

Best neural simultaneous approximation

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Article Info

Article history:

Received Jan 14, 2020

Revised Apr 28, 2020

Accepted Jun 2, 2020

Keywords:

2-normed space

Approximation

Neural networks

ABSTRACT

For many years, approximation concepts has been investigated in view of neural networks for the several applications of the two topics. Researchers studied simultaneous approximation in the 2-normed space and proved essential theorems concern with existence, uniqueness and degree of best approximation. Here, we define a new 2-norm in L_p -space, with $p < 1$, so we call it L_p quasi 2- normed space ($L_{p,2}$). The set of approximations is a space of feedforward neural networks that is constructed in this paper. Existence and uniqueness of best neural approximation for a function from $L_{p,2}$ is proved, describing the rate of best approximation in terms of modulus of smoothness.

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1. INTRODUCTION

The first notes about simultaneous approximation was done by Dunham in [1]. He generated the classical Chebyshev approximation by approximating two continuous functions f^+ and f^- , with $f^+(x) \leq f^-(x)$, for all $x \in [a, b]$, simultaneously. He also proved that his simultaneous approximation is equivalent to the classical one function Chebyshev approximation when $f^+ = f^-$.

For more specification, Diaz and Mclaughlin [2] proved that the above problem is equivalent to the problem of approximating $\frac{1}{2}|f^+ + f^-|$. Also approximating two appropriate functions simultaneously is equivalent to approximating one function by elements of a certain set \mathcal{S} . Moreover, they defined the best simultaneous approximation \mathcal{S} to the set \mathcal{S} in [3] as follow

$$\inf_{\mathcal{S} \in \mathcal{S}} \sup_{f \in \mathcal{F}} \|f - s\| = \sup_{f \in \mathcal{F}} \|f - \mathcal{S}\|,$$

where \mathcal{F} is a set of uniformly bounded functions on $[a, b]$ and \mathcal{S} is a set of functions on $[a, b]$. They proved that \mathcal{S} is equivalent to the best simultaneous approximation of two functions. The set \mathcal{F} varies among researchers, it is $C[a, b]$ in [4] and [5], the space of uniformly bounded functions in [6], Banach space in [7], weighted space [8], L_p spaces [4] or 2-normed space as in [9-16].

The 2-normed space was firstly defined by Gahler in his paper [9], and then generalized by Iseki in his paper [17]. This space provides a tool to deal with 2-structures. For the same porpuse, others defined quasi-normed , quasi-(2;p)-normed space and generalized each one (see [18, 19]). First, we define the 2-normed space generally from [1]

Definition (1) A norm $\|\cdot\| : X \times X \rightarrow R^+$ is 2-norm on X if it satisfies the following conditions:

[C1] $\|x_1, x_2\| = 0$, if and only if x_1, x_2 are linearly dependent from X. .

[C2] $\|x_1, x_2\| = \|x_2, x_1\|$, for all x_1, x_2 from X.

[C3] $\|\alpha x_1, x_2\| = |\alpha| \|x_1, x_2\|$, for all $\alpha \in R$ and $x_1, x_2 \in X$

[C4] $\|x_1 + x_2, z\| \leq \|x_1, z\| + \|x_2, z\|$, for all $x_1, x_2, z \in X$.

The space $(X, \|\cdot, \cdot\|)$ is called 2-normed space. Later, Park [18] substitute [C4] with the following condition

[C4*] $\|x_1 + x_2, z\| \leq C\{\|x_1, z\| + \|x_2, z\|\}$, for all $x_1, x_2, z \in X$.

In this paper, we deal with 2-normed space with a special definition that deals with Lebesgue-integrable space

$$L_p[a, b] = \left\{ f: \int_a^b |f(x)|^p dx < \infty \right\}$$

Through this paper, we refer $L_{p,2}$ to the space $L_p[a, b] \times L_p[a, b]$ in the following manner

$$L_{p,2}[a, b] = \left\{ f: \int_a^b |f(x)\varphi(x)|^p dx < \infty, \text{ for every } \varphi \in L_p \right\} \tag{1}$$

Together with the non-negative function $\|\cdot, \cdot\|_p$ over the vector space $L_{p,2}$ as follow

$$\|f, g\|_p = \left(\int_a^b |f(x)g(x)|^p dx \right)^{1/p}, \tag{2}$$

for any function f and g from $L_{p,2}$. The space $L_{p,2}[a, b]$ is a 2-normed space since it satisfies the following conditions

[C1] $\|f, g\|_p = 0$, if and only if f and g are linearly dependent functions from $L_{p,2}$.

