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A variant of hybrid conjugate gradient methods based on the convex combination for optimization

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ABSTRACT

On some studies a conjugate parameter plays an important role for the conjugate gradient methods. In this paper, a variant of hybrid is provided in the search direction based on the convex combination. This search direction ensures that the descent condition holds. The global convergence of the variant of hybrid is also obtained. Our strong evidence is a numerical analysis showing that the proposed variant of hybrid method is efficient than the Hestenes and Stiefel method.

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1. INTRODUCTION

The conjugate gradient [1] is an adaptation of the optimization method to large-scale problems. Here, we matter with the following problem:

$$Min \;\; f(x) \;,\;\; \mathrm{x} \in \mathrm{R}^{\mathrm{n}} \tag{1}$$

to denote the objective function, and $g(x)$,denote the gradient $f(x)$, respectively.

Many conjugate gradient methods for (1) produce a sequence $\{x_k\}$ by the recurrence:

$$x_0 \in R^n \;, \;\; x_{k+1} = x_k + \alpha_k d_k \tag{2}$$

where d_k is a search direction and α_k is a step size which was calculated by applying line search rules, the most important and famous is the Wolfe conditions:

$$f(x_k + \alpha_k d_k) \le f(x_k) + \delta \alpha_k g_k^T d_k \tag{3}$$

$$d_k^T g(x_k + \alpha_k d_k) \ge \sigma \, d_k^T g_k \tag{4}$$

where $0 < \delta < \sigma$. For details see [2].

Therefore, this study will focus on the CG method whose direction of search is written as:

$$d_{k+1} = -g_{k+1} + \beta_k d_k, \tag{5}$$

where β_k is a scalar known as the CG update. The suitable choice for β_k leads to improve numerical performance conjugate gradient methods. There are some well known and effective conjugate gradient methods, such as, [3-6].

The two most popular formulas know in conjugate gradient methods are as follows:

$$\beta_k^{DY} = \frac{\|g_{k+1}\|^2}{d_k^T y_k} \quad , \quad \beta_k^{HS} = \frac{g_{k+1}^T y_k}{d_k^T y_k} \tag{6}$$

where $y_k = g_{k+1} - g_k$. The first is better in terms of convergence and the second is better in numerical performance. More details can be found in [7, 8].

Yuan [9], claimed that the parameter conjugate gradient method has best numerical performance is given by:

$$\beta_k^{Y} = \frac{g_{k+1}^T y_k}{(f(x_k) - f(x_{k+1}))/\alpha_k - g_k^T d_k / 2} \tag{7}$$

Newly, in [10], presented a modified Yuan method has good convergence property is given by:

$$\beta_k^{B} = \frac{g_{k+1}^T g_{k+1}}{(f(x_k) - f(x_{k+1}))/\alpha_k - g_k^T d_k / 2}. \tag{8}$$

The global convergence property is an important property of conjugate gradient methods, and has as many results as we can see in [1, 11].

The urge of this paper is to get great algorithms we combine the benefits of the two formulas β_k^Y and β_k^B. Provide novel algorithms have fantastic numerical performance with good better convergence and a discussion for the theoretical properties and the numerical results.

2. A CONVEX COMBINATION OF β_k^B AND β_k^Y METHODS

In vision of the gentle convergence property of the β_k^B method, and the efficient performance of the β_k^Y method, we paying attention on designing new methods which possess the above properties simultaneously. The parameter β_k of the hybrid conjugate gradient method of β_k^B and β_k^Y is formulized as:

$$\beta_k^{HYB} = (1 - \vartheta_k)\beta_k^Y + \vartheta_k \beta_k^B \tag{9}$$

So, we get:

$$d_{k+1}^{HYB} = -g_{k+1} + \beta_k^{HYB} s_k, \quad s_k = x_{k+1} - x_k \tag{10}$$

To be determined the scalar parameter ϑ_k, the see it : if $\vartheta_k = 0$, then $\beta_k^{HYB} = \beta_k^Y$ and $\vartheta_k = 1$, then $\beta_k^{HYB} = \beta_k^B$. If $0 < \vartheta_k < 1$, then β_k^{HYB} is a proper convex combination of the parameters is β_k^Y and β_k^B.

