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Analytical solutions of linear and non-linear incommensurate fractional-order coupled systems

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ABSTRACT

In this paper, a new analytical method is developed for solving linear and non-linear fractional-order coupled systems of incommensurate orders. The system consists of two fractional-order differential equations of orders $0 < \alpha, \beta \leq 1$. The proposed approach is performed by decoupling the system into two fractional-order differential equations; the first one is a fractional-order differential equation (FoDE) of one variable of order $(\alpha + \beta)$, while the second one depends on the solution of the first one. The general solution of the coupled system is obtained using the adomian decomposition method (ADM). The main ideas of this work are verified via several examples of linear and nonlinear systems, and the numerical simulations are performed using mathematica.

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1. INTRODUCTION

Fractional-order Differential Equations (FoDEs) are well suited to model physical systems with memory orfractal attributes, and they are increasingly utilized to model many problems in fluid dynamics, viscoelasticity, biology, physics and engineering [1-5]. Several methods have been suggested to solve linear and non-linear FoDEs such as the Predictor Corrector Method (PCM) [6, 7], the Adomain Decomposition Method (ADM) [2, 8-12], the Homotopy Perturbation Method (HPM) [13], the Variational Iteration Method (VIM) [14-16], the Differential Transform Method (DTM) [17], and the Finite difference method [18, 19]. An analytical method for obtaining solutions of linear FoDEs with Jumarie type derivative in terms of the Mittag-Leffler functions and the generalized sine and cosine functions was presented in several papers see [7, 20-22]. Several non-linear phonemena are modeled by coupled FoDEs such as the fractional-order Chuah's, Rössler's, Duffing's, and Stiff's fractional-order systems [7, 23]. The single-term Haar wavelet series (STHW) method were introduced to solve singular stiff delay systems and non-linear singular systems in fluid dynamics [23-25].

In this paper, linear and nonlinear incommensurate fractional-order systems in two variables, $x(t)$ and $y(t)$, of order α and β, $0 < \alpha, \beta \leq 1$, are considered. The linear system is first considered and converted into two differential equations; one equation is of order $\alpha + \beta$, denoted by $(\alpha + \beta)$-FoDE, that only depends on $y(t)$, while the analytical solution of the second variable, $x(t)$, follows from the solution of the first part.

The Adomian Decomposition Method (ADM) is implemented to obtain the analytical solution of the $(\alpha+\beta)$-FoDE. Similarly, the solution of the Non-Linear System of Incommensurate Fractional-order (NLS-IFo) is also obtained by extending the same technique that used for the case of the linear one. Such solutions are obtained using the Adomian Polynomials (AP's), which form the basis for the ADM.

This paper is outlined as follows: Next section introduces necessary definitions and preliminaries. Section 3 introduces a method to convert a LS-IFo into an $(\alpha+\beta)$-FoDE in one variable. Section 4 introduces the ADM as a tool to solve the $(\alpha+\beta)$-FoDE of linear incommensurate order systems. The general solution of the nonlinear one is introduced in Section 5. Section 6 summarizes the main ideas of this work via numerical simulations followed by conclusions and final remarks.

2. BASIC DEFINITIONS AND PRELIMINARIES

The Caputo's definition of fractional-order derivatives is adopted in this work. It is a modification of the Riemann-Liouville definition, with the advantage of only using the initial conditions of the corresponding integer-order derivatives that suites most physical system [26-28]. The following definitions and preliminaries of fractional calculus are presented here for completeness.

Definition 2.1 [11]: Let $f(t)$ be anintegrable piecewise continuous function on any finite subinterval of $(0,+\infty)$, then the fractional integral of $f(t)$ of order α is defined as:

$$J^{\alpha}f(t):=\frac{t^{\alpha-1}}{\Gamma(\alpha)}*f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}f(\tau)d\tau, t>0, \alpha>0. \tag{1}$$

Definition 2.2 [11]: The Caputo fractional-order derivative is defined as:

$$D^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t\frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}d\tau, n-1\leq\alpha<n, n\in\mathbb{N}. \tag{2}$$

Theorem 2.3 [11], [24]: The Caputo fractional-order derivative of the power function satisfies

$$D^{\alpha}t^{\mu}=\begin{cases}\frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}t^{\mu-\alpha}, n-1<\alpha<n, \mu>n-1, \mu\in\mathbb{R},\\ \\ 0, n-1<\alpha<n, \mu\leq n-1, \mu\in\mathbb{N}, n\in\mathbb{N}.\end{cases} \tag{3}$$

Theorem 2.4 [3], [24]: The Riemann Liouville fractional-order integral of the power function satisfies

$$J^{\alpha}t^{\mu}=\frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}t^{\mu+\alpha}, \alpha>0, \mu>-1. \tag{4}$$

Theorem 2.5 [3]: If $x>0$, $\alpha\in\mathbb{R}$ and $n-1<\alpha<n$. Then,

$$D^{\alpha}J^{\alpha}f(t)=f(t), n\in\mathbb{N}. \tag{5}$$

Theorem 2.6 [3]: If f is a continuous function on $[a,b]$and $\alpha\geq 0$. Then,

$$J^{\alpha}D^{\alpha}f(t)=f(t)-\sum_{k=0}^{n-1}\frac{t^k}{k!}f^k(0), n\in\mathbb{N}. \tag{6}$$

3. CONVERTING A LS-IFO INTO AN $(\alpha+\beta)$-FODE

The Integral Transform Methods (ITM) such as Fourier Transform (FT), Laplace Transform (LT), and Mellin Transform (MT) are used to solve a single FoDE [29]. In the case of coupled systems of FoDEs, it is necessary to employ specific techniques that are appropriate to the given problem. There are several methods for solving such problems, see [29] for example. The proposed method in this work presents a new direct technique that is competitive to that of the corresponding ones in which order conversion allows one to simplify the solution method. For completeness, the following lemma outlines the conversion results, which allows one to generate an $(\alpha+\beta)$-FoDE from the coupled one.

Lemma 3.1: The following non-homogeneous LS-IFo:

$$D^{\alpha}x(t)=a_{11}x(t)+a_{12}y(t)+f_1(t), \tag{7(a)}$$

$$D^{\beta}y(t) = a_{21}x(t) + a_{22}y(t) + f_2(t), \tag{7(b)}$$

can be converted into the following equivalent system:

$$x(t) = \frac{1}{a_{21}}\left(D^{\beta}y(t) - a_{22}y(t) - f_2(t)\right), \tag{8(a)}$$

$$D^{\alpha+\beta}y(t) - a_{11}D^{\beta}y(t) - a_{22}D^{\alpha}y(t) + det(A)y(t) = h(t), \tag{8(b)}$$

where $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $h(t) = D^{\alpha}f_2(t) + a_{21}f_1(t) - a_{11}f_2(t)$, D^{α} and D^{β} are the Caputo's fractional-order derivatives; $0 < \alpha, \beta \leq 1$; and a_{ij}'s are constants for $i, j = 1, 2$ with $a_{21} \neq 0$, and where $f_1(t)$ and $f_2(t)$ are continuous functions of t on some interval I.

Corollary 3.2: If $\alpha = \beta$ in system (7), then the system will be equivalent to the following equations:

$$x(t) = \frac{1}{a_{21}}\left(D^{\alpha}y(t) - a_{22}y(t) - f_2(t)\right), \tag{9(a)}$$

$$D^{2\alpha}y(t) - tr(A)D^{\alpha}y(t) + det(A)y(t) = h(t), \tag{9(b)}$$

where $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $h(t) = D^{\alpha}f_2(t) + a_{21}f_1(t) - a_{11}f_2(t)$.

Proof: The proof follows immediately from Lemma 3.1. ■

Thus, a LS-IFo in two variables has been converted into two parts; the first one is an $(\alpha + \beta)$-FoDE in $y(t)$, while the other one is just a direct analytical solution of $x(t)$ that only depends on $y(t)$.