[C2] $\|f, g\|_p = \|g, f\|_p$, for all f and g from $L_{p,2}$.

[C3] $\|\alpha f, g\|_p = |\alpha| \|f, g\|_p$, for all $\alpha \in R$ and $f, g \in L_{p,2}$.

[C4] $\|f + g, \varphi\|_p \leq C\{\|f, \varphi\|_p + \|g, \varphi\|_p\}$, for all $f, g, \varphi \in L_{p,2}$.

The space $L_{p,2}[a, b]$ is a 2-normed space since it satisfies the conditions in Definition(1).

[C1] Let f, g be two linearly dependent functions from $L_{p,2}$, with $f \neq g$ iff $\langle f, g \rangle = 0$, iff $\|f, g\|_p = 0$.

[C2] By (2), we have $\|f, g\|_p = \|g, f\|_p$.

[C3] Let $\alpha \in R$, then $\|\alpha f, g\|_p = \left(\int_a^b |\alpha f(x)g(x)|^p dx \right)^{1/p} = |\alpha| \left(\int_a^b |f(x)g(x)|^p dx \right)^{1/p} = |\alpha| \|f, g\|_p$,

[C4] Let $f, g \in L_{p,2}$, since $0 < p < 1$, then there exists $C > 0$ satisfies

$$\begin{aligned} \|f + g, \varphi\|_p &= \left(\int_a^b |(f + g)(x)\varphi(x)|^p dx \right)^{1/p} \\ &\leq C \left\{ \left(\int_a^b |f(x)\varphi(x)|^p dx \right)^{1/p} + \left(\int_a^b |g(x)\varphi(x)|^p dx \right)^{1/p} \right\} \\ &= C\{\|f, \varphi\|_p + \|g, \varphi\|_p\}. \end{aligned}$$

To continue our investigation for a neural best approximation, we need the following definitions that are related to convergence sequences of functions from $L_{p,2}$.

Definition(2) A sequence of functions $\{f_n\}_{n \geq 1}$ from $L_{p,2}$ is said to be *Cauchy Sequence* if and only if

$$\lim_{n,m \rightarrow \infty} \|f_n - f_m, \varphi_1\|_p = 0,$$

and

$$\lim_{n,m \rightarrow \infty} \|f_n - f_m, \varphi_2\|_p = 0,$$

for some independent functions $\varphi_1, \varphi_2 \in L_{p,2}$.

Definition(3) A sequence of functions $\{f_n\}_{n \geq 1}$ from $L_{p,2}$ is said to be *convergent* to some $f \in L_{p,2}$ if and only if

$$\lim_{n,m \rightarrow \infty} \|f_n - f, \varphi\|_p = 0,$$

for all $\varphi \in L_{p,2}$. The following definitions give some useful properties to the space $L_{p,2}$, that we need later in the main results.

Definition(4) The space $L_{p,2}$ is said to be *complete* if and only if every Cauchy sequence $\{f_n\}_{n \geq 1}$ from $L_{p,2}$ converges to a function that belongs to $L_{p,2}$. To measure the degree of best approximation, we define the modulus of smoothness in $L_{p,2}$ as follow

Definition(5) Let $f \in L_{p,2}$, then the k th symmetric difference of f is given by

$$\Delta_h^k(f, x, [a, b]) = \begin{cases} \sum_{i=0}^k \binom{k}{i} (-1)^{k-i} f\left(x - \frac{kh}{2} + ih\right), & x \pm \frac{kh}{2} \in [a, b] \\ 0, & o.w. \end{cases} \quad (3)$$

So the k th modulus of smoothness of f is given by

$$\omega_k(f, \varphi, \delta, [a, b])_p = \sup_{0 < h \leq \delta} \|\Delta_h^k(f, \cdot), \varphi\|_p, \quad (4)$$

for some $\delta \geq 0$.

2. CONSTRUCTION OF FNN WITH RELU ACTIVATION FUNCTION

We have to talk about the set of approximation. Choosing the target approximation space is as much important as choosing the function space. It is related to the applicable properties and the accurate results to each space. Moreover, sometimes it is preferred to replace a certain function by its approximation from some vital space. Scientists approximate functions by polynomials, wavelets, splines and neural networks. For the wide usage of neural networks and their ability to solve problems from different fields (see [21-38]) a set of functions from L_p space is approximated by neural networks in this work. Many papers contains this topic widely, we mention some of them in the references below (see [39-47]).

