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New reliable modifications of the homotopy methods

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ABSTRACT

In this article, new modifications of the homotopy methods are presented and applied to non-homogeneous fractional Volterra integro-differential equations with boundary conditions. A comparative study between the new modified homotopy perturbation method (MHPM) and the new modified homotopy analysis method (MHAM). Several illustrative examples are given to demonstrate the effectiveness and reliability of the methods.

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1. INTRODUCTION

In this paper, we shall be concerned with the non-homogeneous fractional Volterra integro-differential equations of the second kind of the form:

$$^{c}D^{\alpha}y(x) = f(x)y(x) + \Theta(x) + \int_0^x \Psi(x,t)\Delta(y(t))dt, \tag{1}$$

with the boundary conditions

$$y^{(k)}(0) = d_k, \tag{2}$$
$$y^{(k)}(1) = c_k, \quad k = 1, \cdots, n, \ n-1 < \alpha \leq n, \ 0 \leq x \leq 1, \ n \in \mathbb{N}, \tag{3}$$

where $^{c}D^{\alpha}$ denotes a differential operator with fractional order α, and the $\Theta(x)$, $f(x)$ and $\Psi(x,t)$ are holomorphic functions, $\Delta(y(t))$ is a polynomial of $y(t)$ with constant coefficients.

The homotopy analysis method (HAM) proposed by Liao in 1992 and the homotopy perturbation method (HPM) proposed by He in 1998 are compared through an evolution equation used as the second example in a recent paper by Ganji et al. It is found that the HPM is a special case of the HAM when $\hbar = 1$. The well-known and powerful HAM is based on both Homotopy in topology and the McLaurin series. In one of his pioneering articles, he claimed that the method does not require either small or large parameters comparing with the perturbation techniques. The general concept of this method has been considered by many researchers in their published works [1–5].

The fractional integro-differential equations have attracted much more interest of mathematicians and physicists which provides an efficiency for the description of many practical dynamical arising in engineering and scientific disciplines such as, physics, biology, electrochemistry, chemistry, economy, electromagnetic, control theory and viscoelasticity [1, 2, 6–15]. In recent years, many authors focus on the development of numerical and analytical techniques for fractional integro-differential equations. For instance, we can recall the following works. Al-Samadi and Gumah [16] applied the HAM for fractional SEIR epidemic model, Zurigat et al. [17] applied HAM for system of fractional integro-differential equations. Yang and Hou [18] applied the Laplace decomposition method to solve the fractional integro-differential equations, Mittal and Nigam [19] applied the Adomian decomposition method to approximate solutions for fractional integro-differential equations. Ma and Huang [20] applied hybrid collocation method to study integro-differential equations of fractional order. Moreover, properties of the fractional integro-differential equations have been studied by several authors [7, 21–27]. The main objective of the present paper is to study the behavior of the solution that can be formally determined by analytical approximated methods as the MHAM and MHPM.

2. PRELIMINARIES

In this section, we give f fractional calculus theory which are further used in this paper [2, 7, 25, 28, 29].

Definition 2..1 *A real function $f(x), x > 0$, is said to be in the space $\mathbb{C}\varepsilon, \varepsilon \in \mathbb{R}$, if there exists a real number $p > \varepsilon$ such that $f(x) = x^p f_1(x)$, where $f_1(x) \in \mathbb{C}[0, 1)$. Clearly $\mathbb{C}_\varepsilon \subset \mathbb{C}_\omega$ if $\omega \leq \varepsilon$.*

Definition 2..2 *A function $f(x), x > 0$, is said to be in the space $\mathbb{C}^n_\varepsilon, n \in \mathbb{N} \cup \{0\}$, if $f^{(n)} \in \mathbb{C}_\varepsilon$.*

Definition 2..3 *[2] The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f \in \mathbb{C}_\varepsilon, \varepsilon \geq -1$ is defined as*

$$\begin{aligned} J^\alpha f(x) &= \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \qquad x > 0, \quad \alpha \in \mathbb{R}^+, \\ J^0 f(x) &= f(x), \end{aligned} \tag{4}$$

where $\mathbb{R}^+$ is the set of positive real numbers.