4. THE GENERAL SOLUTION OF $(\boldsymbol{\alpha} + \boldsymbol{\beta})$-FODE USING THE ADM

In this section, we use the ADM to obtain the general solution of an $(\alpha + \beta)$-FoDE. See [30, 31] for an overview of the ADM approach. Theorem 4.1 introduces a new approach for solving the non-homogeneous LS-IFo system, while the case of homogeneous systems is addressed by corollary 4.2, i.e.;

Theorem 4.1: The following LS-IFo:

$$D^{\alpha}x(t) = a_{11}x(t) + a_{12}y(t) + f_1(t), \tag{10(a)}$$

$$D^{\beta}y(t) = a_{21}x(t) + a_{22}y(t) + f_2(t), \tag{10(b)}$$

subject to the initial conditions

$$x(0) = v_1, y(0) = v_2, \tag{10(c)}$$

has a solution of the form $(x(t, \alpha, \beta), y(t, \alpha, \beta))$ such that:

$$\begin{aligned} &y(t, \alpha, \beta) = \\ &\textstyle\sum_{k=0}^{\infty}\left(a_{11}J^{\alpha}y_0 + a_{22}J^{\beta}y_0 - b\, J^{\alpha+\beta}y_0\right)^k = \\ &\textstyle\sum_{k=0}^{\infty}\sum_{k_1+k_2+k_3=k}\binom{k}{k_1, k_2, k_3} {a_{11}}^{k_1}{a_{22}}^{k_2}(-b)^{k_3}J^{k_1\alpha+k_2\beta+k_3(\alpha+\beta)}y_0\,, \end{aligned} \tag{11}$$

and,

$$x(t, \alpha, \beta) = \frac{1}{a_{21}}\left(D^{\beta}y(t, \alpha, \beta) - a_{22}y(t, \alpha, \beta) - f_2(t)\right), \tag{12}$$

$$y_0 = v_2 + \frac{c - f_2(0) - a_{22}v_2}{\Gamma(\beta+1)}t^{\beta} - \frac{a_{11}v_2}{\Gamma(\alpha+1)}t^{\alpha} + J^{\beta}f_2(t) + J^{\alpha+\beta}(a_{21}f_1(t) - a_{11}f_2(t)), \tag{13}$$

$$c = a_{21}v_1 + a_{22}v_2 + f_2(0) = y'(0), \tag{14}$$

and where $b = det(A)$ in which $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ such that $a_{21} \neq 0$.

Proof: Based on Lemma 3.1, system (10) is equivalent to (8), and so (12) is completely identified. Let us, now, employ the ADM to solve (8.b). By applying J^{α}on both sides of such equation, one obtains:

$$D^{\beta} y(t) - y'(0) = a_{11} J^{\alpha} D^{\beta} y(t) + a_{22} y(t) - a_{22} y(0) - bJ^{\alpha} y(t) + J^{\alpha}(D^{\alpha} f_2(t) + a_{21} f_1(t) - a_{11} f_2(t)). \tag{15}$$

That is;

$$D^{\beta} y(t) = c + a_{11} J^{\alpha} D^{\beta} y(t) + a_{22} y(t) - a_{22} v_2 - bJ^{\alpha} y(t) + f_2(t) - f_2(0) + J^{\alpha}(a_{21} f_1(t) - a_{11} f_2(t)). \tag{16}$$

Applying J^{β} on (16) yields;

$$y(t) = v_2 + J^{\beta}\big(c - a_{22} v_2 - f_2(0)\big) + a_{11} J^{\alpha} J^{\beta} D^{\beta} y + a_{22} J^{\beta} y - bJ^{\alpha+\beta} y + J^{\beta} f_2(t) + J^{\alpha+\beta}(a_{21} f_1(t) - a_{11} f_2(t)), \tag{17}$$

which can be written as

$$y(t) = v_2 + \frac{c - f_2(0) - a_{22} v_2}{\Gamma(\beta+1)} t^{\beta} - \frac{a_{11} v_2}{\Gamma(\alpha+1)} t^{\alpha} + a_{11} J^{\alpha} y + a_{22} J^{\beta} y - bJ^{\alpha+\beta} y + J^{\beta} f_2(t) + J^{\alpha+\beta}\big(a_{21} f_1(t) - a_{11} f_2(t)\big). \tag{18}$$

Considering the ADM, we assume that the general solution of (18) takes the following general form:

$$y(t, \alpha, \beta) = \textstyle\sum_{n=0}^{\infty} y_n(t, \alpha, \beta), \tag{19}$$

in which

$$y_0 = v_2 + \frac{c - f_2(0) - a_{22} v_2}{\Gamma(\beta+1)} t^{\beta} - \frac{a_{11} v_2}{\Gamma(\alpha+1)} t^{\alpha} + J^{\beta} f_2(t) + J^{\alpha+\beta}(a_{21} f_1(t) - a_{11} f_2(t)), \tag{20}$$

and,

$$y_k(t, \alpha, \beta) = a_{11} J^{\alpha} y_{k-1}(t) + a_{22} J^{\beta} y_{k-1}(t) - bJ^{\alpha+\beta} y_{k-1}(t), k \geq 1. \tag{21}$$

Now, we have the following claim that we wish to prove:

$$y_k(t, \alpha, \beta) = \big(a_{11} J^{\alpha} y_0 + a_{22} J^{\beta} y_0 - b\, J^{\alpha+\beta} y_0\big)^k = \textstyle\sum_{k_1+k_2+k_3=k} \binom{k}{k_1, k_2, k_3} {a_{11}}^{k_1} {a_{22}}^{k_2} (-b)^{k_3} J^{k_1\alpha + k_2\beta + k_3(\alpha+\beta)} y_0, \tag{22}$$

where $(J^{\alpha} y_0)^p \equiv J^{\alpha p} y_0$.

By using induction on k, one observes that (22) is obvious for the base of induction. That is; when $k = 1$, it's clear that the statement is true. Now, assume that the statement for k is true, and the relation (22) is correct. It is sufficient to show that (22) is also correct for $k + 1$; $k \geq 1$. It follows from (21) that:

$$y_{k+1}(t, \alpha, \beta) = a_{11} J^{\alpha} y_k(t) + a_{22} J^{\beta} y_k(t) - bJ^{\alpha+\beta} y_k(t), \tag{23}$$

$$y_{k+1}(t, \alpha, \beta) = \textstyle\sum_{k_1+k_2+k_3=k} \binom{k}{k_1, k_2, k_3} {a_{11}}^{k_1+1} {a_{22}}^{k_2} (-b)^{k_3} J^{(k_1+1)\alpha + k_2\beta + k_3(\alpha+\beta)} y_0 + \textstyle\sum_{k_1+k_2+k_3=k} \binom{k}{k_1, k_2, k_3} {a_{11}}^{k_1} {a_{22}}^{k_2+1} (-b)^{k_3} J^{k_1\alpha + (k_2+1)\beta + k_3(\alpha+\beta)} y_0 + \textstyle\sum_{k_1+k_2+k_3=k} \binom{k}{k_1, k_2, k_3} {a_{11}}^{k_1} {a_{22}}^{k_2} (-b)^{k_3+1} J^{k_1\alpha + k_2\beta + (k_3+1)(\alpha+\beta)} y_0, \tag{24}$$

or,

$$y_{k+1}(t,\alpha,\beta)=\sum_{k_1+k_2+k_3=k}\left[\binom{k}{k_1,k_2,k_3}a_{11}{}^{k_1}a_{22}{}^{k_2}(-b)^{k_3}J^{k_1\alpha+k_2\beta+k_3(\alpha+\beta)}y_0\left(a_{11}J^{\alpha}y_0+a_{22}J^{\beta}y_0-bJ^{\alpha+\beta}y_0\right)\right]=\left(a_{11}J^{\alpha}y_0\ +a_{22}J^{\beta}y_0-b\,J^{\alpha+\beta}y_0\right)^{k}\left(a_{11}J^{\alpha}y_0\ +a_{22}J^{\beta}y_0-b\,J^{\alpha+\beta}y_0\right)=\left(a_{11}J^{\alpha}y_0\ +a_{22}J^{\beta}y_0-b\,J^{\alpha+\beta}y_0\right)^{k+1}, \quad (25)$$

