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Impulsive Control and Synchronization of Rössler Chaotic System

Yan Yan, Wang Kaihua, Gui Zhanji*
School of Mathematics and Statistics, Hainan Normal University, Haikou, Hainan, China.
*Corresponding author, e-mail: zhanjigui@sohu.com

Abstract

In this paper, synchronization between two Rössler Chaotic Systems with impulsively controlling is established by using the criteria on uniform equi-boundedness and equi-Lagrange stability for impulsive systems. After several theoretical derivations, some simulation results are given to demonstrate our results.

Keywords: *Equi-boundedness, equi-Lagrange stability, impulsive system, impulsive synchronization*

1. Introduction

Impulsive differential equations have been gained considerable attention in science and engineering (eg. [1] - [3]) in recent years, since they provide a natural framework for mathematical modeling of many physical phenomena. Recently, impulsive control, which is based on the theory of impulsive differential equations, has been gained renewed interests for controlling chaotic systems.

The study of impulsive synchronization of two identical chaotic systems is one of the most important applications of impulsive control. In [4] and [5], two autonomous chaotic systems, the drive system and the driven system, have been considered for impulsive synchronization. Further detailed analysis of impulsive control and impulsive synchronization of chaotic systems are presented, e.g. in [6] - [9] etc.

A number of robust communication systems employing the two types of synchronization have been developed in [10]. It has been shown that impulsive synchronization systems may be combined with conventional cryptographic techniques [11] to achieve the two desired properties of increasing the complexity and reducing the redundancy of the transmitted signals. It has been further established that impulsive synchronization achieves efficient bandwidth utilization. However, the proposed impulsive synchronization systems suffer from the transmission time-frame congestion [12]. In [13], a promising application of impulsive synchronization of chaotic systems to a secure communication scheme was presented. In this paper, we will use the method introduced in [13] to establish the synchronization between two Rössler Chaotic Systems.

2. Preliminaries

In this section, we present some sufficient preliminaries from [13] for our main results. To facilitate our discussion, it is convenient to introduce the notations as follows, where $M \geq 0$:

$$\kappa_0 := \{g \in C[\mathbb{R}_+, \mathbb{R}_+] : g(s) > 0 \text{ if } s > 0 \text{ and } g(0) = 0\}$$
$$\kappa := \{g \in \kappa_0 : g(s) \text{ is strictly increasing}\}$$
$$\kappa\Re := \{g \in \kappa : \lim_{s\to\infty} g(s) = \infty\}$$
$$PC := \{p : \mathbb{R}_+ \to \mathbb{R}_+ : p(t) \in C((t_k, t_{k+1}]) \text{ and } p(t_k^+) \text{ exists}, k = 1, 2, ...\}$$
$$S^c(M) := \{x \in \mathbb{R}^n : \| x \| \geq M\}$$
$$S^c(M)^0 := \{x \in \mathbb{R}^n : \| x \| > M\}$$

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$$v_0(M) := \{V : \mathbb{R}_+ \times S^c(M) \to \mathbb{R}_+ : V(t,x) \in C((t_k, t_{k+1}] \times S^c(M)) \mid \text{locally Lipschitz in } x$$
$$\text{and } V(t_k^+, x) \text{ exists for } k = 1, 2, \ldots\}$$

Impulsive differential equations are usually defined as an ordinary differential equation coupled with a difference equation, as expressed in the following:

$$\begin{cases} \dot{x} = f(t,x) & t \neq t_k \\ \Delta x = I(t,x) & t = t_k \end{cases} \tag{1}$$

where $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $x(t_k^+) = \lim_{t \to t_k^+} x(t)$, $x(t_k^-) = \lim_{t \to t_k^-} x(t)$, and the moments of impulse satisfy $0 = t_1 < t_2 < \cdots < t_k < \cdots$ and $\lim_{k\to\infty} t_k = \infty$. Let f, $I : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous on $(t_k, t_{k+1}] \times \mathbb{R}^n$ and $f(t_k^+, x)$, $I(t_k^+, x)$ exist for each $k = 1, 2 \ldots$. This guarantees that, for each $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$, there exists a local solution of (1) satisfying the initial condition $x(t_0^+) = x_0$. Let $x(t) := x(t, t_0, x_0)$ be any solution of (1) satisfying $x(t_0^+) = x_0$ and $x(t)$ be left continuous at each $t_k > t_0$ in its interval of existence, i.e. $x(t_k^-) = x(t_k)$. Then we have the following definitions.