Definition 2..4 *[21] The fractional derivative of $f(x) \in \mathbb{C}^n_{-1},\ n \in \mathbb{N} \cup \{0\}$ in the Caputo sense is defined by*

$$\begin{aligned} {}^cD^\alpha f(x) &= J^{n-\alpha} D^n f(x) \\ &= \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} \frac{d^n f(t)}{dt^n} dt, & n-1 < \alpha < n, \\ \frac{d^n f(x)}{dx^n}, & \alpha = n, \end{cases} \end{aligned} \tag{5}$$

where the parameter α is the order of the derivative, in general it is real or even complex. But in this chapter, we will consider α as positive real.
Hence, we have the following properties:

1. $J^\alpha J^v f = J^{\alpha+v} f, \quad \alpha, v > 0.$
2. $J^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} x^{\alpha+\beta}, \quad \alpha > 0, \beta > -1, \quad x > 0.$
3. $J^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \quad m-1 < \alpha \leq m.$

Definition 2..5 *[8] The Riemann-Liouville fractional derivative of order $\alpha > 0$ is normally defined as*

$$D^\alpha f(x) = D^m J^{m-\alpha} f(x), \qquad m-1 < \alpha \leq m. \tag{6}$$

Theorem 2..1 *[2] **(Banach contraction principle).** Let (X, d) be a complete metric space, then each contraction mapping $T : X \longrightarrow X$ has a unique fixed point x of T in X i.e. $Tx = x$.*

Theorem 2..2 *[2] **(Schauder's fixed point theorem).** Let X be a Banach space and let A a convex, closed subset of X. If $T : A \longrightarrow A$ be the map such that the set $\{Ty : y \in A\}$ is relatively compact in X (or T is continuous and completely continouous). Then T has at least one fixed point $y^* \in A : Ty^* = y^*$.*

3. DESCRIPTION OF THE METHOD

Some powerful methods have been focusing on the development of more advanced and efficient methods for integro-differential equations such as the HAM and HPM [2, 27, 30, 31]. We will describe these methods in this section:

3.1. Homotopy analysis method (HAM)

The basic concept behind the HAM is illustrated by using the following nonlinear equation:

$$N[y] = 0,$$

where N is a nonlinear operator, $y(x)$ is unknown function and x is an independent variable. Let $y_0(x)$ denote an initial guess of the exact solution $y(x), \hbar \neq 0$ an auxiliary parameter, $H_1(x) \neq 0$ an auxiliary function, and L an auxiliary linear operator with the property $L[s(x)] = 0$ when $s(x) = 0$. Then using $q \in [0, 1]$ as an embedding parameter, we can construct a homotopy when consider, $N[y] = 0$, as follows [2]:

$$(1-q)L[\phi(x;q) - y_0(x)] - q\hbar H_1(x)N[\phi(x;q)] \\ = \hat{H}[\phi(x;q); y_0(x), H_1(x), \hbar, q]. \quad (7)$$

It should be emphasized that we have great freedom to choose the initial guess $y_0(x)$, the auxiliary linear operator L, the non-zero auxiliary parameter $\hbar$, and the auxiliary function $H_1(x)$. Enforcing the homotopy (7) to be zero, i.e.,

$$\hat{H} \quad {}_1[\phi(x;q); y_0(x), H_1(x), \hbar, q] = 0, \quad (8)$$

we have the so-called zero-order deformation equation

(1 − q)L[φ(x; q) − y0(x)] = qÿH1(x)N[φ(x; q)], (9)

when q = 0, the zero-order deformation (9) becomes

$$\phi(x;0) = y_0(x), \quad (10)$$

and when $q = 1$, since $\hbar \neq 0$ and $H_1(x) \neq 0$, the zero-order deformation (9) is equivalent to

$$\phi(x;1) = y(x). \quad (11)$$

Thus, according to Eqs.(10) and (11), as the embedding parameter q increases from 0 to 1, $\phi(x;q)$ varies continuously from the initial approximation $y_0(x)$ to the exact solution $y(x)$. Such a kind of continuous variation is called deformation in homotopy [31]. Due to Taylor's theorem, $\phi(x;q)$ can be expanded in a power series of q as follows:

$$\phi(x;q) = y_0(x) + \sum_{m=1}^{\infty} y_m(x)q^m, \quad (12)$$

where,

$$y_m(x) = \frac{1}{m!}\frac{\partial^m \phi(x;q)}{\partial q^m}|_{q=0}. \quad (13)$$