which implies that (22) is also true for $k+1$. Now, since (19) and (25) yields the general solution described by (11), one has to verify (14). For this purpose, consider (12) again, and let $x(0)=v_1$, then observe that all terms of (12) will be zero except three terms; $\frac{1}{a_{21}}c$,$\frac{-1}{a_{21}}f_2(0)$ and $\frac{-a_{22}}{a_{21}}v_2$, i.e.;

$$x(0)=v_1=\frac{1}{a_{21}}(c-f_2(0)-a_{22}v_2), \quad (26)$$

which yields (14). ■

Corollary 4.2: If $\alpha=\beta$ in system (10), then the solution $(x(t,\alpha),y(t,\alpha))$ of this system will be of the following form:

$$y(t,\alpha)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}a^{n-k}(-b)^{k}J^{n\alpha+k\alpha}y_0, \quad (27)$$

and,

$$x(t,\alpha)=\frac{1}{a_{21}}\left(D^{\alpha}y(t,\alpha)-a_{22}y(t,\alpha)-f_2(t)\right), \quad (28)$$

where,

$$y_0=v_2+\frac{c-f_2(0)-av_2}{\Gamma(\alpha+1)}t^{\alpha}+J^{\alpha}f_2(t)+J^{2\alpha}[a_{21}f_1(t)-a_{11}f_2(t)], \quad (29)$$

$$c=a_{21}v_1+a_{22}v_2+f_2(0)=y'(0), \quad (30)$$

and where $a=\mathrm{tr}(A)$ and $b=\det(A)$ in which $A=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}$ such that $a_{21}\neq 0$.
Proof: The proof is similar to Theorem 4.1. ■

Corollary 4.3: The following homogeneous LS-IFo:

$$D^{\alpha}x(t)=a_{11}x(t)+a_{12}y(t), \quad (31(a))$$

$$D^{\beta}y(t)=a_{21}x(t)+a_{22}y(t), \quad (31(b))$$

subject to the initial conditions

$$x(0)=v_1, y(0)=v_2, \quad (31(c))$$

has a solution of the form $(x(t,\alpha,\beta),y(t,\alpha,\beta))$ such that:

$$y(t,\alpha,\beta)=\sum_{n=0}^{\infty}\sum_{k_1+k_2+k_3=n}\binom{k}{k_1,k_2,k_3}a_{11}{}^{k_1}a_{22}{}^{k_2}(-b)^{k_3}J^{k_1\alpha+k_2\beta+k_3(\alpha+\beta)}y_0, \quad (32)$$

and,

$$x(t,\alpha,\beta)=\frac{1}{a_{21}}\left(D^{\beta}y(t,\alpha,\beta)-a_{22}y(t,\alpha,\beta)\right), \quad (33)$$

where

$$y_0 = v_2 + \frac{c-a_{22}v_2}{\Gamma(\beta+1)}t^{\beta} - \frac{a_{11}v_2}{\Gamma(\alpha+1)}t^{\alpha}, \quad (34)$$
$$c = a_{21}v_1 + a_{22}v_2 = y'(0), \quad (35)$$

and where $b = det(A)$ in which $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ such that $a_{21} \neq 0$.
Proof: The proof can be directly deduced from Theorem 4.1 by assuming $f_1(t) = f_2(t) = 0$.■

Corollary 4.4: The following homogeneous system:

$$D^{\alpha}x(t) = a_{11}x(t) + a_{12}y(t), \quad (36(a))$$

$$D^{\alpha}y(t) = a_{21}x(t) + a_{22}y(t), \quad (36(b))$$

subject to the initial conditions

$$x(0) = v_1, y(0) = v_2, \quad (36(c))$$

has a solution of the form $(\mathrm{x(t)}, \mathrm{y(t)})$ such that

$$y(t,\alpha) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}a^{n-k}(-b)^k J^{n\alpha+k\alpha}y_0, \quad (37)$$

and,

$$x(t,\alpha) = \frac{1}{a_{21}}\left(D^{\alpha}y(t,\alpha) - a_{22}y(t,\alpha)\right), \quad (38)$$

where,

$$y_0 = v_2 + \frac{c-av_2}{\Gamma(\alpha+1)}t^{\alpha}, \quad (39)$$

$$c = a_{21}v_1 + a_{22}v_2 = y'(0), \quad (40)$$

and where $a = tr(A)$ and $b = det(A)$ in which $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ such that $a_{21} \neq 0$.
Proof: The proof follows directly from Corollary 4.2 when $f_1(t) = f_2(t) = 0$. ■

5. THE GENERAL SOLUTION OF NLS-IFO USING ADM

This section introduces the general solution of the NLS-IFo using ADM. This can be made by extending the same technique used for handling the linear one. Such solution depends on the AP's, which form the basis for the ADM approach. In particular, the non-linear term, in this method, is usually identified using the AP's [19, 32], i.e., whenever the nonlinear term $\mathrm{N(v) = g(v)}$; where v is an unknown function that appears in the system, the AP ($\mathrm{A_n}$) yields an analytical function that is used to generate the general solution of the system [18]. These $\mathrm{A_n}$'s are generated to be analytical functions [18], and can be obtained by the following formula [19, 33]:

$$A_n = \frac{1}{n!}\frac{d^n}{d\boldsymbol{\lambda}^n}\left(N\left[\sum_{k=0}^{n}\boldsymbol{\lambda}^k v_k\right]\right)\Big|_{\boldsymbol{\lambda}=0}, \quad (41)$$

where $\boldsymbol{\lambda}$ is a parameter introduced for convenience. However, Theorem 5.1 employs the ADM approach to solve NLS-IFo's.

Theorem 5.1: The following NLS-IFo:

$$D^{\alpha}x(t) = a_{11}x(t) + a_{12}y(t) + g_1\big(x(t), y(t)\big), \quad (42(a))$$

$$D^{\beta}y(t) = a_{21}x(t) + a_{22}y(t) + g_2\big(x(t), y(t)\big), \quad (42(b))$$

subject to the initial conditions

$$x(0) = v_1, y(0) = v_2, \tag{42(c)}$$

has a solution of the form $(x(t,\alpha,\beta), y(t,\alpha,\beta))$ such that

$$y(t,\alpha,\beta) = \sum_{n=0}^{\infty} y_n(t), \tag{43}$$

and,

$$x(t,\alpha,\beta) = \frac{1}{a_{21}}\Big(D^\beta y(t,\alpha,\beta) - a_{22}y(t,\alpha,\beta) - g_2\big(x(t),y(t)\big)\Big), \tag{44}$$

where,

$$y_0 = v_2 + \frac{c-g_2\big(x(0),y(0)\big)-a_{22}v_2}{\Gamma(\beta+1)}t^\beta - \frac{a_{11}v_2}{\Gamma(\alpha+1)}t^\alpha, \tag{45(a)}$$

$$y_n(t) = a_{11}J^\alpha y_{n-1}(t) + a_{22}J^\beta y_{n-1}(t) - bJ^{\alpha+\beta}y_{n-1}(t) + J^\beta B_{n-1} + J^{\alpha+\beta}(a_{21}A_{n-1} - a_{11}B_{n-1}), n \geq 1, \tag{45(b)}$$

and,

$$c = a_{21}v_1 + a_{22}v_2 + g_2\big(x(0),y(0)\big) = y'(0), \tag{46}$$

and where $0 < \alpha,\beta \leq 1$, $g_1(x(t),y(t)) = \sum_{n=0}^{\infty} A_n$ and $g_2(x(t),y(t)) = \sum_{n=0}^{\infty} B_n$ such that A_n and B_n are the AP's corresponding to $g_1(\cdot)$ and $g_2(\cdot)$ respectively, and $b = \det(A)$ in which $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ such that $a_{21} \neq 0$.