Definition 1: Solutions of the impulsive system (1) are said to be
(S1) equi-attractive in the large if for each $\varepsilon > 0$, $\alpha > 0$ and $t_0 \in \mathbb{R}_+$, there exists a number $T := T(t_0, \varepsilon, \alpha) > 0$ such that $\| x_0 \| < \alpha$ implies $\| x(t) \| < \varepsilon$, for $t \geq t_0 + T$;
(S2) uniformly equi-attractive in the large if T in (S1) is independent of t_0.

Definition 2: Solutions of the impulsive system (1) are said to be
(B1) equi-bounded if for each $\alpha > 0$, $t_0 \in \mathbb{R}_+$, there exists a constant $\beta := \beta(t_0, \alpha) > 0$ such that $\| x_0 \| \leq \alpha$ implies that $\| x(t) \| < \beta$, for $t > t_0$;
(B2) uniformly equi-bounded if β in (B1) is independent of t_0;
(B3) equi-Lagrange stable if (S1) and (B1) hold together;
(B4) uniformly equi-Lagrange stable if (S2) and (B2) hold together.
Now we shall need the following results [13].

Lemma 1: The solution of (1) are uniformly equi-bounded if
(a) $V \in v_0(M)$, for some $M \geq 0$ and there exist functions $a, b \in \kappa\Re$ such that $b(\| x \|) \leq V(t,x) \leq a(\| x \|)$, $(t,x) \in \mathbb{R}_+ \times S^c(M)$,
(b) there exist functions $p \in PC$ and $c_k \in \kappa_0$ such that

$$D^+ V(t,x) \leq p(t) c_k(V(t,x)) \tag{2}$$

$(t,x) \in (t_k, t_{k+1}) \times S^c(M)^0$ for $k = 1, 2, \ldots$,
(c) there exists a constant $N \geq 0$ such that if $\| x(t_k) \| \leq M$, then $\| x + I(t_k, x) \| \leq N$, for $k = 1, 2, \ldots$,
(d) there exist functions $\psi_k \in \kappa\Re$ and $\psi_k \in \kappa_0$ such that $\psi(s) \leq \psi_k(s) \leq s$, $s \in \mathbb{R}_+$ and

$$V(t_k^+, x + I(t_k, x)) \leq \psi_k(V(t_k, x)) \tag{3}$$

whenever (t_k, x), $(t_k, x + I(t_k, x)) \in \mathbb{R}_+ \times S^c(M)^0$, for $k = 1, 2, \ldots$
(e) there exist constants $\lambda > 0$ and $\gamma_k \geq 0$ such that

$$\int_{t_k}^{t_{k+1}} p(s)ds + \int_y^{\psi_k(y)} \frac{ds}{c_k(s)} \leq -\gamma_k \tag{4}$$

where $y \geq \lambda$, $k = 1, 2, \ldots$.

Lemma 2: The solution of (1) are equi-Lagrange stable if
(a) (1) is equi-bounded;
(b) Condition (b) of lemma 1 holds for $V \in v_0(0)$ with inequality (2) being true for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$;
(c) There exist functions $\Psi_k \in \kappa_0$ such that inequality (3) holds, for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$ and for all $k = 1, 2, \ldots$;
(d) There exists a constant $\rho > 0$ and functions $C_k \in \kappa$ such that $C_k(s) \leq c_k(s)$, for all $s \in \mathbb{R}_+$ and for all $k = 1, 2, \ldots$;
(e) (4) holds, for all $y > 0$ and $\sum_{k=1}^{\infty} \gamma_k = \infty$.

3. Synchronization of Rössler chaotic system

In this section, we will show that two Rössler chaotic systems can be synchronized by impulsive control. The Rössler chaotic system is described by the following differential equation:

$$\begin{cases} \dot{x}_1 = -(x_2 + x_3) \\ \dot{x}_2 = x_1 + a x_2 \\ \dot{x}_3 = b + x_3(x_1 - c) \end{cases} \tag{5}$$

where a, b, c are positive parameter. When we choose the parameters $a = 0.2, b = 0.2, c = 5.7$, and the initial condition $(x_1(0), x_2(0), x_3(0))^T = (0.1, 0.1, 0.1)^T$, then system (5) has an notable Rössler attractor, as shown in Figure 1.