Let the approximation neural operator

$$\mathcal{N} = \sum_{i=1}^n c_i \mathcal{R}(w_i x + \vartheta_i), \quad (5)$$

where

$$\mathcal{R}(x) = x^+ = \max(0, x) = \begin{cases} 0, & x \leq 0 \\ x, & x > 0 \end{cases}, \quad (6)$$

is the Relu activation function. For its simplicity and efficiency, scientists use Relu function to activate the neural network. In comparison with other activation functions, it gives faster and more acceptable results, it solves the problem of vanishing gradient that most activation functions suffer from. In the field of function approximation, [48-50] are some papers that dealt with neural approximation with Relu activation function. Now, we are ready to discuss the essential point in this paper. Here is the definition of the best simultaneous approximation of the set $L_{p,2}$ by elements of \mathfrak{N} under the norm (2).

Definition(6) The simultaneous best approximation of a subset \mathcal{F} of $L_{p,2}$ is $N^* \in \mathfrak{N}$ in the expression

$$\inf_{N \in \mathfrak{N}} \left\{ \sup_{f \in \mathcal{F}} \|f - N, \varphi\|_p \right\} = \sup_{f \in \mathcal{F}} \|f - N^*, \varphi\|_p \quad (7)$$

In the next section, we construct our neural approximation of type (5) simultaneously to $L_{p,2}$.

3. EXISTENCE THEOREM

Let $f \in L_{p,2}$, then there exists a FNN of the form:

$$\mathcal{N}_n = \sum_{i=1}^n c_i \mathcal{R}(w_i x + \vartheta_i),$$

where \mathcal{R} is the Relu activation function on $[a, b]$ and the parameters c_i , w_i , and ϑ_i are chosen as follow:

$$\begin{aligned}
 w_i &= -2 \frac{hn}{|b-a|^i} \\
 \vartheta_i &= \frac{hn}{|b-a|^i} \left(2a + (2i-1) \frac{b-a}{n} \right), \\
 c_0 &= f(a) - \sum_{i=1}^n c_i \mathcal{R}(w_i a + \vartheta_i), \\
 c_i &= \frac{1}{2b} \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} f \left(x - \frac{kh}{2} + lh \right),
 \end{aligned}$$

where $h = \frac{b}{2n}$.

Proof:

Since $\mathcal{R}(x) = x^+$, $\forall x \in [a, b]$, then by (6), $\sup_{x \in [a,b]} |\mathcal{R}(x)| = b$.

Let the partition $a < x_1 < x_2 < \dots < x_n = b$, such that for all $1 \leq i \leq n$, and let $x_i = a + i \frac{b-a}{n}$,

Choosing $c_0 = f(a) - \sum_{i=1}^n c_i \mathcal{R}(w_i a + \vartheta_i)$, gives the guaranty that $f(a) = \mathcal{N}_n(a)$.

For all $x \in [a, b]$, there is $j \in \mathbb{N}$, $0 \leq j \leq n$, such that $x \in [x_{j-1}, x_j]$, and that

$$\begin{aligned}
 \mathcal{N}_n(x) &= f(a) + \sum_{i=1}^n \frac{1}{2b} \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} f \left(x - \frac{kh}{2} + lh \right) [\mathcal{R}(w_i x + \vartheta_i) - \mathcal{R}(w_i a + \vartheta_i)] \\
 &= f(a) + \sum_{i=1}^{j-1} \frac{1}{2b} \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} f \left(x - \frac{kh}{2} + lh \right) [\mathcal{R}(w_i x + \vartheta_i) - \mathcal{R}(w_i a + \vartheta_i)] \\
 &\quad + \frac{1}{2b} \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} f \left(x_j - \frac{kh}{2} + lh \right) [\mathcal{R}(w_i x + \vartheta_i) - \mathcal{R}(w_i a + \vartheta_i)] \\
 &\quad + \sum_{i=j+1}^n \frac{1}{2b} \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} f \left(x - \frac{kh}{2} + lh \right) [\mathcal{R}(w_i x + \vartheta_i) - \mathcal{R}(w_i a + \vartheta_i)] \\
 &= f(a) + S_1 + S_2 + S_3
 \end{aligned}$$

To estimate $|\mathcal{R}(w_i x + \vartheta_i) - \mathcal{R}(w_i a + \vartheta_i)|$, we have two cases. For $i > j$, we have, $x \leq x_j \leq x_{i-1}$, so by monotonicity of \mathcal{R} and our choices of the parameters w_i, ϑ_i and x_i , we get