Proof: From Lemma 3.1, system (42) is equivalent to (8), and so (44) is completely identified. Now, applying J^α and J^β on both sides of system (8.b) respectively, yields:

$$y(t) = v_2 + \frac{c-g_2\big(x(0),y(0)\big)-a_{22}v_2}{\Gamma(\beta+1)}t^\beta - \frac{a_{11}v_2}{\Gamma(\alpha+1)}t^\alpha + a_{11}J^\alpha y(t) + a_{22}J^\beta y(t) - bJ^{\alpha+\beta}y(t) + J^\beta g_2\big(x(t),y(t)\big) + J^{\alpha+\beta}\Big(a_{21}g_1\big(x(t),y(t)\big) - a_{11}g_2\big(x(t),y(t)\big)\Big). \tag{47}$$

Considering the ADM, the general solution of (47) is assumed as in (19) in which:

$$y_0 = v_2 + \frac{c-g_2\big(x(0),y(0)\big)-a_{22}v_2}{\Gamma(\beta+1)}t^\beta - \frac{a_{11}v_2}{\Gamma(\alpha+1)}t^\alpha, \tag{48(a)}$$

and,

$$y_n(t) = a_{11}J^\alpha y_{n-1}(t) + a_{22}J^\beta y_{n-1}(t) - bJ^{\alpha+\beta}y_{n-1}(t) + J^\beta B_{n-1} + J^{\alpha+\beta}(a_{21}A_{n-1} - a_{11}B_{n-1}), n \geq 1, \tag{48(b)}$$

which yields (43). Now, consider (44) and let the initial condition be $x(0) = v_1$ then it follows from Theorem 4.1 that (46) is verified. ■

Corollary 5.2: If $\alpha = \beta$ in system (42), then the solution $(x(t,\alpha), y(t,\alpha))$ will be of the following form:

$$y(t,\alpha) = \sum_{n=0}^{\infty} y_n(t), \tag{49}$$

and,

$$x(t,\alpha) = \frac{1}{a_{21}}(D^\alpha y(t,\alpha) - a_{22}y(t,\alpha) - g_2(x(t),y(t))), \tag{50}$$

where,

$$y_0 = v_2 + \frac{c-av_2-g_2\big(x(0),y(0)\big)}{\Gamma(\alpha+1)}t^\alpha, \tag{51(a)}$$

$$y_n(t) = aJ^{\alpha}y_{n-1}(t) - bJ^{2\alpha}y_{n-1}(t) + J^{\alpha}B_{n-1} + J^{2\alpha}[a_{21}A_{n-1} - a_{11}B_{n-1}], n \geq 1, \tag{51(b)}$$

and,

$$c = a_{21}v_1 + a_{22}v_2 + g_2\big(x(0), y(0)\big) = y'(0), \tag{52}$$

and where $0 < \alpha \leq 1$, $g_1(x(t), y(t)) = \sum_{n=0}^{\infty} A_n$and $g_2(x(t), y(t)) = \sum_{n=0}^{\infty} B_n$ such that A_n and B_n are the AP's corresponding to g_1 and g_2 respectively, $a = tr(A)$ and $b = det(A)$ in which $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ such that $a_{21} \neq 0$.

Proof: The proof is similar to that of Theorem 5.1. ■

Remark 5.3: Observe that in the case when $a_{21} \neq 0$, one can reverse the order of solution by solving $y(t, \alpha, \beta)$ in terms of $x(t, \alpha, \beta)$, and proceed as discussed in Theorem 4.1.

6. NUMERICAL EXAMPLES

To highlight the main results of this work and to show the effectiveness of solving both linear and nonlinear fractional-order systems of incommensurate orders using the proposed method, three numerical examples in Electrical and Biomedical Engineering are investigated.

Example 6.1: The human malady of ventricular arrhythmia or irregular heartbeat is treated clinically using the drug lidocaine. The model for the dynamics of the drug therapy that is valid for a special body weight can be described by the following homogeneous LS-IFo [34]:

$$D^{\alpha}x(t) = -0.09x(t) + 0.038y(t), \tag{53(a)}$$

$$D^{\beta}y(t) = 0.066x(t) - 0.038y(t), \tag{53(b)}$$

subject to the following physically significant initial data:

The drug in the blood stream= $x(0) = v_1 = 0\,, \mathit{The\ injection\ dosage} = y(0) = v_2 = y^*$. (53(c))

where $x(t)$ is the amount of lidocaine in the blood stream, and $y(t)$ is the amount of lidocaine in body tissue. The exact solutions of (53) for $\alpha = \beta = 1$, and $y^* = 2$ is:

$$x(t) = -0.6735(e^{-0.1204\,t} - e^{-0.0076\,t}), \tag{54(a)}$$

$$y(t) = 0.5393e^{-0.1204\,t} + 1.4608\,e^{-0.0076\,t}. \tag{54(b)}$$

In order to obtain the solutions of (53) using the proposed technique; we may rewrite this system in the following form:

$$\begin{bmatrix} D^{\alpha}x(t) \\ D^{\beta}y(t) \end{bmatrix} = A \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \tag{55}$$

where $A = \begin{bmatrix} -0.09 & 0.038 \\ 0.066 & -0.038 \end{bmatrix}$, and consequently $a = tr(A) = -0.128$ and $b = det(A) = 0.000912$. Obviously, using (35) yields $c = y'(0) = -0.076$, and by using (34), one obtains y_0 in the following form:

$$y_0 = 2 + \frac{0.18}{\Gamma(\alpha+1)}t^{\alpha}, \tag{56}$$

and $y(t, \alpha, \beta)$ can be obtained using (32) as follows:

$$y(t, \alpha, \beta) = \sum_{n=0}^{\infty} \sum_{k_1+k_2+k_3=n} \binom{k}{k_1, k_2, k_3} (-0.09)^{k_1} (-0.038)^{k_2} (-0.000912)^{k_3} J^{k_1\alpha + k_2\beta + k_3(\alpha+\beta)} \left[2 + \frac{0.18t^{\alpha}}{\Gamma(\alpha+1)}\right], \tag{57}$$

i.e.;

$$y(t,\alpha,\beta) = 2 - \frac{19t^{\beta}}{250\Gamma(1+\beta)} + \frac{627t^{\alpha+\beta}}{125\times10^{3}\Gamma(1+\alpha+\beta)} - \frac{5643t^{2\alpha+\beta}}{125\times10^{4}\Gamma(1+2\alpha+\beta)} + \frac{50787t^{3\alpha+\beta}}{125\times10^{7}\Gamma(1+3\alpha+\beta)} - \frac{457083t^{4\alpha+\beta}}{125\times10^{9}\Gamma(1+4\alpha+\beta)} + \frac{4113747t^{5\alpha+\beta}}{125\times10^{11}\Gamma(1+5\alpha+\beta)} - \frac{37023723t^{6\alpha+\beta}}{125\times10^{13}\Gamma(1+6\alpha+\beta)} + \dots, \tag{58}$$

and from (33), $x(t,\alpha,\beta)$ can be expressed as:

$$x(t,\alpha,\beta) = \frac{1}{0.066}\left(D^{\beta}y(t,\alpha,\beta) + 0.038y(t,\alpha,\beta)\right), \tag{59}$$

i.e.;

$$x(t,\alpha,\beta) = \frac{19t^{\alpha}}{250\Gamma(\alpha+1)} - \frac{171t^{2\alpha}}{25\times10^{3}\,\Gamma(2\alpha+1)} - \frac{361t^{\alpha+\beta}}{125\times10^{3}\,\Gamma(\alpha+\beta+1)} + \frac{29241t^{2\alpha+\beta}}{6875\times10^{4}\,\Gamma(2\alpha+\beta+1)} + \frac{1539t^{3\alpha}}{25\times10^{5}\,\Gamma(3\alpha+1)} - \frac{13851t^{4\alpha}}{25\times10^{6}\,\Gamma(4\alpha+1)} - \frac{29241t^{3\alpha+\beta}}{55\times10^{7}\,\Gamma(3\alpha+\beta+1)} + \dots. \tag{60}$$

In order to demonstrate the efficiency of the above results, Table 1 shows the error between the exact solution and our analytical solution using the proposed method over the time t, where $t \in [0,15]$, $\alpha = \beta = 1$, and $n = 20$. The behavior of the solution for system (53) for different values of α and β using our proposed method is shown in Figures 1 and 2. Actually, this solution can be used to estimate the maximum possible safe dosage y^{*} and the duration of time that the drug lidocaine is effective [34].