Let the initial guess $y_0(x)$, the auxiliary linear parameter L, the nonzero auxiliary parameter $\hbar$ and the auxiliary function $H_1(x)$ be properly chosen so that the power series (12) of $\phi(x;q)$ converges at $q = 1$, then, we have under these assumptions the solution series,

$$y(x) = \phi(x;1) = y_0(x) + \sum_{m=1}^{\infty} y_m(x). \quad (14)$$

From (12), we can write (9) as follows:

$$\begin{aligned} (1-q)L[\phi(x;q)-y_0(x)] &= (1-q)L[\sum_{m=1}^{\infty} y_m(x)q^m] \\ &= q\hbar H_1(x)N[\phi(x;q)], \end{aligned} \tag{15}$$

then,

$$L[\sum_{m=1}^{\infty} y_m(x)q^m] - qL[\sum_{m=1}^{\infty} y_m(x)q^m] = q\hbar H_1(x)N[\phi(x;q)]. \tag{16}$$

By differentiating (16) m times with respect to q, we obtain,

$$\begin{aligned} \{L[\sum_{m=1}^{\infty} y_m(x)q^m] - qL[\sum_{m=1}^{\infty} y_m(x)q^m]\}^{(m)} &= q\hbar H_1(x)N[\phi(x;q)]^{(m)} \\ &= m!L[y_m(x)-y_{m-1}(x)] \\ &= \hbar H_1(x)m\frac{\partial^{m-1}N[\phi(x;q)]}{\partial q^{m-1}}|_{q=0}. \end{aligned}$$

Therefore,

$$L[y_m(x)-\chi_m y_{m-1}(x)] = \hbar H_1(x)\Re_m(\overrightarrow{y_{m-1}}(x)), \tag{17}$$

where,

$$\Re_m(\overrightarrow{y_{m-1}}(x)) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N[\varphi(x;q)]}{\partial q^{m-1}}|_{q=0}, \tag{18}$$

and

$$\chi_m = \begin{cases} 0 & m \le 1, \\ 1 & m > 1. \end{cases}$$

4. HOMOTOPY PERTURBATION METHOD (HPM)

The homotopy perturbation method first proposed by He [1]. To illustrate the basic idea of this method, we consider the following nonlinear differential equation

$$A(y) - f(r) = 0, \quad r \in \Omega, \tag{19}$$

under the boundary conditions

$$B\left(y, \frac{\partial y}{\partial n}\right) = 0, \quad r \in \Gamma, \tag{20}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function, Γ is the boundary of the domain Ω. In general, the operator A can be divided into two parts L and N, where L is linear, while N is nonlinear. (19) therefore can be rewritten as follows

$$L(y) + N(y) - f(r) = 0. \tag{21}$$

By the homotopy technique (Liao 1992, 1997). We construct a homotopy $v(r,p) : \Omega \times [0,1] \longrightarrow \mathbb{R}$ which satisfies

$$H(v,p) = (1-p)[L(v) - L(y_0)] + p[A(v) - f(r)] = 0, p \in [0,1]. \tag{22}$$

or

$$H(v,p) = L(v) - L(y_0) + pL(y_0)] + p[N(v) - f(r)] = 0, \tag{23}$$

where $p \in [0,1]$ is an embedding parameter, y_0 is an initial approximation of (19) which satisfies the boundary conditions. From (22), (23) we have

$$H(v,0) = L(v) - L(y_0) = 0, \tag{24}$$
$$H(v,1) = A(v) - f(r) = 0. \tag{25}$$

The changing in the process of p from zero to unity is just that of $v(r,p)$ from $y_0(r)$ to $y(r)$. In topology this is called deformation and $L(v) - L(y_0)$, and $A(v) - f(r)$ are called homotopic. Now, assume that the solution of Eqs. (22), (23) can be expressed as

v = v0 + pv1 + p2v2 + · · · (26)

The approximate solution of (19) can be obtained by Setting $p = 1$.

