TELKOMNIKA, Vol. 11, No. 6, June 2013, pp. 2976 ~ 2982
e-ISSN: 2087-278X

Leader-following Consensus of Multi-agent in Switching Networks with Time-delay

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Abstract

This paper is devoted to the study of multi-agent consensus with a time-varying reference state in directed networks with both switching topology and constant time delay. Stability analysis is performed based on a proposed Lyapunov–Krasovskii function. Sufficient conditions based on linear matrix inequalities (LMIs) are given to guarantee multi-agent consensus on a time-vary reference state under arbitrary switching of the network topology even if the network communication is affected by time delay. Finally, simulation examples are given to validate the theoretical results.

Keywords: *consensus problem, multi-agent, time-delay, switching topology, directed network*

1. Introduction

The problem of distributed coordinated control of a group of autonomous agents is of recent interest in control and robotics. This is partly due to broad applications of multi-agent systems in many areas including multi-vehicle rendezvous, formation control of multi-robot, flocking, swarming, distributed sensor fusion, attitude alignment, and congestion control in communication networks.

An important problem in coordinated control of a group of agents is to find a distributed control law so that all agents can reach a common consensus value. This problem is usually called the consensus problem. In 1995, Vicsek et al. [1] proposed a simple but interesting discrete-time model of autonomous agents all moving in the plane with the same speed but with different headings. Simulation results provided in [1] show that all agents can eventually move in the same direction without centralized coordination. The first paper providing a theoretical explanation for these observed behaviors in Vicsek model is [2]. The results in [2] are extended to the case of directed graph in [3] using matrix analysis [4] and algebraic graph theory [5], and in [6] using set-valued Lyapunov theory. When network communication is affected by time-delays, the consensus problem is investigated in [7-10]. Other research works related to this topic have recently addressed in [11-14], to name a few.

Recently, consensus problem in network of agents with time-delay is studied using linear Matrix inequality method [15-17], which has been extensively used in delay system. In [15], Lin et al. studied average-consensus problem for continuous-time networks of agents with both switching topology and time-delay. In [16], Sun et al. discussed average consensus problem in undirected networks of dynamic agents with fixed and switching topologies as well as multiple time-varying communication delays. Also, in [17], Lin et al. investigated consensus problems of the multi-agent system on directed graphs with external disturbances and model uncertainty in absence and presence of time-delay.

For most consensus algorithms studied in the literature, the final consensus value to be reached is usually constant [3, 6], which might not be applicable in some practical applications. Therefore, it is essential to investigate the consensus algorithm where the final consensus value is specified by a time-varying reference state.

The aim of this paper is to investigate consensus algorithm in directed networks of agents with both switching topology and constant communication delay so that all agents can reach consensus on a time-varying reference state that evolves over time. In [18], Ren proposed a consensus algorithm so that each vehicle in the team reaches consensus on a time-

Received December 13, 2012; Revised March 10, 2013; Accepted April 10, 2013

varying reference state, but the case when network communication is affected by time-delay is not considered.

In this paper, we will extend the results in [18] to the case that network communication is affected by time-delay. We study the consensus problem for continuous-time directed networks of agents with switching topology and time-delay as well as a time-varying reference state evolving according to a nonlinear model. Only the case with constant time delay is considered here, the case with time-varying delay is considered in another paper. For the case of constant time-delay, by constructing a common Lyapunov-Krasovskii function, a sufficient condition expressed as linear matrix inequalities (LMIs) is proposed to guarantee all agents can reach consensus on a time-varying reference state under arbitrary switching of the network topology. Furthermore, we prove that if the network topology of agents is balanced and in another graph, the $n+1$th node representing the time-varying reference state is globally reachable, then the consensus problem is exactly solvable.

This paper is organized as follows. In section 2, we establish the notation and formally state the problem. We present our main results in section 3, show simulation results supporting the objectives of the paper in section 4 and offer concluding remarks in section 5.

2. Problem Statement

Consider a group of n agents with dynamics given by:

$$\dot{x}_i(t) = u_i(t) \quad i \in I = \{1, 2, \cdots, n\}, \tag{1}$$

Where $x_i(t) \in R$ is the state of the ith agent at time t, which might represent physical quantities such as attitude, position, temperature, voltage, and so on, and $u_i(t) \in R$ is the control input (or protocol) at time t.