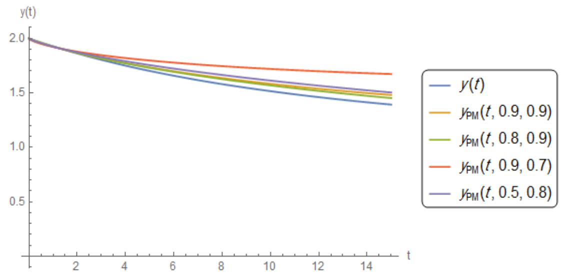

Figure 1. The behavior of the solution $y_{PM}(t,\alpha,\beta)$ using our proposed method for system (53) for different values of α and β

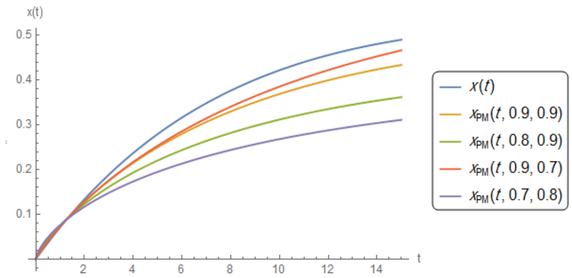

Figure 2. The behavior of the solution $x_{PM}(t,\alpha,\beta)$ using our proposed method for system (53) for different values of α and β

Table 1. The error between the exact and our analytical solutions for system (53) over $t \in [0,15]$, where $\alpha = \beta = 1$, and $n = 20$

t	$y_{PM}(t,\alpha,\beta)$	$y_{Exact}(t)$	$\lvert y_{PM} - y_{Exact}\rvert$	$x_{PM}(t)$	$x_{Exact}(t)$	$\lvert x_{PM} - x_{Exact}\rvert$
1	1.9277995480	1.9277995480	2.22040×10^{-16}	0.0713262164	0.0713262164	1.24900×10^{-16}
5	1.7017974400	1.7017974400	2.22040×10^{-16}	0.2796119909	0.2796119909	5.55110×10^{-17}
10	1.5159529100	1.5159529100	2.22040×10^{-16}	0.4223494308	0.4223494308	1.66530×10^{-16}
15	1.3924776760	1.3924776760	3.81920×10^{-14}	0.4905147616	0.4905147616	8.22950×10^{-13}

Example 6.2: Consider the following non-homogeneous LS-IFo:

$$D^{\alpha}x(t) = x(t) - 2y(t) + 2t, \tag{61(a)}$$

$$D^{\beta}y(t) = 2x(t) - 0.9\,y(t) - 3, \tag{61(b)}$$

subject to the initial conditions,

$$x(0) = v_1 = 1, y(0) = v_2 = 0. \tag{61(c)}$$

Here, the exact solutions of (63) for $\alpha = \beta = 1$ are of the form:

$$x(t) = \frac{-90683e^{t/20}Cos[\frac{\sqrt{1239}t}{20}]+147382Cos[\frac{\sqrt{1239}t}{20}]^2+32922tCos[\frac{\sqrt{1239}t}{20}]^2+457\sqrt{1239}e^{t/20}Sin[\frac{\sqrt{1239}t}{20}]+147382Sin[\frac{\sqrt{1239}t}{20}]^2+32922tSin[\frac{\sqrt{1239}t}{20}]^2}{56699}. \tag{62(a)}$$

$$y(t) = -\frac{10}{56699}\Big(5723e^{t/20}Cos\left[\frac{\sqrt{1239}t}{20}\right] - 5723Cos\left[\frac{\sqrt{1239}t}{20}\right]^2 - 7316tCos\left[\frac{\sqrt{1239}t}{20}\right]^2 + 205\sqrt{1239}e^{t/20}Sin\left[\frac{\sqrt{1239}t}{20}\right] - 5723Sin\left[\frac{\sqrt{1239}t}{20}\right]^2 - 7316tSin\left[\frac{\sqrt{1239}t}{20}\right]^2\Big), \tag{62(b)}$$

One might rewrite system (61) in the following matrix form:

$$\begin{bmatrix} D^{\alpha}x(t) \\ D^{\beta}y(t) \end{bmatrix} = A\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}, \tag{63}$$

where $A = \begin{bmatrix} 1 & -2 \\ 2 & 0.9 \end{bmatrix}$ and $\begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} = \begin{bmatrix} 2t \\ -3 \end{bmatrix}$. Then $a = tr(A) = 1.9$ and $b = det(\mathrm{A}) = 4.9$. Using (14) yields $c = -2.1$, and then by using (13), one might obtain y_0 in the following form:

$$y_0 = 1 - \frac{1}{\Gamma(\alpha+1)}t^{\alpha} - J^{\beta}(3) + J^{\alpha+\beta}(4t+3). \tag{66}$$

Thus, $y(t,\alpha,\beta)$ can be obtained using (11) to be as follows:

$$y(t,\alpha,\beta) = \sum_{k=0}^{\infty}\sum_{k_1+k_2+k_3=k}\binom{k}{k_1,k_2,k_3}(0.9)^{k_2}(-4.9)^{k_3}J^{k_1\alpha+k_2\beta+k_3(\alpha+\beta)}\left[1 - \frac{1}{\Gamma(\alpha+1)}t^{\alpha} - J^{\beta}(3) + J^{\alpha+\beta}(4t+3)\right] \tag{65}$$

$$y(t,\alpha,\beta) = -\frac{t^{\beta}}{\Gamma(1+\beta)} + \frac{2t^{\alpha+\beta}}{\Gamma(1+\alpha+\beta)} + \frac{4t^{1+\alpha+\beta}}{\Gamma(2+\alpha+\beta)} + \frac{2t^{2\alpha+\beta}}{\Gamma(1+2\alpha+\beta)} + \frac{4t^{1+2\alpha+\beta}}{\Gamma(2+2\alpha+\beta)} + \frac{2t^{3\alpha+\beta}}{\Gamma(1+3\alpha+\beta)} + \frac{4t^{1+3\alpha+\beta}}{\Gamma(2+3\alpha+\beta)} + \frac{2t^{4\alpha+\beta}}{\Gamma(1+4\alpha+\beta)} + \frac{4t^{1+4\alpha+\beta}}{\Gamma(2+4\alpha+\beta)} + \frac{2t^{5\alpha+\beta}}{\Gamma(1+5\alpha+\beta)} + \frac{4t^{1+5\alpha+\beta}}{\Gamma(2+5\alpha+\beta)} + \dots \tag{66}$$