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \cdots \tag{27}$$

5. THE MAIN RESULTS

In this section, we shall give an uniqueness result of (1), with the condition (2) and prove it. Before starting and proving the main results, we introduce the following hypotheses:

(A1) There exists a constant $L_\Delta > 0$ such that, for any $y_1, y_2 \in C(J,\mathbb{R})$

$$|\Delta(y_1) - \Delta(y_2)| \le L_\Delta |y_1 - y_2|$$

(A2) There exists a function $\Psi^* \in C(D,\mathbb{R}^+)$, the set of all positive function continuous on $D = \{(x,t) \in \mathbb{R}\times\mathbb{R} : 0 \le t \le x \le 1\}$ such that

$$\Psi^* = \sup_{x,t\in[0,1]} \int_0^x |\Psi(x,t)|\, dt < \infty.$$

(A3) The two functions $f, \Theta : J \to \mathbb{R}$ are continuous.

Lemma 5..1 *If $y_0(x) \in C(J,\mathbb{R})$, then $y(x) \in C(J,\mathbb{R}^+)$ is a solution of the problem $(1)-(2)$ iff y satisfying*

$$\begin{aligned} y(x) &= y_0 + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} f(s)y(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\Theta(s)ds \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left(\int_0^s \Psi(s,\tau)\Delta(y(\tau))d\tau\right) ds, \end{aligned}$$

where $y_0 = \sum_{k=0}^{n-1} d_k \frac{x^k}{k!}$.

Our result is based on the Banach contraction principle.

Theorem 5..2 *Assume that (A1), (A2) and (A3) hold. If*

$$\left(\frac{\|f\|_\infty + \Psi^* L_\Delta}{\Gamma(\alpha + 1)}\right) < 1. \tag{28}$$

Then there exists a unique solution $y(x) \in C(J)$ to $(1)-(2)$.

Proof 5..3 *By Lemma 5..1. we know that a function y is a solution to $(1)-(2)$ iff y satisfies*

$$\begin{aligned} y(x) &= y_0 + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} f(s)y(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\Theta(s)ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left(\int_0^s \Psi(s,\tau)\Delta(y(\tau))d\tau\right)ds. \end{aligned}$$

Let the operator $T : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ be defined by

$$\begin{aligned} (Ty)(x) &= y_0 + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} f(s)y(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\Theta(s)ds + \frac{1}{\Gamma(\alpha)} \\ &\times \int_0^x (x-s)^{\alpha-1}\left(\int_0^s \Psi(s,\tau)\Delta(y(\tau))d\tau\right)ds, \end{aligned}$$

we can see that, If $y \in C(J,\mathbb{R})$ is a fixed point of T, then y is a solution of $(1)-(2)$.

Now we prove T has a fixed point y in $C(J,\mathbb{R})$. For that, let $y_1, y_2 \in C(J,\mathbb{R})$ and for any $x \in [0,1]$ such that

$$\begin{aligned} y_1(x) &= y_0 + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} f(s)y_1(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\Theta(s)ds + \frac{1}{\Gamma(\alpha)} \\ &\times \int_0^x (x-s)^{\alpha-1}\left(\int_0^s \Psi(s,\tau)\Delta(y_1(\tau))d\tau\right)ds, \end{aligned}$$

and,

$$\begin{aligned} y_2(x) &= y_0 + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} f(s)y_2(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\Theta(s)ds + \frac{1}{\Gamma(\alpha)} \\ &\times \int_0^x (x-s)^{\alpha-1}\left(\int_0^s \Psi(s,\tau)\Delta(y_2(\tau))d\tau\right)ds. \end{aligned}$$

Consequently, we get

$$\begin{aligned} & |(Ty_1)(x) - (Ty_2)(x)| \\ \leq\ & \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}|f(s)|\,|y_1(s)-y_2(s)|\,ds \\ & + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left(\ \int_0^s |\Psi(s,\tau)|\,|\Delta(y_1(\tau)) - \Delta(y_2(\tau))|\,d\tau\ \right)ds \\ \leq\ & \frac{\|f\|_\infty}{\Gamma(\alpha+1)}|y_1(x)-y_2(x)| + \frac{\Psi^* L_\Delta}{\Gamma(\alpha+1)}|y_1(x)-y_2(x)|\,\Gamma(\alpha+1)\,|y_1(x)-y_2(x)| \\ =\ & \left(\frac{\|f\|_\infty + \Psi^* L_\Delta}{\Gamma(\alpha+1)}\right)|y_1(x)-y_2(x)|. \end{aligned}$$

