},\ y_{t-1} := J(e_t) - J(e_{t-1}),\ \vartheta_t := \frac{J(e_t) p_{t-1}}{\|p_{t-1}\|^2}. \tag{4}$$

Lemma 2.1: consider the search direction p_t generated by (3). It can be established that p_t corresponds to a sufficient descent direction. In other words, for all $t \geq 0$, the following condition holds:

$$J(e_t)^T p_t = -\|J(e_t)\|^2. \tag{5}$$

Proof: by direct computation, we can see that:

$$\begin{aligned} J(e_t)^T p_t &= -\|J(e_t)\|^2 + \frac{J(e_t)^T y_{t-1}}{\|p_{t-1}\|^2} J(e_t)^T p_{t-1} - \frac{J(e_t)^T p_{t-1}}{\|p_{t-1}\|^2} J(e_t)^T y_{t-1} \\ &= -\|J(e_t)\|. \end{aligned}$$

Definition 2.1: let $\mathcal{D} \subseteq \mathbb{R}^n$ be a nonempty closed convex set. Then for any $y \in \mathbb{R}^n$, its projection onto $\mathcal{D}$, denoted by $P_{\mathcal{D}}[y]$, is defined by :

$$P_{\mathcal{D}}[y] := \arg\min\{\|y - x\| \ : x \in \mathcal{D}\}.$$

The projection operator $P_{\mathcal{D}}$ has a well-known property, that is, for any $y, x \in \mathbb{R}^n$ the following nonexpansive property hold

$$\|P_{\mathcal{D}}[y] - P_{\mathcal{D}}[x]\| \leq \|y - x\|. \quad (6)$$

In what follows, we state the iterative procedures/steps of our method.

Algorithm 1

Input: set an initial point $e_0 \in \mathcal{D}$, the positive constants: $Tol > 0,\ r \in (0,1),\ x \in (0,2),\ a > 0, \mu > 0$. Set $t = 0$.
Step 0: compute $J(e_t)$. If $\|J(e_t)\| \leq Tol$, stop. Otherwise, generate the search direction p_t using (3).
Step 1: determine the step-size $\alpha_t = \max\{ar^m | m \geq 0\}$ such that

$$J(e_t + \alpha_t p_t)^T p_t \geq \mu\alpha_t \|p_t\|^2. \quad (7)$$

Step 2: compute $k_t = e_t + \alpha_t p_t$, where k_t is a trial point.
Step 3: if $k_t \in \mathcal{D}$ and $\|J(k_t)\| = 0$, stop. Otherwise, compute the next iterate by

$$e_{t+1} = P_{\mathcal{D}}\left[e_t - x\frac{J(k_t)^T(e_t - k_t)}{\|J(k_t)\|^2}J(k_t)\right], \quad (8)$$

Step 4: finally we set $t = t + 1$ and return to step 1.

3. CONVERGENCE ANALYSIS

In this section, we establish the global convergence property of Algorithm 1. To analyze its convergence behavior, we impose the following assumptions on the mapping J. Assumption 1:

i) The solution set of the constrained nonlinear (1), denoted by $\mathcal{D}^*$, is nonempty.
ii) The mapping J is Lipschitz continuous on $\mathbb{R}^n$. That is, there exists a constant $L > 0$ such that

$$\|J(\alpha) - J(\beta)\| \leq L\|\alpha - \beta\|\ \forall \alpha, \beta \in \mathbb{R}^n \quad (9)$$

iii) For any $\beta \in \mathcal{D}^*$ and $\alpha \in \mathbb{R}^n$, it holds that

$$J(\alpha)^T(\alpha - \beta) \geq 0. \quad (10)$$

Lemma 3.1: consider two sequences $\{p_t\}$ and $\{e_t\}$ generated by Algorithm 1. We can guarantee the existence of a step size α_t that satisfies the line search (7) for all $t \geq 0$. Proof: for any $m \geq 0$, suppose (7) does not hold for the iterate t_0-th, then we have,

$$-J(e_{t_0} + ar^m p_{t_0})^T p_{t_0} < \mu ar^m \|p_{t_0}\|^2.$$

Thus, by the continuity of J and with $0 < r < 1$, it follows that by letting $m \to \infty$, we have,

$$-J(e_{t_0})^T p_{t_0} \leq 0,$$

which contradicts (5).