Using (12) yields $x(t,\alpha,\beta)$ which will be as:

$$x(t,\alpha,\beta) = \frac{1}{2}\left(D^{\beta}y(t,\alpha,\beta) - 0.9y(t,\alpha,\beta) + 3\right), \tag{67}$$

i.e.;

$$x(t,\alpha,\beta) = 1 + \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + \frac{t^{5\alpha}}{\Gamma(5\alpha+1)} + \frac{t^{6\alpha}}{\Gamma(6\alpha+1)} + \frac{t^{7\alpha}}{\Gamma(7\alpha+1)} + \frac{t^{8\alpha}}{\Gamma(8\alpha+1)} + \frac{2t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{2t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{2t^{3\alpha+1}}{\Gamma(3\alpha+2)} + \frac{2t^{4\alpha+1}}{\Gamma(4\alpha+2)} + \frac{2t^{5\alpha+1}}{\Gamma(5\alpha+2)} + \cdots. \tag{68}$$

For $\alpha = \beta = 1$, (66) and (68) will be, respectively, as:

$$y(t,1,1) = -t + \frac{29t^2}{20} + \frac{739t^3}{600} - \frac{8251t^4}{24\times10^3} - \frac{237341t^5}{12\times10^5} + \frac{2320469t^6}{72\times10^6} + \frac{75896179t^7}{504\times10^7} - \frac{643449211t^8}{4032\times10^8} + \cdots, \tag{69}$$

and,

$$x(t,1,1) = 1 + t + \frac{5t^2}{2} - \frac{2t^3}{15} - \frac{779t^4}{1200} + \frac{461t^5}{6\times10^4} + \frac{241951t^6}{36\times10^5} + \frac{99041t^7}{252\times10^6} - \frac{74905769t^8}{2016\times10^7} + \cdots. \tag{70}$$

Figure 3, however, shows a graphical comparison between the exact solution and our analytical solution using the proposed method, when $\alpha = \beta = 1$ and $n = 30$. Furthermore, the behavior of the solutions for system (61) for different values of α and β using our proposed method is shown in Figures 4 and 5.

Example 6.3: Consider the following NLS-IFo:

$$D^{\alpha}x(t) = -4y(t) - \frac{5}{4}y^3(t), \tag{71(a)}$$

$$D^{\beta}y(t) = -x(t) - y(t) + y^2(t) - 5t^3 + 2t^{2.5}, \tag{71(b)}$$

subject to the initial conditions

$$x(0) = v_1 = 2, y(0) = v_2 = 1. \tag{71(c)}$$

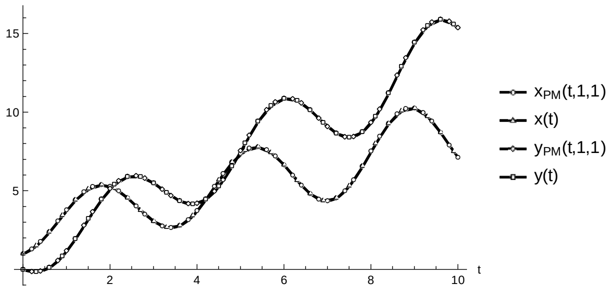

Figure 3. A comparison between the exact solutions $(x(t), y(t))$ for system (61) and our analytical solutions $(x_{PM}(t,1,1), y_{PM}(t,1,1))$, when $\alpha = \beta = 1$ and $n = 30$

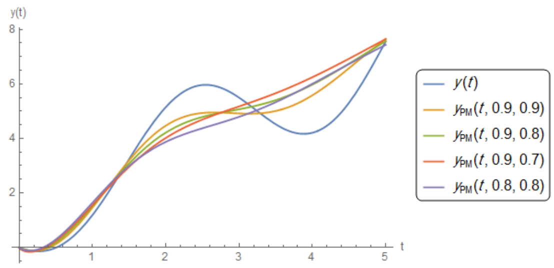

Figure 4. The behavior of the solution $y_{PM}(t, \alpha, \beta)$ using our proposed method for system (61) for different values of α and β

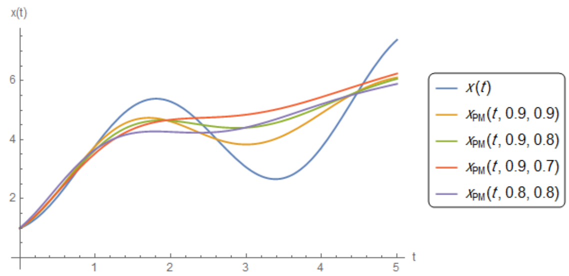

Figure 5. The behavior of the solution $x_{PM}(t, \alpha, \beta)$ using our proposed method for system (61) for different values of α and β

System (71) can be rewritten in the following matrix form:

$$D^{\alpha}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = A\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} g_1(x(t), y(t)) \\ g_2(x(t), y(t)) \end{bmatrix}, \tag{72}$$

where $A = \begin{bmatrix} 0 & -4 \\ -1 & -1 \end{bmatrix}$ and $\begin{bmatrix} g_1(x(t), y(t)) \\ g_2(x(t), y(t)) \end{bmatrix} = \begin{bmatrix} -\frac{5}{4}y^3(t) \\ y^2(t) - 5t^3 + 2t^{2.5} \end{bmatrix}$. Then, $a = tr(A) = -1$ and $b = det(A) = -4$. Using (46) yields $c = -2$, and then y_0 can be obtained from (45.a) to be as follows:

$$\mathrm{y}_0 = 1 - \frac{2t^{\beta}}{\Gamma(\beta+1)}. \tag{73}$$

Now, we can deal with the two non-linear terms; $g_1(x(t), y(t)) = -\frac{5}{4}y^3(t) = \sum_{n=0}^{\infty} A_n$and $g_2(x(t), y(t)) = y^2(t) - 5t^3 + 2t^{2.5} = \sum_{n=0}^{\infty} B_n$, by utilizing the AP's. This can be done by considering (41) again. However, some of these AP's for such nonlinear terms are given below:

$$A_0 = -\frac{15t^{2\beta}}{\Gamma(\beta+1)^2} + \frac{10t^{3\beta}}{\Gamma(\beta+1)^3} + \frac{15t^{\beta}}{2\Gamma(\beta+1)} - \frac{5}{4}, \tag{74}$$

$$A_1 = -\frac{315t^{\alpha+\beta}}{16\Gamma(\alpha+\beta+1)} + \frac{315t^{\alpha+2\beta}}{4\Gamma(\beta+1)\Gamma(\alpha+\beta+1)} + \frac{465t^{\alpha+2\beta}}{8\Gamma(\alpha+2\beta+1)} - \frac{315t^{\alpha+3\beta}}{4\Gamma(\beta+1)^2\Gamma(\alpha+\beta+1)} - \frac{465t^{\alpha+3\beta}}{2\Gamma(\beta+1)\Gamma(\alpha+2\beta+1)} - \frac{225\Gamma(2\beta+1)t^{\alpha+3\beta}}{4\Gamma(\beta+1)^2\Gamma(\alpha+3\beta+1)} + \frac{465t^{\alpha+4\beta}}{2\Gamma(\beta+1)^2\Gamma(\alpha+2\beta+1)} + \cdots, \tag{75}$$

$$A_2 = \frac{15t^{3\beta}}{2\,\Gamma(3\beta+1)} - \frac{30t^{4\beta}}{\Gamma(\beta+1)\,\Gamma(3\beta+1)} - \frac{15\,\Gamma(2\beta+1)t^{4\beta}}{\Gamma(\beta+1)^2\,\Gamma(4\beta+1)} - \frac{30\,\Gamma(3\beta+1)t^{4\beta}}{\Gamma(\beta+1)\,\Gamma(2\beta+1)\,\Gamma(4\beta+1)} - \frac{15t^{4\beta}}{\Gamma(2\beta+1)^2} + \frac{90t^{5\beta}}{\Gamma(\beta+1)^2\,\Gamma(3\beta+1)} + \frac{60\,\Gamma(2\beta+1)t^{5\beta}}{\Gamma(\beta+1)^3\,\Gamma(4\beta+1)} + \frac{120\,\Gamma(3\beta+1)t^{5\beta}}{\Gamma(\beta+1)^2\,\Gamma(2\beta+1)\,\Gamma(4\beta+1)} + \cdots, \tag{76}$$