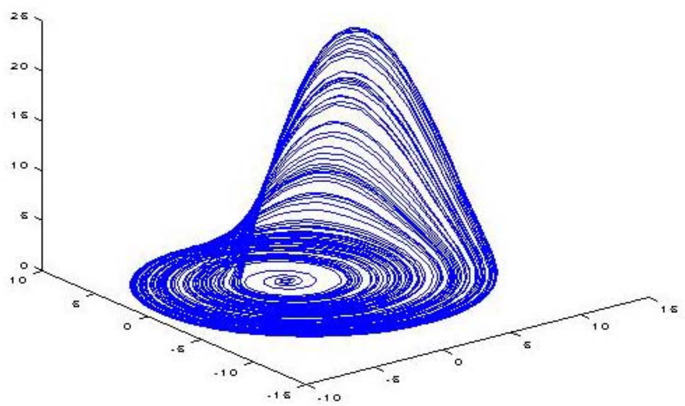

Figure 1. Rössler attractor

System (5) can be rewritten into the following form as our first driving system:

$$\dot{\mathbf{x}} = A\mathbf{x} + g(\mathbf{x}) \tag{6}$$

where $\mathbf{x} = (x_1, x_2, x_3)^T$ is state variable, and

$$A = \begin{bmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ 0 & 0 & -c \end{bmatrix}, \quad g(\mathbf{x}) = \begin{bmatrix} 0 \\ 0 \\ x_1 x_3 + b \end{bmatrix}.$$

The second driving system is the derivative of the Rössler chaotic system give by (6), which is expressed by

$$\ddot{\mathbf{x}} = A\dot{\mathbf{x}} + \begin{bmatrix} 0 \\ 0 \\ \dot{x}_1 x_3 + \dot{x}_3 x_1 \end{bmatrix} \tag{7}$$

Let the first driven Rössler chaotic system be

$$\begin{cases} \dot{\mathbf{u}} = A\mathbf{u} + \begin{bmatrix} 0 \\ 0 \\ u_1 u_3 + b \end{bmatrix} & t \neq t_k \\ \Delta\mathbf{u} = -B_k \mathbf{e} & t = t_k \end{cases} \tag{8}$$

where B_k are $n \times n$ constant matrices, for all $k = 1, 2, ...$ and $\mathbf{e} = \mathbf{x} - \mathbf{u}$. The second driven system is the derivative of the first driven system and is given by

$$\begin{cases} \ddot{\mathbf{u}} = A\dot{\mathbf{u}} + \begin{bmatrix} 0 \\ 0 \\ \dot{u}_1 u_3 + \dot{u}_3 u_1 \end{bmatrix} & t \neq t_k \\ \Delta\dot{\mathbf{u}} = -B_k \dot{\mathbf{e}} & t = t_k \end{cases} \tag{9}$$

where $\dot{\mathbf{e}} = \dot{\mathbf{x}} - \dot{\mathbf{u}}$. With (6)-(9), the error dynamics $\mathbf{e}$ and $\dot{\mathbf{e}}$ can be expressed as

$$\begin{cases} \dot{\mathbf{e}} = A\mathbf{e} + \begin{bmatrix} 0 \\ 0 \\ x_1 x_3 - u_1 u_3 \end{bmatrix} & t \neq t_k \\ \Delta\mathbf{e} = B_k \mathbf{e} & t = t_k \end{cases} \tag{10}$$

and

$$\ddot{\mathbf{e}} = A\dot{\mathbf{e}} + \begin{bmatrix} 0 \\ 0 \\ \dot{x}_1 x_3 + x_1 \dot{x}_3 - \dot{u}_1 u_3 - \dot{u}_3 u_1 \end{bmatrix} \tag{11}$$

Now we have the following theorem.

Theorem 1. System (11) is uniformly equi-bounded if the largest eigenvalue of $(I + B_k^T)(I + B_k)$, denoted by λ_k, satisfies

$$\lambda_k \leq \exp\{-2\alpha_k\} \tag{12}$$

for all $k = 1, 2, ...$ and for $\| \tilde{e}(t_k) \|$, $\| \tilde{e}(t_k) + B_k \tilde{e}(t_k) \| > M$ for some $M > 0$, where $\tilde{e} = (e, \dot{\bar{e}})$, $\alpha_k := (1/2)[\gamma_k + \ell\Delta_{k+1}] + \delta$, $\gamma_k > 0$ are constants and $\{\gamma_k\}_{k=1}^{\infty}$ has an upper bound, $0 < \delta \ll \inf_k (\gamma_k + \ell\Delta_{k+1})$, $\Delta_k := t_k - t_{k-1} > r > 0$, for all $k = 2, 3, ...$ and $\ell := 2L + d + |\dot{x}_1| + |u_3| + |x_1|$, where d is the largest eigenvalue of $Q := A + A^T$. Moreover, if $\gamma_k = 1/k$, for all $k = 1, 2, ...,$ and for all $\| \tilde{e}(t_k) \| > 0$, then (11) is uniformly equi-Lagrange stable.