Lemma 3.2: suppose the sequences $\{e_t\}$ and $\{k_t\}$ are generated by Algorithm 1 under assumption 3. Then, we can observe the following property:

$$\alpha_t \geq \max\left\{a, \frac{rc\|J(e_t)\|^2}{(L+\mu)\|p_t\|^2}\right\}. \quad (11)$$

Proof: let $\hat{\alpha}_t = \alpha_t r^{-1}$. Assume $\alpha_t \neq a$, from (7), $\hat{\alpha}_t$ does not satisfy (7). That is,

$$-J(e_t + \hat{\alpha}_t p_t)^T p_t < \mu\hat{\alpha}_t \|p_t\|^2.$$

From (9) and (5), it can be obviously seen that,

$$\begin{aligned}c\|J(e_t)\|^2 &\le -J_t^T p_t \\ &= (J(e_t+\hat{\alpha}_t p_t) - J(e_t))^T p_t - J(e_t+\hat{\alpha}_t p_t)^T p_t \\ &\le L\hat{\alpha}_t\|p_t\|^2 + \mu\hat{\alpha}_t\|p_t\|^2 \\ &\le \hat{\alpha}_t(L+\mu)\|p_t\|^2.\end{aligned}$$

This gives the desired inequality (11).

Lemma 3.3: assuming assumption 3 holds, let $\{e_t\}$ and $\{k_t\}$ be sequences generated by Algorithm 1. For any solution e^* within the solution set $\mathcal{D}^*$, we have the inequality,

$$\|e_{t+1} - e^*\|^2 \le \|e_t - e^*\|^2 - \mu^2\|e_t - k_t\|^4. \tag{12}$$

In addition, $\{e_t\}$ is bounded and:

$$\sum_{t=0}^{\infty} \|e_t - k_t\|^4 < +\infty. \tag{13}$$

Proof: we start by utilizing the weak monotonicity assumption (assumption 3 (iii)) on the mapping J. Consequently, for any solution $e^* \in \mathcal{D}^*$, we have the following inequality:

$$J(k_t)^T(e_t - e^*) \ge J(k_t)^T(e_t - k_t).$$

the above inequality together with (7) gives:

$$J(e_t + \alpha_t p_t)^T(e_t - k_t) \ge \mu\alpha_t^2\|p_t\|^2 \ge 0. \tag{14}$$

from (6) and (14), we have the following:

$$\begin{aligned}\|e_{t+1} - e^*\|^2 &= \left\| P_{\mathcal{D}}\left[e_t - x\frac{J(k_t)^T(e_t-k_t)}{\|J(k_t)\|^2}J(k_t)\right] - e^*\right\|^2 \\ &\le \left\| \left[e_t - x\frac{J(k_t)^T(e_t-k_t)}{\|J(k_t)\|^2}J(k_t)\right] - e^*\right\|^2 \\ &= \|e_t - e^*\|^2 - 2x\left(\frac{J(k_t)^T(e_t-k_t)}{\|J(k_t)\|^2}\right)J(k_t)^T(e_t - e^*) + x^2\left(\frac{J(k_t)^T(e_t-k_t)}{\|J(k_t)\|}\right)^2 \\ &= \|e_t - e^*\|^2 - 2x\left(\frac{J(k_t)^T(e_t-k_t)}{\|J(k_t)\|^2}\right)J(k_t)^T(e_t - k_t) + x^2\left(\frac{J(k_t)^T(e_t-k_t)}{\|J(k_t)\|}\right)^2 \\ &= \|e_t - e^*\|^2 - x(2-x)\left(\frac{J(k_t)^T(e_t-k_t)}{\|J(k_t)\|}\right)^2 \\ &\le \|e_t - e^*\|^2.\end{aligned}$$

thus, the sequence $\{\|e_t - e^*\|\}$ has a nonincreasing and convergent property. Therefore, this makes $\{e_t\}$ to be bounded and therefore the following holds.

$$\mu^2\sum_{t=0}^{\infty}\|e_t - k_t\|^4 < \|e_0 - e^*\|^2 < +\infty.$$

Remark 3.1: taking into account of the definition of k_t and also by (13), it can be deduced that:

$$\lim_{t\to\infty} \alpha_t\|p_t\| = 0. \tag{15}$$

Theorem 3.2: suppose assumption 3 holds. Let $\{e_t\}$ and $\{k_t\}$ be sequences generated by Algorithm 1, then:

$$\liminf_{t\to\infty} \|J(e_t)\| = 0. \tag{16}$$

Proof: it can be obviously seen from lemma 3 and remark 3 that the sequences $\{e_t\}$ and $\{k_t\}$ are bounded by a positive constant say e_b and k_b respectively. In addition with the continuity of J, it further implies that $\{\|J(e_t)\|\}$ is bounded by a constant say u. Also, by Lipschitz continuity, we have that:

$$\|y_{t-1}\| = \|J(e_t) - J(e_{t-1})\| \leq L\|e_t - e_{t-1}\| \leq 2Le_b \tag{17}$$