From the inequality (28) we have

$$\|Ty_1 - Ty_2\|_\infty \leq \left(\frac{\|f\|_\infty + \Psi^* L_\Delta}{\Gamma(\alpha+1)}\right)\|y_1 - y_2\|_\infty.$$

This means that T is contraction map. By the Banach contraction principle, we can conclude that T has a unique fixed point y in $C(J,\mathbb{R})$.

6. NUMERICAL EXAMPLE

In this section, we proposed a numerical solution for nonlinear fractional Volterra integro-differential equations by using the MHAM and MHPM, as shown in Tables 1-3 and Figure 1.

Example 1. Consider the following fractional Volterra integro-differential equation with the boundary conditions:

$$^{c}D^{\alpha}y(x) = y(x) + x(1+e^{x}) + 3e^{x} - \int_{0}^{x} y(t)dt, \;\; 3 < \alpha \leq 4, \;\; x \in [0,1], \tag{29}$$

with the boundary conditions:

$$\begin{aligned} &y(0) = 1, \;\; y(1) = 1+e, \\ &y''(0) = 2, \;\; y''(1) = 3e. \end{aligned} \tag{30}$$

Therefore the exact solution is

$$y(x) = 1 + xe^{x}, \text{ for } \alpha = 4$$

Table 1. Numerical results of the example 1

x	Exact	MHAM	MHPM
0.0	1.000000000	1.000000000	1.000000000
0.1	1.110517092	1.107047479	1.102647336
0.2	1.244280552	1.237721115	1.229286556
0.3	1.404957642	1.395985741	1.384226779
0.4	1.596729879	1.586238703	1.572164823
0.5	1.824360635	1.813384918	1.798234919
0.6	2.093271280	2.082918367	2.068063429
0.7	2.409626895	2.401010221	2.387829229
0.8	2.780432743	2.774603867	2.784330528
0.9	3.213642800	3.211517152	3.205058944
1.0	3.718281828	3.718552210	3.718281829

Table 2. Values of A and B for different values of α by MHAM

	$\alpha = 4$	$\alpha = 3.5$
A	0.96461853025138	1.06172444793295
B	3.43572350417823	1.60468681403168

Table 3. Values of A and B for different values of α by MHPM

	$\alpha = 4$	$\alpha = 3.5$
A	0.9200006577	0.9855569381
B	3.806600712	2.306799253

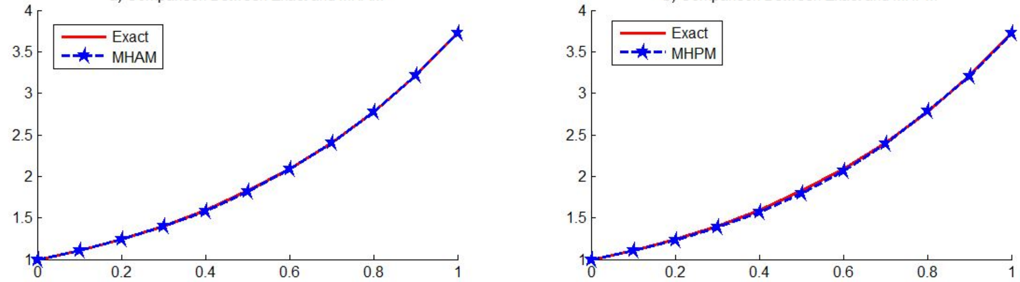

Figure 1. Numerical Results of the Example 1

7. CONCLUSION

In this paper, new modifications of the homotopy perturbation method (HPM) and the homotopy analysis method (HAM) are presented and applied to non-homogeneous fractional Volterra integro-differential equations with boundary conditions. A comparative study between the new modified of homotopy perturbation method (MHPM) and the new modified of homotopy analysis method (MHAM). For this purpose, we showed that the MHAM is more rapid convergence than the MHPM.

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