Proof: To prove the result, we will prove that (11) satisfies all the conditions described by Lemma 2. Let $B := \sup_k(\alpha_k)$, and $V(t, \tilde{e}) := V(\tilde{e}) = \tilde{e}^T \tilde{e} = \| \tilde{e} \|^2$. Choose $b(\| \tilde{e} \|) = a(\| \tilde{e} \|) = \| \tilde{e} \|^2$. The Upper-right derivative of V is given by

$$
\begin{aligned}
D^+V(\tilde{e}) &= \dot{\tilde{e}}^T \tilde{e} + \tilde{e}^T \dot{\tilde{e}} \\
&= (\dot{e}^T, \ddot{\bar{e}}^T)(e, \dot{\bar{e}})^T + (e^T, \dot{\bar{e}}^T)(\dot{e}, \ddot{\bar{e}}^T) \\
&= \dot{e}^T e + \ddot{\bar{e}}^T \dot{\bar{e}} + e^T \dot{e} + \dot{\bar{e}}^T \ddot{\bar{e}} \\
&\le 2L \| e \|^2 + \dot{\bar{e}}^T (A^T + A) \dot{\bar{e}} + 2(\dot{x}_1 x_3 + x_1 \dot{x}_3 - \dot{\bar{u}}_1 u_3 - \dot{\bar{u}}_3 u_1) \dot{\bar{e}}_3
\end{aligned}
$$

where $\dot{e}^T e \le L \| e \|^2$, for some $L > 0$, which follows form (10). Note that $\dot{\bar{e}}^T (A^T + A) \dot{\bar{e}} < d \dot{\bar{e}}^T \dot{\bar{e}}$, where d is the largest eigenvalue of $A^T + A$, and

$$
\begin{aligned}
&(\dot{x}_1 x_3 + x_1 \dot{x}_3 - \dot{\bar{u}}_1 u_3 - \dot{\bar{u}}_3 u_1) \dot{\bar{e}}_3 \\
&\le | \dot{x}_1 x_3 + x_1 \dot{x}_3 - \dot{\bar{u}}_1 u_3 - \dot{\bar{u}}_3 u_1 | \cdot | \dot{\bar{e}}_3 | \\
&= | \dot{x}_1 (x_3 - u_3) + u_3 (\dot{x}_1 - \dot{\bar{u}}_1) + x_1 (\dot{x}_3 - \dot{\bar{u}}_3) + \dot{\bar{u}}_3 (x_1 - u_1) | \cdot | \dot{\bar{e}}_3 | \\
&= | \dot{x}_1 e_3 + u_3 \dot{\bar{e}}_1 + x_1 \dot{\bar{e}}_3 + \dot{\bar{u}}_3 e_1 | \cdot | \dot{\bar{e}}_3 | \\
&\le \frac{1}{2} (| \dot{x}_1 | + | u_3 | + | x_1 | + | \dot{\bar{u}}_3 |) \| \tilde{e} \|^2
\end{aligned}
$$

Thus, we can conclude that

$$
\begin{aligned}
D^+V(\tilde{e}) &\le (2L + d) \| \tilde{e} \|^2 + 2(\dot{x}_1 x_3 + x_1 \dot{x}_3 - \dot{\bar{u}}_1 u_3 - \dot{\bar{u}}_3 u_1) \dot{\bar{e}}_3 \\
&\le (2L + d + | \dot{x}_1 | + | u_3 | + | x_1 | + | \dot{\bar{u}}_3 |) V(\tilde{e})
\end{aligned},
$$

Set $p(t) := 2L + d + | \dot{x}_1 | + | u_3 | + | x_1 | + | \dot{\bar{u}}_1 |$ and $c(s) := s$. Clearly, $p \in PC$ and $c \in \kappa$. Thus, conditions (a), (b) of Lemma 1 are satisfied over $\mathbb{R}_+ \times \mathbb{R}^n$. If $\| \tilde{e}(t_k) \| \le M, k = 1, 2, ...,$ then, by inequality (12), we have

$$
\begin{aligned}
\| \tilde{e}(t_k) + B_k \tilde{e}(t_k) \| &\le \| I + B_k \| \| \tilde{e}(t_k) \| \\
&\le \exp(-\alpha_k) M \\
&< \exp(-\delta) M := N
\end{aligned}
$$

for $k = 1, 2, ...,$ and thus, condition (c) of Lemma1 is also satisfied.