A weighted directed graph (or digraph) G of order n will be used to model the interaction topology among these agents, where $V = \{v_1, v_2, \cdots, v_n\}$ is the set of nodes, $E \subseteq V \times V$ is the set of edges, and $A = [a_{ij}] \in R^{n\times n}$ is a symmetric weighted adjacency matrix with nonnegative adjacency elements a_{ij}. In G, the ith node represents the ith agent and a directed edge from agent i to agent j denoted as $e_{ij} = (v_i, v_j)$ represents a directional information exchange link from agent i to agent j, that is, agent i can receive or obtain information from agent j. $e_{ij} \in E$ if and only if $a_{ij} > 0$. Moreover, we assume $a_{ii} = 0$ for all $i \in I$. The set of neighbors of node v_i is denoted by $N_i = \{v_j \in V : (v_i, v_j) \in E\}$. A directed path in graph $G(V, E, A)$ is a sequence of ordered edges (v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), $(v_{i_3}, v_{i_4}) \cdots$ in that graph. Graph G is called strongly connected if there is a directed path from v_i to v_j and v_j to v_i between any pair of distinct nodes v_i and v_j, $\forall (i, j) \in I \times I$.

The in-degree and out-degree of node v_i are, respectively, defined as follows:

$$d_{in}(v_i) = \sum_{j=1}^{n} a_{ji}, \; d_{out}(v_i) = \sum_{j=1}^{n} a_{ij}. \tag{2}$$

The degree matrix of graph G is a diagonal matrix $D = \left[d_{ij} \right]$ where $d_{ij} = 0$ for all $i \neq j$ and $d_{ii} = d_{out}(v_i)$. The Laplacian matrix associated with the graph is defined as $L = D - A$.

A consensus algorithm is proposed in [7] as

$$u_i(t) = \sum_{v_j \in N_i} a_{ij}(x_j(t) - x_i(t)), \quad i = 1, 2, \cdots, n. \tag{3}$$

We say protocol (3) asymptotically solves the consensus problem, if and only if for all $i, j = 1, 2, \cdots, n$, $x_i(t) \to x_j(t)$ as $t \to \infty$. The final consensus value, which depends on both the information exchange topologies and the weights a_{ij}, is a constant and might be a priori unknown. However, in some applications, it might be desirable that each state $x_i(t)$ approaches a time-varying reference state $x^r(t)$.

In this paper, we are interested in discussing consensus problem with time-varying reference state $x^r(t)$ in directed networks of multi-agents with both switching topology and constant time-delay, where the information pass through the edge (v_i, v_j) (from v_j to v_i) with time-delay $\tau(t)$. In the case of constant time delay, $\tau(t) = \tau$. We say the consensus problem with time-varying reference state is solved if $\lim_{t\to\infty}\left|x_i(t) - x^r(t)\right| = 0, i = 1, 2, \cdots, n.$

To study the consensus problem with time-varying reference state, we need to consider another graph $\tilde{G}$ with set of nodes $\tilde{V} = \{v_1, v_2, \cdots, v_{n+1}\}$. The time-varying reference state $x^r(t)$ can be viewed as a virtual agent and we let node v_{n+1} to present this virtual agent. Obviously, graph G is the subgraph obtained from $\tilde{G}$ by deleting node v_{n+1} and all edges incident on node v_{n+1}. We define a diagonal matrix $B \in R^{n\times n}$ with diagonal elements $b_i = a_{i0}, i \in I$, where $b_i > 0$ if the virtual agent is a neighbor of agent i and $b_i = 0$, otherwise. In $\tilde{G}$, if there exists a directed path from every node $v_i, i \in I$, to node v_{n+1}, we say that node v_{n+1} is globally reachable in $\tilde{G}$, which is much weaker then strong connectedness.

To solve this problem, we use following consensus protocol:

$$u_i(t) = \sum_{v_j \in N_i(t)} a_{ij}(t)(x_j(t-\tau) - x_i(t-\tau)) + b_i(t)(x^r(t-\tau) - x_i(t-\tau)) + \dot{x}^r(t) \quad (4)$$

where the time delay $\tau \geq 0$ is constant. In (4), we allow the weight factors $a_{ij}(t)$ and $b_i(t)$ to be dynamically changing to represent possible time-varying relative confidence of each agent's information state.