Now, suppose (16) is not valid, that is, there exist a constant say $s > 0$ such that $s \leq \|J(e_t)\|$, $t \geq 0$. Then this along with (5) implies that:

$$\|p_t\| \geq cs, \quad \forall t \geq 0. \tag{18}$$

From (3), it follows that for all $t \geq 1$,

$$\begin{aligned}
\|p_t\| &= \left\| -J(e_t) + \frac{J(e_t)^T y_{t-1}}{\|p_{t-1}\|^2} p_{t-1} - \frac{J(e_t)^T p_{t-1}}{\|p_{t-1}\|^2} y_{t-1} \right\| \\
&\leq \|J(e_t)\| + \frac{\|J(e_t)\|\|y_{t-1}\|}{\|p_{t-1}\|^2}\|p_{t-1}\| + \frac{\|J(e_t)\|\|p_{t-1}\|}{\|p_{t-1}\|^2}\|y_{t-1}\| \\
&= \|J(e_t)\| + 2\frac{\|J(e_t)\|\|y_{t-1}\|}{\|p_{t-1}\|} \\
&\leq u + 4L\frac{ue_b}{cs} \triangleq \gamma.
\end{aligned}$$

From (11), we have:

$$\begin{aligned}
\alpha_t\|p_t\| &\geq \max\left\{a, \frac{rc\|J(e_t)\|^2}{(L+\mu)\|p_t\|^2}\right\}\|p_t\| \\
&\geq \max\left\{acs, \frac{rcs^2}{(L+\mu)\gamma}\right\} > 0,
\end{aligned}$$

which contradicts (15). Hence (16) is valid.

4. NUMERICAL EXPERIMENTS

In this section, we assess the performance of Algorithm 1 (referred to as efficient method of three-term (EMTT)) using the Dolan and Moré performance profile [24]. The performance profile takes into account the number of iterations, the number of function evaluations, and the central processing unit (CPU) running time. We evaluate the efficiency of EMTT by applying it to solve several nonlinear monotone functions with convex constraints. To analyze its computational efficiency, we compare EMTT with the following algorithms:

- Xiao and Zhu conjugate gradient method for convex constrained monotone equations [25] (denoted by conjugate gradient descent (CGD) with $a = 1,\ r = 0.1,\ x = 1,\ \mu = 10^{-4}$;
- Liu and Feng derivative-free iterative method for nonlinear monotone equations with convex constraints [26] (denoted by projected Dai-Yuan (PDY)) with $a = 1,\ r = 0.5,\ c = 1,\ \mu = 0.01,\ x = 1$;

We note that all codes were coded and implemented in MATLAB environment using:

- Control parameters: for Algorithm 1, we select $a = 1,\ r = 0.6,\ \mu = 10^{-4},\ x = 1.8,\ Tol = 10^{-6}$. As for Algorithm 1, we select all parameters as in [26].
- Dimensions: $1,000,\ 5,000,\ 10,000,\ 50,000,\ 100,000$.
- Initial points: $e_1 = (0.1, 0.1, \cdots, 0.1)^T$, $e_2 = (0.2, 0.2, \cdots, 0.2)^T$, $e_3 = (0.5, 0.5, \ldots, 0.5)^T$, $e_4 = (1.2, 1.2, \cdots, 1.2)^T$, $e_5 = (1.5, 1.5, \cdots, 1.5)^T$, $e_6 = (2, 2, \ldots, 2)^T$, $e_7 = \text{rand}(0, 1)$.