Define the mapping $\Psi_k(s) := \exp(-2\alpha_k)s$ for all $s \ge 0$. Clearly, $\Psi_k(s) \le s$, for all $s \ge 0$ and $\Psi_k(s) \ge (1/2)\exp(-2B)s =: \Psi(s)$ for all $s \ge 0$. i.e. $\Psi(s) \le \Psi_k(s) \le s$. Furthermore, by inequality (12), for $\| \tilde{e}(t_k) + B_k \tilde{e}(t_k) \| > M$, $k = 1, 2, ...,$ we have

$$
\begin{aligned}
V(\tilde{e}(t_k) + B_k \tilde{e}(t_k)) &= (\tilde{e}(t_k) + B_k \tilde{e}(t_k))^T \cdot (\tilde{e}(t_k) + B_k \tilde{e}(t_k)) \\
&= \tilde{e}^T(t_k)(I + B_k^T)(I + B_k)\tilde{e}(t_k) \\
&\le \lambda_k \tilde{e}(t_k)^T \tilde{e}(t_k) \\
&\le \exp(-2\alpha_k) \| \tilde{e}(t_k) \|^2 \\
&= \Psi_k(V(\tilde{e}(t_k)))
\end{aligned}
$$

This implies that condition (d) of Lemma 1 is satisfied. In addition, it is easy to check that

$$\int \frac{ds}{s} = \ln(s) .$$

It follows that, for $k = 1, 2, ...$

$$\int_{t_k}^{t_{k+1}} p(s)ds + \int_{y}^{\Psi_k(y)} \frac{ds}{s} = \ell\Delta_{k+1} + \ln\frac{\Psi_k(y)}{y}$$
$$= -\gamma_k - 2\delta \qquad .$$
$$\leq -\gamma_k$$

Thus, condition (e) of Lemma 1 is also satisfied. Therefore we conclude that (11) is uniformly equi-bounded as desired. This implies, by choosing $\gamma_k = 1/k$, for all $k = 1, 2, ...,$ and applying Lemma 1 and Lemma 2, that solution to system (11) are also equi-Lagrange stable, as desired.

4. Numerical Simulation

In this section, we shall discuss an example to illustrate our main results. We take

$a = 0.2, b = 0.2, c = 5.7$,

$B_k = B = -diag(0.2, 0.2, 0.2)$,

$\Delta_K = \Delta = 0.002$,

$(x_1(0), x_2(0), x_3(0)) = (1.5, -1.7, 1.8)$,

$(\dot{x}_1(0), \dot{x}_2(0), \dot{x}_3(0)) = (1.2, 1.5, 1.7)$,

$(u_1(0), u_2(0), u_3(0)) = (1.46, -1.87, 2.5)$,

$(\dot{u}_1(0), \dot{u}_2(0), \dot{u}_3(0)) = (3.9, 1.74, -1.62)$.

Then all conditions of Theorem 1 are satisfied and Figure 2 represents the simulation results which shows that the impulsive control synchronization is realized.

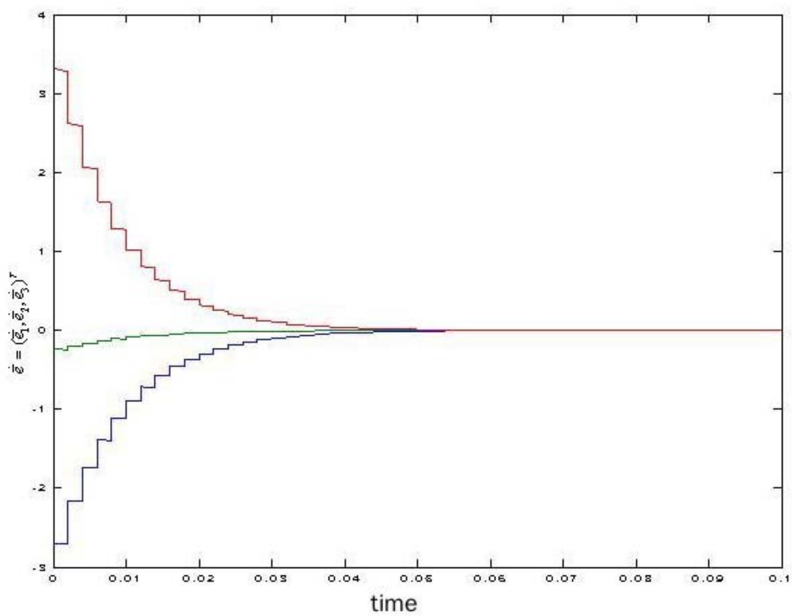

Figure 2. Uniform equi-Lagrange stability of system (11)

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