With (4), (1) can be written in a matrix form:

$$\dot{x}(t) = -(L_k + B_k)x(t-\tau) + B_k\mathbf{1}x^r(t-\tau) + \mathbf{1}\dot{x}^r(t), k = s(t). \quad (5)$$

Where, $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^{\mathrm{T}}$, the map $s(t): [0, +\infty] \to I_\Gamma = \{1, 2, \cdots, N\}$ ($N \in Z^+$ denotes the total number of all possible directed graphs) is a switching signal that determines the network topology. $L_k = L(G_k)$ is the Laplacian of graph G_k that belongs to a set $\Gamma = \{G_k : k \in I_\Gamma\}$, which is obviously finite. B_k is a diagonal matrix with diagonal elements $b_1(t), b_2(t), \cdots, b_n(t)$ and $\mathbf{1} = (1,1,\cdots,1)^{\mathrm{T}}$.

The following lemma shows an important property of Laplacian of graph.

Lemma 1[19] The digraph G has a globally reachable node if and only if Laplacian L of graph G has a simple zero eigenvalue with eigenvector $\mathbf{1} = (1, \cdots, 1)^{\mathrm{T}} \in R^n$.

Let $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^{\mathrm{T}}$, $e(t) = x(t) - \mathbf{1}x^r(t)$. From Lemma 1,

$$-(L_k + B_k)x(t-\tau) + B_k\mathbf{1}x^r(t-\tau) = -(L_k + B_k)e(t-\tau),$$

Then system (5) can be rewritten as:

$$\dot{e}(t) = -H_k e(t-\tau) \quad (6)$$

Where $H_k = L_k + B_k$.

In the following section, we will discuss the convergence of dynamical system (6), that is, $\lim_{t\to\infty} e(t)=0$.

3. Main Results

In this section, we provide the convergence analysis of the consensus problem in directed network with switching topology and time-delay.

Before giving the main results, we first introduce some lemmas that play an important role in the proof of the main results.

Lemma 2 [20] (Schur complement) Let $S=\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$ be given partitioned matrix, then, $S<0$ if and only if $S_{11}<0$, $S_{22}-S_{21}S_{11}^{-1}S_{12}<0$ or $S_{22}<0$, $S_{11}-S_{12}S_{22}^{-1}S_{21}<0$.

Lemma 3 [21] The matrix $H_k=L_k+B_k$ is positive stable if and only if node $n+1$ is globally reachable in $\tilde{G}_k$.

Lemma 4 [21] Suppose G_k is balance graph. Then $H_k+H_k^{\mathrm{T}}$ is positive definite if and only if node $n+1$ is globally reachable in $\tilde{G}_k$.

In the following, we present our main results.

Theorem 1 Consider a directed network of multi-agent with both switching topology and constant time-delay τ. Protocol (4) asymptotically solves the consensus problem with time-varying reference state $x^r(t)$, if for all $k\in I_\Gamma$, there exists positive defined matrix $P,Q,R\in R^{n\times n}$ satisfying.

$$\begin{pmatrix} -PH_k-H_k^TP+Q & PH_k & 0_{n\times n} \\ H_k^TP & -\dfrac{R}{\tau} & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & -Q+\tau H_k^{\mathrm{T}}RH_k \end{pmatrix}<0 \tag{7}$$

Proof. Define a common Lyapunov-Krasovskii function for system (6) as follows:

$$V(t)=e^{\mathrm{T}}(t)Pe(t)+\int_{t-\tau}^{t}e^{\mathrm{T}}(s)Qe(s)ds+\int_{-\tau}^{0}\int_{t+\theta}^{t}\dot{e}^{\mathrm{T}}(s)R\dot{e}(s)dsd\theta.$$

Along the trajectory of system (6), we have

$$\begin{aligned}\dot{V}(t)&=2e^{\mathrm{T}}(t)P\dot{e}(t)+e^{\mathrm{T}}(t)Qe(t)-e^{\mathrm{T}}(t-\tau)Qe(t-\tau)-\int_{t-\tau}^{t}\dot{e}^{\mathrm{T}}(s)R\dot{e}(s)ds+\tau\dot{e}^{\mathrm{T}}(t)R\dot{e}(t)\\ &=-2e^{\mathrm{T}}(t)PH_ke(t-\tau)+e^{\mathrm{T}}(t)Qe(t)-e^{\mathrm{T}}(t-\tau)Qe(t-\tau)\\ &\quad-\int_{t-\tau}^{t}\dot{e}^{\mathrm{T}}(s)R\dot{e}(s)ds+\tau e^{\mathrm{T}}(t-\tau)H_k^{\mathrm{T}}RH_ke(t-\tau).\end{aligned}$$