The test problems with $J = (J_1, J_2, \cdots, J_n)$ are given below:

Problem 1: exponential function [27].

$$J_1(e) = e^{e_1} - 1,$$
$$J_i(e) = e^{e_i} + e_i - 1, \text{ for } i = 2, 3, ..., n,$$
$$\text{and } \mathcal{D} = \mathbb{R}^n_+.$$

Problem 2: modified logarithmic function [27].

$$J_i(e) = \ln(e_i + 1) - \frac{e_i}{n}, \text{ for } i = 1, 2, 3, ..., n,$$
$$\text{and } \mathcal{D} = \left\{ e \in \mathbb{R}^n : \sum_{i=1}^{n} e_i \leq n, e_i > -1, i = 1, 2, \cdots, n \right\}.$$

Problem 3: [28]:

$$J_i(e) = \min\left(\min(|e_i|, e_i^2), \max(|e_i|, e_i^3)\right) \text{ for } i = 2, 3, ..., n,$$
$$\text{and } \mathcal{D} = \mathbb{R}^n_+.$$

Problem 4: strictly convex function 1 [27].

$$J_i(e) = e^{e_i} - 1, \text{ for } i = 1, 2, ..., n,$$
$$\text{and } \mathcal{D} = \mathbb{R}^n_+.$$

Problem 5: strictly convex function 2 [27] .

$$J_i(e) = \frac{i}{n} e^{e_i} - 1, \text{ for } i = 1, 2, ..., n,$$
$$\text{and } \mathcal{D} = \mathbb{R}^n_+.$$

Problem 6: tridiagonal exponential function [29] .

$$J_1(e) = e_1 - e^{\cos(h(e_1+e_2))},$$
$$J_i(e) = e_i - e^{\cos(h(e_{i-1}+e_i+e_{i+1}))}, \text{ for } i = 2, ..., n-1,$$
$$J_n(z) = e_n - e^{\cos(h(e_{n-1}+e_n))},$$
$$h = \frac{1}{n+1}$$

Problem 7: nonsmooth function [30] .

$$J_i(e) = e_i - \sin|e_i - 1|, \ i = 1, 2, 3, ..., n,$$
$$\text{and } \mathcal{D} = \left\{ c \in \mathbb{R}^n : \sum_{i=1}^{n} e_i \leq n, e_i \geq -1, i = 1, 2, \cdots, n \right\}.$$

Problem 8: the trig exp function [27]:

$$J_1(e) = 3e_1^3 + 2e_2 - 5 + \sin(e_1 - e_2)\sin(e_1 + e_2)$$
$$J_i(e) = 3e_i^3 + 2e_{i+1} - 5 + \sin(e_i - e_{i+1})\sin(e_i + e_{i+1}) + 4e_i - e_{i-1}e^{e_{i-1}-e_i} - 3 \text{ for } i = 2, 3, ..., n-1$$
$$J_n(z) = e_{n-1}e^{e_{n-1}-e_n} - 4e_n - 3, \text{ where h} = \frac{1}{m+1} \text{ and } \mathcal{D} = \mathbb{R}^n_+..$$

Problem 9 [31]:

$$t_i = \sum_{i=1}^{n} e_i^2, \ d = 10^{-5}$$
$$J_i(e) = 2d(e_i - 1) + 4(t_i - 0.25)e_i, \ i = 1, 2, 3, ..., n.\text{and} \quad \mathcal{D} = \mathbb{R}^n_+.$$

Figure 1 displays the performance profile in terms of the number of iterations. EMTT is depicted as the top curve, indicating superior performance. EMTT solves approximately 51% of the test problems with fewer iterations, while CGD and PDY solve around 30% and 30% respectively. Similarly, in Figure 2, EMTT demonstrates a lower number of function evaluations compared to CGD and PDY. Figure 3 presents the performance profile based on CPU time. Once again, EMTT emerges as the top curve, solving the highest percentage of problems within a factor τ of the best time. Specifically, EMTT solves around 61% of the test problems with the least CPU time, while CGD and PDY solve approximately 21% and 11% respectively. Based on these comparisons, EMTT outperforms CGD and PDY according to the Dolan and Moré metric [24], encompassing the number of iterations, the total number of function evaluations, and the CPU time.