$$A_3 = \frac{15t^{4\beta}}{2\Gamma(4\beta+1)} - \frac{30t^{5\beta}}{\Gamma(2\beta+1)\,\Gamma(3\beta+1)} - \frac{30t^{5\beta}}{\Gamma(\beta+1)\,\Gamma(4\beta+1)} - \frac{15\,\Gamma(2\beta+1)t^{5\beta}}{\Gamma(\beta+1)^2\,\Gamma(5\beta+1)} - \frac{30\,\Gamma(3\beta+1)t^{5\beta}}{\Gamma(\beta+1)\,\Gamma(2\beta+1)\,\Gamma(5\beta+1)} - \frac{30\,\Gamma(4\beta+1)t^{5\beta}}{\Gamma(\beta+1)\,\Gamma(3\beta+1)\,\Gamma(5\beta+1)} - \cdots, \tag{77}$$

and,

$$B_0 = \frac{4t^{2\beta}}{\Gamma(\beta+1)^2} + 2t^{5/2} - \frac{4t^{\beta}}{\Gamma(\beta+1)} - 5t^3 + 1, \tag{78}$$

$$B_1 = \frac{21t^{\alpha+\beta}}{2\Gamma(\alpha+\beta+1)} - \frac{21t^{\alpha+2\beta}}{\Gamma(\beta+1)\Gamma(\alpha+\beta+1)} - \frac{31t^{\alpha+2\beta}}{\Gamma(\alpha+2\beta+1)} + \frac{62t^{\alpha+3\beta}}{\Gamma(\beta+1)\Gamma(\alpha+2\beta+1)} + \frac{30\Gamma(2\beta+1)t^{\alpha+3\beta}}{\Gamma(\beta+1)^2\Gamma(\alpha+3\beta+1)} - \frac{60\Gamma(2\beta+1)t^{\alpha+4\beta}}{\Gamma(\beta+1)^3\Gamma(\alpha+3\beta+1)} - \frac{20\Gamma(3\beta+1)t^{\alpha+4\beta}}{\Gamma(\beta+1)^3\Gamma(\alpha+4\beta+1)} + \frac{40\Gamma(3\beta+1)t^{\alpha+5\beta}}{\Gamma(\beta+1)^4\Gamma(\alpha+4\beta+1)} - \frac{4t^{2\beta}}{\Gamma(2\beta+1)} + \frac{8t^{3\beta}}{\Gamma(\beta+1)\Gamma(2\beta+1)} + \frac{8\Gamma(2\beta+1)t^{3\beta}}{\Gamma(\beta+1)^2\Gamma(3\beta+1)} - \frac{16\Gamma(2\beta+1)t^{4\beta}}{\Gamma(\beta+1)^3\Gamma(3\beta+1)} + \frac{15\sqrt{\pi}t^{\beta+\frac{5}{2}}}{2\Gamma(\beta+\frac{7}{2})} - \frac{60t^{\beta+3}}{\Gamma(\beta+4)} - \frac{15\sqrt{\pi}t^{2\beta+\frac{5}{2}}}{\Gamma(\beta+1)\Gamma(\beta+\frac{7}{2})} + \frac{120t^{2\beta+3}}{\Gamma(\beta+1)\Gamma(\beta+4)}, \tag{79}$$

$$B_2 = -\frac{4t^{3\beta}}{\Gamma(3\beta+1)} + \frac{8t^{4\beta}}{\Gamma(\beta+1)\Gamma(3\beta+1)} + \frac{8\Gamma(2\beta+1)t^{4\beta}}{\Gamma(\beta+1)^2\Gamma(4\beta+1)} + \frac{16\Gamma(3\beta+1)t^{4\beta}}{\Gamma(\beta+1)\Gamma(2\beta+1)\Gamma(4\beta+1)} + \frac{4t^{4\beta}}{\Gamma(2\beta+1)^2} - \frac{16t^{5\beta}}{\Gamma(\beta+1)^2\Gamma(3\beta+1)} - \frac{16\Gamma(2\beta+1)t^{5\beta}}{\Gamma(\beta+1)^3\Gamma(4\beta+1)} - \frac{32\Gamma(3\beta+1)t^{5\beta}}{\Gamma(\beta+1)^2\Gamma(2\beta+1)\Gamma(4\beta+1)} + \cdots, \tag{80}$$

$$B_3 = -\frac{4t^{4\beta}}{\Gamma(4\beta+1)} + \frac{8t^{5\beta}}{\Gamma(2\beta+1)\Gamma(3\beta+1)} + \frac{8t^{5\beta}}{\Gamma(\beta+1)\Gamma(4\beta+1)} + \frac{8\Gamma(2\beta+1)t^{5\beta}}{\Gamma(\beta+1)^2\Gamma(5\beta+1)} + \frac{16\Gamma(3\beta+1)t^{5\beta}}{\Gamma(\beta+1)\Gamma(2\beta+1)\Gamma(5\beta+1)} + \frac{16\Gamma(4\beta+1)t^{5\beta}}{\Gamma(\beta+1)\Gamma(3\beta+1)\Gamma(5\beta+1)} + \frac{8\Gamma(4\beta+1)t^{5\beta}}{\Gamma(2\beta+1)^2\Gamma(5\beta+1)} - \frac{32\Gamma(3\beta+1)t^{6\beta}}{\Gamma(\beta+1)\Gamma(2\beta+1)^2\Gamma(4\beta+1)} + \cdots. \tag{81}$$

For $n \geq 1$, (45.b) will be:

$$y_n(t) = -J^{\beta}y_{n-1}(t) + 4J^{\alpha+\beta}y_{n-1}(t) + J^{\beta}B_{n-1} - J^{\alpha+\beta}(A_{n-1}), n \geq 1 \tag{82}$$

Substituting (73) and (82) in (43), we obtain:

$$y(t,\alpha,\beta) = 1 - \frac{2t^{\beta}}{\Gamma(\beta+1)} + \sum_{n=1}^{\infty}\left[-J^{\beta}y_{n-1}(t) + 4J^{\alpha+\beta}y_{n-1}(t) + J^{\beta}B_{n-1} - J^{\alpha+\beta}(A_{n-1})\right]. \tag{83}$$

Based on y_0 given in (73); one can reach to the following general solution:

$$y(t,\alpha,\beta) = 1 - \frac{2t^{\beta}}{\Gamma[1+\beta]} + \frac{15\sqrt{\pi}t^{\frac{5}{2}+\beta}}{4\Gamma\left[\frac{7}{2}+\beta\right]} - \frac{30t^{3+\beta}}{\Gamma[4+\beta]} + \frac{21t^{\alpha+\beta}}{4\Gamma[1+\alpha+\beta]} - \frac{2t^{2\beta}}{\Gamma[1+2\beta]} + \frac{15\sqrt{\pi}t^{\frac{5}{2}+2\beta}}{4\Gamma\left[\frac{7}{2}+2\beta\right]} - \frac{30t^{3+2\beta}}{\Gamma[4+2\beta]} - \frac{41t^{\alpha+2\beta}}{4\Gamma[1+\alpha+2\beta]} + \frac{465\sqrt{\pi}t^{\frac{5}{2}+\alpha+2\beta}}{16\Gamma\left[\frac{7}{2}+\alpha+2\beta\right]} - \frac{465t^{3+\alpha+2\beta}}{2\Gamma[4+\alpha+2\beta]} + \frac{651t^{2\alpha+2\beta}}{16\Gamma[1+2\alpha+2\beta]} + \cdots. \tag{84}$$