By Newton-Leibniz formula,

$$e(t-\tau)=e(t)-\int_{t-\tau}^{t}\dot{e}(s)ds$$

and note that $2x^{\mathrm{T}}y\le x^{\mathrm{T}}M^{-1}x+y^{\mathrm{T}}My$ hold for any appropriate positive definite matrix M, we have:

$$\begin{aligned}-2e^{\mathrm{T}}(t)PH_ke(t-\tau)&=-2e^{\mathrm{T}}(t)PH_ke(t)+\int_{t-\tau}^{t}2(H_k^{\mathrm{T}}Pe(t))^{\mathrm{T}}\dot{e}(s)ds\\ &\le-2e^{\mathrm{T}}(t)PH_ke(t)+\tau e^{\mathrm{T}}(t)PH_kR^{-1}H_k^{\mathrm{T}}Pe(t)+\int_{t-\tau}^{t}\dot{e}^{\mathrm{T}}(s)R\dot{e}(s)ds.\end{aligned}$$

Consequently,

$$\begin{aligned}\dot{V}(t)&\le-2e^{\mathrm{T}}(t)PH_ke(t)+\tau e^{\mathrm{T}}(t)PH_kR^{-1}H_k^{\mathrm{T}}Pe(t)\\ &\quad+e^{\mathrm{T}}(t)Qe(t)-e^{\mathrm{T}}(t-\tau)Qe(t-\tau)+\tau e^{\mathrm{T}}(t-\tau)H_k^{\mathrm{T}}RH_ke(t-\tau).\end{aligned}$$

Then, a sufficient condition for $\dot{V}(t)<0$ is

$$e^{\mathrm{T}}(t)(-PH_k - H_k^{\mathrm{T}}P + \tau PH_k R^{-1} H_k^{\mathrm{T}} P + Q)e(t) < 0 \tag{8}$$

and

$$e^{\mathrm{T}}(t-\tau)(-Q + \tau H_k^{\mathrm{T}} R H_k)e(t-\tau) < 0\ . \tag{9}$$

As a result, Equations (8) and (9) hold, if and only if:

$$-PH_k - H_k^{\mathrm{T}}P + \tau PH_k R^{-1} H_k^{\mathrm{T}} P + Q < 0 \tag{10}$$

And

$$-Q + \tau H_k^{\mathrm{T}} R H_k < 0\ . \tag{11}$$

Then, by Schur complement formula, the matrix inequality (10) is equivalent to:

$$\begin{bmatrix} -PH_k - H_k^T P + Q & PH_k \\ H_k^T P & -\dfrac{R}{\tau} \end{bmatrix} < 0$$

Therefore, multi-agent consensus with time-varying reference state $x^r(t)$ can be achieved if the matrix inequality (7) holds. This completes the proof.□

Theorem 2 Suppose that graph G_k ($k \in I_\Gamma$) is balanced, node $n+1$ is kept globally reachable in graph $\tilde{G}_k$ ($k \in I_\Gamma$), then linear matrix inequality (LMI) (7) is solvable.

Proof. Let $P = R = I_n$, and $Q = \alpha I_n$ with $\alpha > 0$. From Lemma 3, 4 and the fact that the set I_Γ is finite, both $\lambda_{min} = \min_{k \in I_\Gamma} \min_i \lambda_i (H_k + H_k^{\mathrm{T}}) > 0$ and $\lambda_{\max} = \max_{k \in I_\Gamma} \max_i \lambda_i (H_k H_k^{\mathrm{T}}) > 0$ can be well defined. Then matrix inequalities (10) and (11) can be transformed into the following matrix inequalities:

$$-(H_k + H_k^{\mathrm{T}}) + \tau H_k H_k^{\mathrm{T}} + \alpha I_n \le -\lambda_{\min} I_n + \tau \lambda_{\max} I_n + \alpha I_n < 0$$

$$-\alpha I_n + \tau H_k^{\mathrm{T}} H_k \le -\alpha I_n + \tau \lambda_{\max} I_n < 0$$

Choosing $\tau\lambda_{max} < \alpha < \dfrac{\lambda_{min}}{2}$, we have $\dot{V} < 0$ if $\tau < \dfrac{\lambda_{min}}{2\lambda_{max}}$. This completes the proof.□