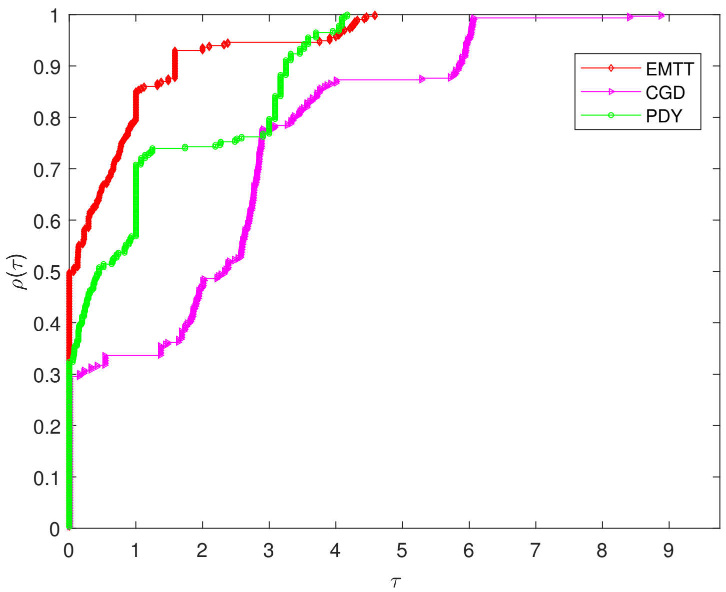

Figure 1. Performance profiles based on number of iterations

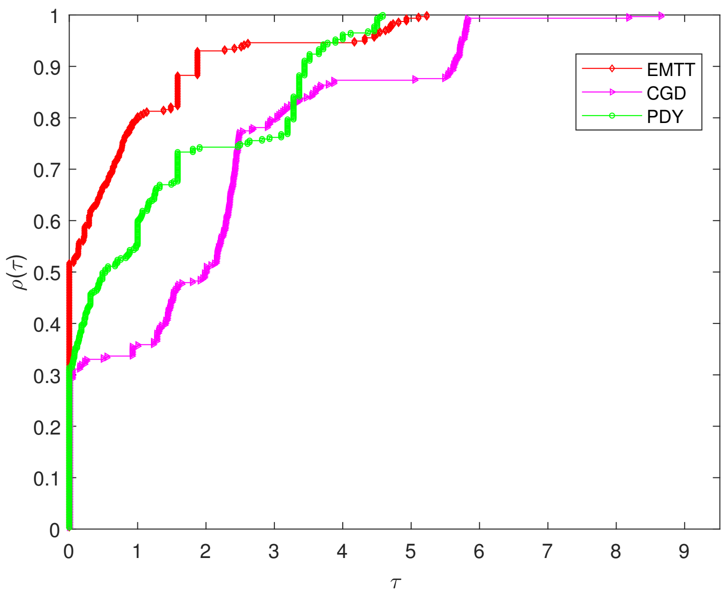

Figure 2. Performance profiles based on number of function evaluations

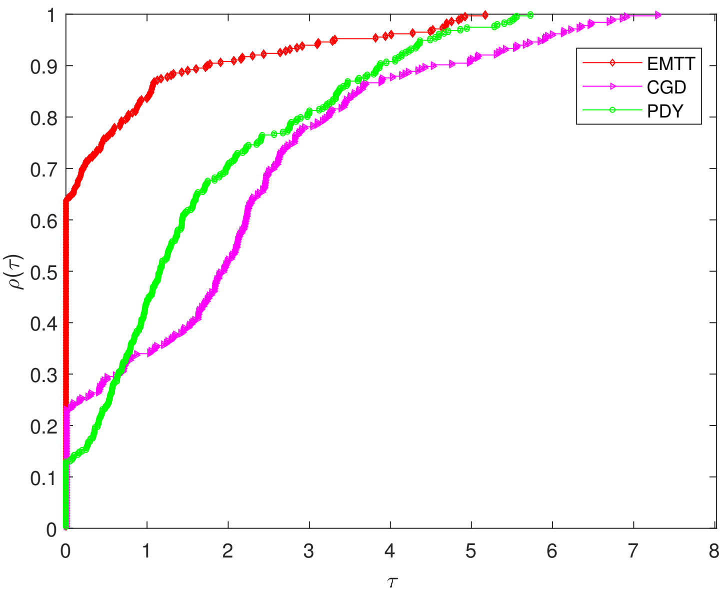

Figure 3. Performance profiles based on CPU time (in seconds)

5. CONCLUSION

In this study, we have expanded upon the modified three-term conjugate gradient method originally developed for solving $\mathcal{M}$-tensor systems and nonsmooth optimization problems involving the ℓ_1-norm. Our objective was to apply this method to address nonlinear equations with convex constraints. Our proposed method ensures that the search direction satisfies the sufficient descent condition, thereby facilitating efficient convergence. Through rigorous analysis, we establish global convergence of the method under the assumption that the underlying operator is Lipschitz continuous and satisfies a weaker monotonicity condition. To validate the effectiveness of our approach, we conducted numerical experiments that demonstrate its efficiency in practice.

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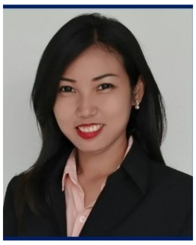

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