Theorem 2.1:

If the relationships (9) and (10) holds, then:

$$d_{k+1}^{HYB} = (1-\vartheta_k)d_{k+1}^{Y} + \vartheta_k d_{k+1}^{B} \tag{11}$$

Proof :

Possessing perspective relationships β_k^B and β_k^Y, the relationship (9) becomes:

$$\beta_k^{HYB} = \frac{(1-\vartheta_k)g_{k+1}^T y_k + \vartheta_k g_{k+1}^T g_{k+1}}{(f(x_k)-f(x_{k+1}))/\alpha_k - g_k^T d_k/2} \tag{12}$$

So, the relation (10) becomes:

$$d_0^{HYB} = -g_0,\; d_{k+1}^{HYB} = -g_{k+1} + \frac{(1-\vartheta_k)g_{k+1}^T y_k + \vartheta_k g_{k+1}^T g_{k+1}}{(f(x_k)-f(x_{k+1}))/\alpha_k - g_k^T d_k/2} s_k \tag{13}$$

In the other mind of a relationship (13) we have:

$$d_{k+1}^{HYB} = -(\vartheta_k g_{k+1} + (1-\vartheta_k)g_{k+1}) + \beta_k^{HYB} s_k, \tag{14}$$

Implies that:

$$d_{k+1}^{HYB} = -(\vartheta_k g_{k+1} + (1-\vartheta_k)g_{k+1}) + ((1-\vartheta_k)\beta_k^Y + \vartheta_k \beta_k^B)s_k \tag{15}$$

The last relation yields:

$$d_{k+1}^{HYB} = \vartheta_k(-g_{k+1} + \beta_k^B s_k) + (1-\vartheta_k)(-g_{k+1} + \beta_k^Y s_k) \tag{16}$$

From (16) we finally conclude:

$$d_{k+1}^{HYB} = (1-\vartheta_k)d_{k+1}^{Y} + \vartheta_k d_{k+1}^{B} \tag{17}$$

Apply conjugacy condition to find value ϑ_k in our method:

$$y_k^T d_k^{HYB} = 0 \tag{18}$$

Multiplying (13) by y_k^T and applying (18) we get:

$$y_k^T\left[-g_{k+1} + \frac{(1-\vartheta_k)g_{k+1}^T y_k + \vartheta_k g_{k+1}^T g_{k+1}}{(f(x_k)-f(x_{k+1}))/\alpha_k - g_k^T d_k/2} s_k\right] = 0 \tag{19}$$

$$-y_k^T g_{k+1} + \frac{(1-\vartheta_k)g_{k+1}^T y_k (y_k^T s_k) + \vartheta_k g_{k+1}^T g_{k+1}(y_k^T s_k)}{(f(x_k)-f(x_{k+1}))/\alpha_k - g_k^T d_k/2} = 0, \tag{20}$$

So,

$$y_k^T g_{k+1} - \frac{g_{k+1}^T y_k}{(f(x_k)-f(x_{k+1}))/\alpha_k - g_k^T d_k/2}(y_k^T s_k) = \vartheta_k\left[\frac{g_{k+1}^T g_{k+1}(y_k^T s_k) - g_{k+1}^T y_k (y_k^T s_k)}{(f(x_k)-f(x_{k+1}))/\alpha_k - g_k^T d_k/2}\right], \tag{21}$$

i.e.

$$\frac{((f(x_k)-f(x_{k+1}))/\alpha_k - g_k^T d_k/2 - y_k^T s_k)}{(f(x_k)-f(x_{k+1}))/\alpha_k - g_k^T d_k/2}(y_k^T g_{k+1}) = \vartheta_k \frac{g_{k+1}^T g_k}{(f(x_k)-f(x_{k+1}))/\alpha_k - g_k^T d_k/2}(y_k^T s_k) \tag{22}$$

Finally,

$$\vartheta_k = \frac{((f(x_k)-f(x_{k+1}))/\alpha_k - g_k^T d_k/2 - y_k^T s_k)(y_k^T g_{k+1})}{g_{k+1}^T g_k (y_k^T s_k)} \tag{23}$$

The appropriate value for ϑ_k ,is in the interval [0, 1]. We get $\beta_k^{HYB} = \beta_k^{Y}$ if $\vartheta_k \le 0$, then set $\vartheta_k = 0$ in (9), and we get $\beta_k^{HYB} = \beta_k^{B}$ if $\vartheta_k \ge 1$, then set $\vartheta_k = 1$ in (9). If it's a value of ϑ_k is specified, β_k^{HYB} it is combines the properties of the Y and the B algorithms in a convex way.