Case(1)

$$\begin{aligned}
 &0 < \mathcal{R}(w_i x + \vartheta_i) - \mathcal{R}(w_i a + \vartheta_i) \\
 &\leq \mathcal{R}(w_i x_j + \vartheta_i) - \mathcal{R}(w_i a + \vartheta_i) \\
 &\leq \mathcal{R}(w_i x_{i-1} + \vartheta_i) - \mathcal{R}(w_i a + \vartheta_i) \\
 &= \mathcal{R}(h) - \mathcal{R}(2hi - h) \\
 &= -2hi = -\frac{bi}{n}.
 \end{aligned}$$

Case(2)

For $i < j$, we have, $x_i \leq x_{j-1} \leq x$, then

$$\begin{aligned}
 &\mathcal{R}(w_i x + \vartheta_i) - \mathcal{R}(w_i a + \vartheta_i) \\
 &\geq \mathcal{R}(w_i x_i + \vartheta_i) - \mathcal{R}(w_i x_{i-1} + \vartheta_i) \\
 &= \mathcal{R}(-h) - \mathcal{R}(h) \\
 &= -h = -\frac{b}{2n}.
 \end{aligned}$$

For the two cases, we conclude that

$$|\mathcal{R}(w_i x + \vartheta_i) - \mathcal{R}(w_i a + \vartheta_i)| \leq h = \frac{b}{2n}$$

Now, we are ready to estimate S_1 , S_2 and S_3

$$\begin{aligned} |S_1| &\leq \frac{1}{2b} \sum_{i=1}^{j-1} \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \left| f\left(x - \frac{kh}{2} + lh\right) \right| |\mathcal{R}(w_i x + \vartheta_i) - \mathcal{R}(w_i a + \vartheta_i)| \\ &\leq \frac{1}{4} \Delta_h^k(f, x, [a, b]) \end{aligned}$$

$$\begin{aligned} |S_2| &\leq \frac{1}{2b} \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \left| f\left(x_j - \frac{kh}{2} + lh\right) \right| |\mathcal{R}(w_i x + \vartheta_i) - \mathcal{R}(w_i a + \vartheta_i)| \\ &\leq \frac{1}{4} \Delta_h^k(f, x, [a, b]) \end{aligned}$$

$$\begin{aligned} |S_3| &\leq \frac{1}{2b} \sum_{i=j+1}^n \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \left| f\left(x - \frac{kh}{2} + lh\right) \right| |\mathcal{R}(w_i x + \vartheta_i) - \mathcal{R}(w_i a + \vartheta_i)| \\ &\leq \frac{1}{4} \Delta_h^k(f, x, [a, b]) \end{aligned}$$

Finally, let $\varphi \in L_{p,2}$, then

$$\begin{aligned} \|\mathcal{N}_n(x) - f(x), \varphi\|_p^p &\leq \int_a^b |\mathcal{N}_n(x) - f(x)|^p |\varphi(x)|^p dx \\ &\leq \frac{1}{2b} \sum_{i=1}^n [\mathcal{R}(w_i x + \vartheta_i) - \mathcal{R}(w_i a + \vartheta_i)] \int_a^b \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \left| f\left(x - \frac{kh}{2} + lh\right) \right|^p |\varphi(x)|^p dx \\ &\leq \frac{3}{4} \omega_k\left(f, \varphi, \frac{1}{n}\right)_p. \blacksquare \end{aligned}$$

4. UNIQUENESS THEOREM

The simultaneous best approximation $N^* \in \mathfrak{N}$ of a subset F of $L_{p,2}$ is unique.

Proof:

To prove that $N^* \in \mathfrak{N}$ is unique, suppose that $N_1, N_2 \in \mathfrak{N}$ be two simultaneous approximations to F , then by Definition(3)

$$\lim_{n \rightarrow \infty} \|N_1 - f, \varphi\|_p = 0,$$

and

$$\lim_{n \rightarrow \infty} \|N_2 - f, \varphi\|_p = 0,$$

So by condition [C4*] of Definition(1), there exists $k \geq 1$,

$$\|N_1 - N_2, \varphi\|_p \leq k(\|N_1 - f, \varphi\|_p + \|N_2 - f, \varphi\|_p)$$

By taking limits to both sides as n tends to infinity, then for all $\varphi \in L_{p,2}$

$$\lim_{n \rightarrow \infty} \|N_1 - N_2, \varphi\|_p = 0,$$

So $N_1 = N_2$, and the best simultaneous approximation to F out of \mathfrak{N} is unique.

5. CONCLUSION

Simultaneous approximation in the $L_{p,2}$ space is defined in details in this paper. Construction of neural networks with rectified activation function that approximates a subset of $L_{p,2}$ simultaneously is obtained too. In spite of its various applications, it gives accurate results that depends on modulus of smoothness. It would be interesting to discuss vital applications in 2-structure spaces for the constructed neural network.

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