4. Simulation

In this section, simulation example will be given to validate the theoretical results obtained in the previous sections. Consider a group of 12 agents labeled 1 through 12 and the time-varying reference state x^r can be viewed as a virtual agent. Figure 1 shows four examples of directed graphs each with 13 nodes. Obviously, node x^r is globally reachable in graph $\tilde{G}_1$, $\tilde{G}_2$, $\tilde{G}_3$ and $\tilde{G}_4$, and the corresponding subgraphs G_1, G_2, G_3, G_4 obtained from $\tilde{G}_1, \tilde{G}_2, \tilde{G}_3, \tilde{G}_4$ by deleting node x^r and all edges incident on node x^r respectively, are balanced. The adjacency matrices of all graphs in this figure have $0-1$ weights. A finite automation with set of states $\{\tilde{G}_1, \tilde{G}_2, \tilde{G}_3, \tilde{G}_4\}$ is shown in Figure 2 representing the discrete-states of a network with switching topology as a hybrid system. The hybrid system starts at the discrete-state $\tilde{G}_1$ and switches every simulation time step to the next state according to the state machine in Figure 2. Let the time-varying reference state be $x^r(t) = 5\sin(t)$ and the initial states of 12 agents be normally distributed random numbers with mean 0 and standard deviation 10.

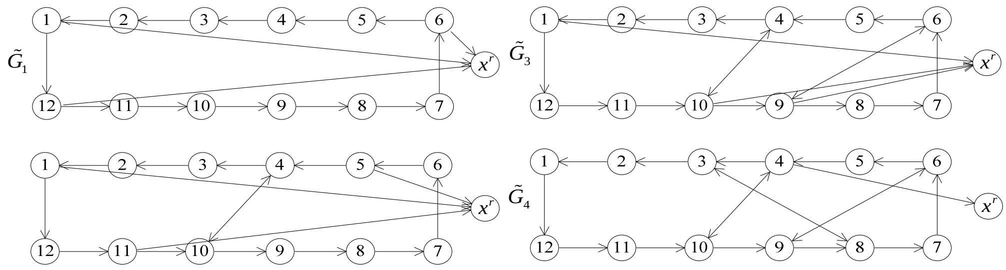

Figure 1. Examples of Directed Graph in which Node x^r is Globally Reachable.

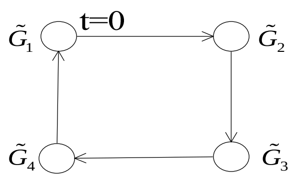

Figure 2. A Finite Automation with Four States.

For constant communication time-delay, from theorem 1, the maximum delay bound $\tau = 0.095s$ and the corresponding feasible solution P, Q, R can be obtained by employing the LMI Toolbox in Matlab. Figure 3 shows the state trajectories of the above network system with switching topology and constant time-delay $\tau = 0.095s$, and the states of 12 agents converge to the time-varying reference state x^r asymptotically. Figure 4 shows the corresponding error system converges to zero asymptotically.

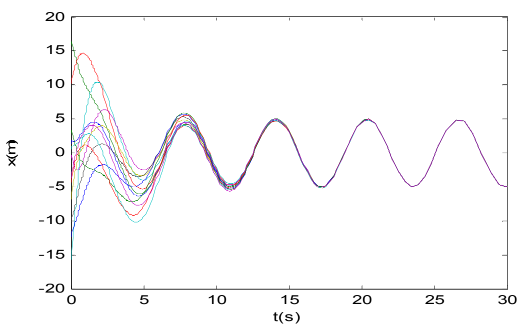

Figure 3. State Trajectories of the System with Switching Topology and Constant Time-delay $\tau = 0.095s$ Converge to Time-varying Reference State x^r Asymptotically

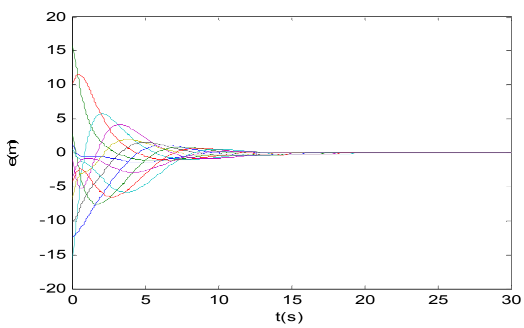

Figure 4. Error System with Switching Topology and Constant Time-delay $\tau = 0.095s$ Converges to Zero Asymptotically

5. Conclusion

This paper addresses a consensus problem of multi-agent systems with a time-varying reference state. The time-varying reference state is a general nonlinear model. Directed switching network topology and constant communication delay are considered in this paper. A sufficient condition in terms of LMIs is given to guarantee the system can reach consensus on a time-varying reference state. Moreover, numerical simulation is shown to verify the theoretical analysis.

Acknowledgements

This work is supported in part by Natural Science Foundation of China (61273183, 61074091 and 61174216), Nature Science Foundation of Hubei Province (2011CDB187), Scientific Innovation Team Project of Hubei Provincial College (T200809 and T201103).

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