Now we formally state our algorithm and call by HYB:

Algorithm (HYB):

Stage 1. Initialization : $x_1 \in R^n$, $0 < \delta_1 < \delta_2 < 1$, $d_1 = -g_1$ and $\alpha_1 = 1/\|g_1\|$.

Stage 2. If $\|g_{k+1}\| \le 10^{-6}$, then stop.

Stage 3. Compute : α_k by using the Wolfe condition and let $x_{k+1} = x_k + \alpha_k d_k$.

Stage 4. Computation ϑ_k , if $g_{k+1}^T g_k (y_k^T s_k) = 0$, then set $\vartheta_k = 0$, else set ϑ_k as in (23) respectively.

Stage 5. Compute : β_k^{HYB} as in (9) .

Stage 6. Compute : $d_{k+1} = -g_{k+1} + \beta_k s_k$. Put $k = k+1$ go to step 2.

3. CONVERGENCE ANALYSIS

We adopt the following assumption used often in the literature to analyze the global convergence.

Assumption 3.1:

suppose that the gradient is “Lipschitz continuous”. Then for some positive constant L , we have:

$$\|g(n) - g(m)\| \le L\|n - m\|, \ \forall\ n, m \in U\ . \tag{24}$$

where we assume the level set:

$$S = \{x \in R^n \,|\, f(x) \le f(x_0)\} \tag{25}$$

is bounded set. Under these assumptions of $f(x)$, there exists a constant $\Gamma \ge 0$ such that:

$$\|g_{k+1}\| \le \Psi \tag{26}$$

More details can be found in [12, 13].
Now, we will focus the proof of the descent property.

Theorem (3.1)

Assume that (24) and (26) hold and let Wolfe conditions hold. Also, let $\{\|s_k\|\}$ tend to zero, and consider there exist two positive constants ξ_1 , ξ_2 we satisfies:

$$(f(x_k) - f(x_{k+1}))/\alpha_k - g_k^T d_k/2 \ge \xi_1 \|s_k\|^2 , \tag{27}$$

$$\|g_{k+1}\|^2 \leq \xi_2 \|s_k\| \tag{28}$$

then d_k^{HYB} satisfies the $g_{k+1}^T d_{k+1}^{HYB} \leq 0$, for all k.

Proof :

It fists $d_0 = -g_0$. If $k = 0,$ it holds $g_0^T d_0 = -\|g_0\|^2 < 0$. Multiplying (11) by g_{k+1}^T, we obtained:

$$g_{k+1}^T d_{k+1}^{HYB} = (1-\vartheta_k) g_{k+1}^T d_{k+1}^Y + \vartheta_k g_{k+1}^T d_{k+1}^B \tag{29}$$

If $\vartheta_k = 0,$ the relation (29) becomes:

$$g_{k+1}^T d_{k+1}^{HYB} = g_{k+1}^T d_{k+1}^Y \tag{30}$$

So, if $\vartheta_k = 0,$ the hybrid method satisfies the descent condition, if it fists for Y method. By using the conditions of Theorem 3.1, we can prove the descent for Y method. It fists:

$$d_{k+1}^Y = -g_{k+1} + \beta_k^Y s_k \tag{31}$$

Multiplying (31) by g_{k+1}^T, we get:

$$g_{k+1}^T d_{k+1}^Y = -g_{k+1}^T g_{k+1} + \beta_k^Y g_{k+1}^T s_k \tag{32}$$

Using definition β_k^Y, we get:

$$g_{k+1}^T d_{k+1}^Y = -g_{k+1}^T g_{k+1} + \frac{g_{k+1}^T y_k}{(f(x_k) - f(x_{k+1}))/\alpha_k - g_k^T d_k / 2} g_{k+1}^T s_k \tag{33}$$