Hence; $x(t,\alpha,\beta)$ can be obtained using (44) to be as:

$$x(t,\alpha,\beta) = -\left(D^{\beta}y(t,\alpha,\beta) + y(t,\alpha,\beta) - y^2(t,\alpha,\beta) + 5t^3 - 2t^{2.5}\right), \tag{85}$$

or;

$$x(t,\alpha,\beta) = 2 - \frac{21t^{\alpha}}{4\Gamma[1+\alpha]} - \frac{4t^{2\beta}}{\Gamma[1+\beta]^2} + \frac{4t^{\beta}}{\Gamma[1+\beta]} - \frac{225\pi t^{5+2\beta}}{16\Gamma[\frac{7}{2}+\beta]^2} - \frac{15\sqrt{\pi}t^{\frac{5}{2}+\beta}}{2\Gamma[\frac{7}{2}+\beta]} + \frac{15\sqrt{\pi}t^{\frac{5}{2}+2\beta}}{\Gamma[1+\beta]\Gamma[\frac{7}{2}+\beta]} - \frac{900t^{6+2\beta}}{\Gamma[4+\beta]^2} + \frac{60t^{3+\beta}}{\Gamma[4+\beta]} - \frac{120t^{3+2\beta}}{\Gamma[1+\beta]\Gamma[4+\beta]} + \frac{225\sqrt{\pi}t^{\frac{11}{2}+2\beta}}{\Gamma[\frac{7}{2}+\beta]\Gamma[4+\beta]} + \dots \tag{86}$$

The two solutions, (84) and (86), can be written, for $\alpha = \beta = 1$, as follow:

$$y(t,1,1) = 1 - 2t + \frac{13t^2}{8} - \frac{17t^3}{24} + \frac{4t^{7/2}}{7} - \frac{289t^4}{384} + \frac{8t^{9/2}}{63} - \frac{1007t^5}{960} - \frac{148t^{11/2}}{693} + \frac{11479t^6}{9216} - \frac{26t^{13/2}}{693} + \cdots, \tag{87}$$

$$x(t,1,1) = 2 - \frac{21t}{4} + \frac{31t^2}{4} - \frac{883t^3}{96} + \frac{3827t^4}{384} - \frac{62t^{9/2}}{63} - \frac{60011t^5}{7680} + \frac{956t^{11/2}}{693} + \frac{137867t^6}{23040} - \dots. \tag{88}$$

For more insight; a comparison is graphically made in Figure 6 between $(x_{MS}(t,1,1), y_{MS}(t,1,1))$ obtained by MS and our analytical solutions $(x_{PM}(t,1,1), y_{PM}(t,1,1))$, followed by Figures 7-10 that illustrate the behavior of the solutions of system (71) for different values of α and β. Finally; Table 2 shows the error between such two solutions for $\alpha = \beta = 1$and $n = 20$.

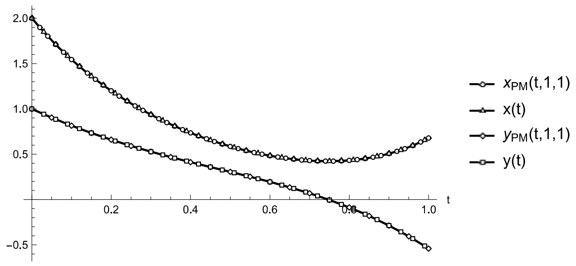

Figure 6. A comparison between the solutions $(x_{MS}(t,1,1), y_{MS}(t,1,1))$ for system (71) and our analytical solutions $(x_{PM}(t,1,1), y_{PM}(t,1,1))$, when $\alpha = \beta = 1$ and $n = 20$

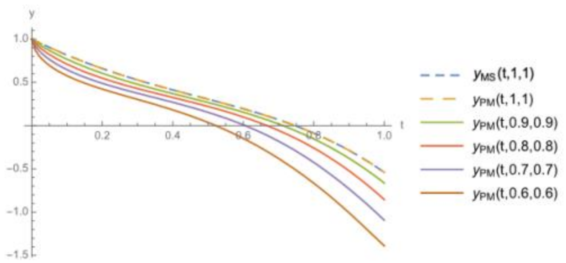

Figure 7. The behavior of the solution $y_{PM}(t,\alpha,\beta)$ using our proposed method for system (71) for different values of α and β

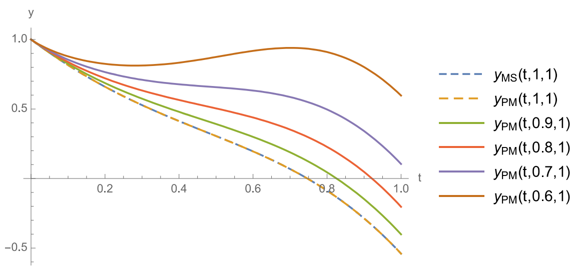

Figure 8. The behavior of the solution $y_{PM}(t,\alpha,\beta)$ using our proposed method for system (71) for other different values of α and β

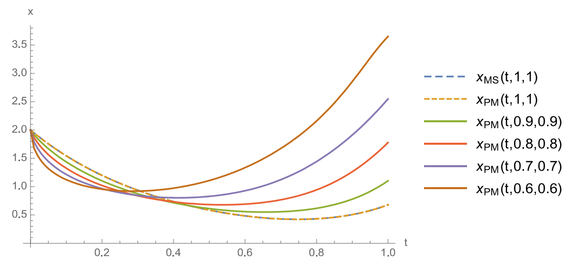

Figure 9. The behavior of the solution $x_{PM}(t,\alpha,\beta)$ using our proposed method for system (71) for different values of α and β.

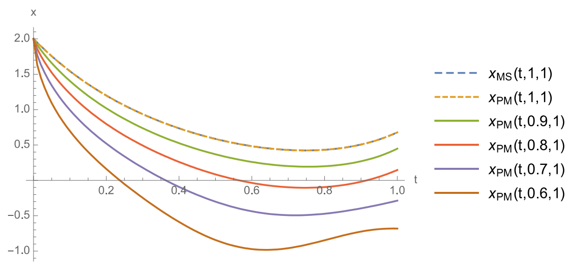

Figure 10. The behavior of the solution $x_{PM}(t,\alpha,\beta)$ using our proposed method for system (71) for other different values of α and β

Table 2. The error between the exact solution and our analytical solution for system (71), where $\alpha = \beta = 1$ and $n = 20$

t	$y_{PM}(t,\alpha,\beta)$	$y_{MS}(t)$	$\lvert y_{PM} - y_{MS}\rvert$	$x_{PM}(t)$	$x_{MS}(t)$	$\lvert x_{PM} - x_{MS}\rvert$
0	1.0000000000	1.0000000000	0.000000	2.0000000000	2.0000000000	0.000000
0.2	0.6599728109	0.6599728165	5.65×10^{-9}	1.1996487780	1.1996488610	8.31×10^{-8}
0.4	0.4129012636	0.4129012782	1.46×10^{-8}	0.7355566706	0.7355566359	3.47×10^{-8}
0.6	0.1949926708	0.1949926852	1.44×10^{-8}	0.4835460881	0.4835459540	1.34×10^{-7}
0.8	-0.0873501663	-0.0873502275	6.12×10^{-8}	0.4322677755	0.4322677252	5.03×10^{-8}
1.0	-0.5401748463	-0.5401746710	1.75×10^{-7}	0.6789543701	0.6789497251	4.65×10^{-6}

7. CONCLUSION

A new analytical method is constructed for solving linear and non-linear systems of incommensurate fractional-order. This proposed method is different from all other analytical methods; it can be applied by backward converting a coupled system into two equations; the first one is an explanation of one variable in terms of another, while the second equation is just a fractional-order differential equation (FoDE) of order $\alpha + \beta$. The Adomian Decomposition Method (ADM) is implemented successfully to construct the solution of an $(\alpha + \beta)$-FoDE. Finally, it was shown, by illustrating several graphical comparisons, that the proposed technique is simple and yet powerful analytic method for handling the systems under consideration.

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