From (33) we get:

$$g_{k+1}^T d_{k+1}^Y \leq -\|g_{k+1}\|^2 + \frac{\|g_{k+1}\|^2 \|y_k\| \|s_k\|}{(f(x_k) - f(x_{k+1}))/\alpha_k - g_k^T d_k / 2} \tag{34}$$

From Lipschitz condition we have $\|y_k\| \leq L \|s_k\|,$ so:

$$g_{k+1}^T d_{k+1}^Y \leq -\|g_{k+1}\|^2 + \frac{\|g_{k+1}\|^2 L \|s_k\|}{(f(x_k) - f(x_{k+1}))/\alpha_k - g_k^T d_k / 2} \tag{35}$$

But, using (27)-(28) we get:

$$g_{k+1}^T d_{k+1}^Y \leq -\|g_{k+1}\|^2 + \frac{\xi_2 L \|s_k\|}{\xi_1} \tag{36}$$

But, benefit from $\|s_k\| \Rightarrow 0,$ the second part in (36) tends to zero, so there exists a $0 < \delta \leq 1,$ such that:

$$\frac{1}{\xi_1} \xi_2 L \|s_k\| \leq \delta \|g_{k+1}\|^2 \tag{37}$$

Now, from (36) we have:

$$g_{k+1}^T d_{k+1}^Y \le -\|g_{k+1}\|^2 + \delta\|g_{k+1}\|^2, \tag{38}$$

i.e.

$$g_{k+1}^T d_{k+1}^Y \le -(1-\delta)\|g_{k+1}\|^2 < 0 \tag{39}$$

If either $\vartheta_k = 1$, the relationship (29) we have:

$$g_{k+1}^T d_{k+1}^{HYB} = g_{k+1}^T d_{k+1}^B \tag{40}$$

But, the descent property of B-method is proved in [10], by using Wolfe conditions.

Now, let $0 < \vartheta_k < 1$ and from (29), we get:

$$g_{k+1}^T d_{k+1}^{HYB} \le (1-\vartheta_k) g_{k+1}^T d_{k+1}^Y + \vartheta_k g_{k+1}^T d_{k+1}^B \tag{41}$$

We obviously can conclude now:

$$g_{k+1}^T d_{k+1}^{HYB} \le 0 \tag{42}$$

We adopt the next theorem used commonly in the research literatures.

Theorem (3.2)

Let the iterative method of the form (2) and (5), where d_k satisfies a $g_k^T d_k < 0$ and α_k satisfies strong Wolfe conditions. If (24) holds, then either,

$$\liminf_{k\to\infty} \|g_k\| = 0 \tag{43}$$

Or,

$$\sum_{k\ge 1} \frac{\|g_{k+1}\|^4}{\|d_{k+1}\|^2} < \infty \tag{44}$$

It was originally given by Zoutendijk [14].

Theorem (3.3)

Let conditions of Theorem 3.2 holds. Then either $g_k = 0$ for some k, or,

$$\liminf_{k\to\infty} \|g_k\| = 0 \tag{45}$$

Proof :

Let $g_k \ne 0$, $\forall k$. Using, a contrary to prove (45), that there exists a number $c > 0$, such that:

$$g_{k+1} \ge c, \ \forall\ k \tag{46}$$

From (11) we get:

$$\left\|d_{k+1}^{HYB}\right\| \leq \left\|d_{k+1}^{Y}\right\| + \left\|d_{k+1}^{B}\right\| \tag{47}$$

Next, it holds:

$$\left\|d_{k+1}^{B}\right\| \leq \left\|g_{k+1}\right\| + \left|\beta_k^B\right| \left\|s_k\right\| \tag{48}$$

From (9), (26)-(28), and (48) we get:

$$\left\|d_{k+1}^{B}\right\| \leq \Psi + \frac{\xi_2}{\xi_1} \tag{49}$$

Also,

$$\left\|d_{k+1}^{Y}\right\| \leq \left\|g_{k+1}\right\| + \left|\beta_k^Y\right| \left\|s_k\right\| \tag{50}$$

Using (26)-(28)and (50) we get:

$$\left\|d_{k+1}^{Y}\right\| \leq \Psi + \frac{\Psi L}{\xi_1} \tag{51}$$

So, using (47), (49) and (51) we get:

$$\left\|d_{k+1}^{HYB}\right\| \leq 2\Psi + \frac{\Psi L}{\xi_1} + \frac{\xi_2}{\xi_1} \tag{52}$$

We obtained:

$$\frac{\left\|g_{k+1}\right\|^4}{\left\|d_{k+1}\right\|^2} \geq \frac{c^4}{\left[2\Psi + \frac{\Psi L}{\xi_1} + \frac{\xi_2}{\xi_1}\right]^2} \tag{53}$$

where from:

$$\sum_{k\geq 1}^{\infty} \frac{\left\|g_{k+1}\right\|^4}{\left\|d_{k+1}\right\|^2} = \infty \tag{54}$$

Applying the Theorem 3.2, we obtain a contradiction. The proof is finish.

4. NUMERICAL RESULTS AND DISCUSSION

In this section, we report some numerical results with the proposed method and HS-method. We test the performance of Algorithm 2 on the following (15) problems with various sizes. Using Fortran 90 to encrypt these methods. In our application, we choose the following parameters : $\delta_1 = 0.001$ and $\delta_2 = 0.9$ Are selected examination problems of references [15]. Stop state is $\left\|g_{k+1}\right\| \leq 10^{-6}$.

It has been reported numerical results in Table 1. Represents the first column and the name of the second problem and its dimensions in [15], respectively. Other test functions have been used in various researches such as [16-25] "NI and NR and NF in the table indicates the number of iterations and the number of evaluations of jobs and the number of replay restart, respectively". Relativa efficiency of the variant of hybrid Algorithm as shown in Table 2.

Table 1. Comparing different conjugate gradient methods with different test functions

		HS algorithm			HYB algorithm		
P. No.	n	NI	NR	NF	NI	NR	NF
Trigonometric	100	19	10	35	20	11	37
	1000	39	22	67	34	22	63
Perturbed Quadratic	100	102	33	155	93	26	148
	1000	326	91	509	317	92	509
Extended PSC1	100	8	6	17	8	6	17
	1000	26	25	505	7	5	15
Q. Diagonal Perturbed	100	47	6	84	53	9	95
	1000	188	29	335	173	24	309
Extended Wood	100	32	13	60	34	11	64
	1000	28	11	54	28	9	49
ARWHEAD (CUTE)	100	12	7	23	8	5	16
	1000	17	11	148	8	6	56
NONDIA (CUTE)	100	17	9	34	13	7	26
	1000	12	7	25	13	7	27
Partial Perturbed Q.	100	82	23	124	76	21	118
	1000	260	57	429	259	65	429
LIARWHD (CUTE)	100	21	12	40	19	11	34
	1000	20	12	45	19	11	44
DENSCHNF (CUTE)	100	20	17	36	18	17	32
	1000	22	17	41	18	16	33
SINCOS	100	8	6	17	8	6	17
	1000	26	25	505	7	5	15
Generalized quartic GQ2	100	34	10	55	31	7	54
	1000	38	12	63	36	9	63
Diagonal 2	100	61	18	103	66	26	105
	1000	192	57	321	180	63	303
Extended Three Expo Terms	100	13	7	20	8	5	13
	1000	26	19	260	28	21	365
Quadratic QF1	100	100	29	152	95	31	144
	1000	371	105	586	375	110	604
Total		2095	684	4730	1985	648	3687

From Table 1, it is completed that the more efficient algorithm in terms of the number of iterations is our method, being the best for 6% of the problems, followed by number of function evaluations, that was the efficient for nearly 22% of the problems and 6% in terms of the number of restart calls .

Table 2. Relativa efficiency of the variant of hybrid algorithm

	NI	NR	NF
HS algorithm	100 %	100 %	100 %
HYB algorithm	94.74 %	94.73 %	77.94 %

5. CONCLUSIONS

The variant of hybrid of the conjugate gradient direction is based on the convex combination. The search direction in the variant of hybrid satisfies as a descent condition and the global convergence is also obtained. Show that variant of hybrid method are the best for solving unconstrained optimization problems.

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