

Independent Number and Dominating Number of (n,k) -Star Graphs

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Abstract

In Graph Theory, independent number and, dominating number are three of the important parameters to measure the resilience of graphs, respectively denoted by $\alpha(G)$ and $\gamma(G)$ for a graph G . But predecessors have proved that computing them are very hard. So computing $\alpha(G)$ and $\gamma(G)$ of some particular known graphs is extremely valuable. In this paper, we determine $\alpha(G)$ and $\gamma(G)$ of (n,k) -star graphs, denoted by $S_{n,k}$, followed by some relative conclusions of n -star, denoted by S_n as the isomorphism of $S_{n,n-1}$. In addition, our method giving dominating set of $S_{n,k}$ is more easily understood than [7], which presented a broadcast algorithm to determine dominating set of $S_{n,k}$.

Keywords: (n,k) -star graph, independent number, dominating number

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1. Introduction

It is widely known that independent number, dominating number and bondage number are the important parameters to measure the resilience of graphs. Next, we see their conception:

Definition 1.1 Let G be a graph, and I be a nonempty subset of $V(G)$, then I is one independent set of G if any two nodes of I is not adjacent in G . Moreover, we call that $|I|$ is independent number of G if $|I|$ is maximum in all independent sets of G , denoted by $\alpha(G)$.

Definition 1.2 Let G be a graph, and S be a nonempty subset of $V(G)$, then S is one dominating set of G if all nodes of G is either in S , or adjacent to a node of S . Moreover, we call that $|S|$ is dominating number of G if $|S|$ is minimum in all dominating sets of G , denoted by $\gamma(G)$.

Clearly, a maximum independent set of graph G is a dominating set of G by Definition 1.1 and Definition 1.2, so it is easy to get $\gamma(G) \leq \alpha(G)$.

In a graph, predecessors have shown that computing $\alpha(G)$ and $\gamma(G)$ are extremely difficult. So computing $\alpha(G)$ and $\gamma(G)$ of some particular known graphs is very valuable. For example, the (n,k) -star graphs was first proposed in 1995 by W.K Chiang et al [1]. Because of good topological properties of $S_{n,k}$, its many properties have been researched such as diameter and connectivity [1] [8], pancyclicity [2], $k_s^{(1)}(G)$ and $k_s^{(2)}(G)$ [3] [5] [6], fault hamiltonicity and fault hamiltonicity connectivity [4] and the others issue [9][10]. In this paper,

we determine $\alpha(G)$ and $\gamma(G)$ of (n, k) -star graphs, so that can get $\alpha(S_n)$ and $\gamma(S_n)$ and of n -star, denoted by S_n .

2. Preliminaries

For given integers n and k , where $1 \leq k \leq n-1$, let $J_n = \{1, 2, \dots, n\}$ and let $P(n, k)$ be the set of k -permutations on J_n for $1 \leq k \leq n-1$, that is, $P(n, k) = \{p_1 p_2 \dots p_k : p_i \in J_n, p_i \neq p_j, 1 \leq i \neq j \leq k\}$.

Definition 2.1 The (n, k) -star graph, denoted by $S_{n,k}$, is an undirected graph with vertex-set $P(n, k)$. The adjacency is defined as follows: a vertex $p_1 p_2 \dots p_i \dots p_k$ is adjacent to a vertex

- (1) $p_1 p_2 \dots p_{i-1} p_i p_{i+1} \dots p_k$, where $2 \leq i \leq k$ (swap p_1 with p_i).
- (2) $x p_2 \dots p_k$, where $x \in J_n - \{p_i : 1 \leq i \leq k\}$ (replace p_1 by x).

Figure 1 shows a $(4,2)$ -star graph $S_{4,2}$.

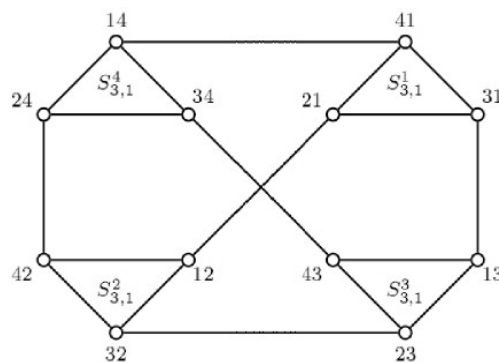


Figure 1. The structure of a $(4,2)$ -star graph $S_{4,2}$

The edges of type (1) are referred to as i -edges ($2 \leq i \leq k$), and the edges of type (2) are referred to as 1-edge. The vertices of type (1) are referred to as swap-adjacent vertices, and the vertices of type (2) are referred to as unswap-adjacent vertices. We also call i -edge as swap-edge, and call 1-edges as unswap-edge. Clearly, every vertex in $S_{n,k}$ has $(k-1)$ swap-adjacent vertices and $(n-k)$ unswap-adjacent vertices. Usually, if $v = p_1 p_2 \dots p_k$ is a vertex in $S_{n,k}$, we call that p_i is the i -th bit for each $i = 1, 2, \dots, k$.

By Definition 2.1, we know $S_{n,n-1} \cong S_n$ and $S_{n,1} \cong K_n$ where S_n is n -star graph and K_n is complete graph with order n . So $S_{n,k}$ is a generalization of S_n . It has been shown by Chiang and Chen [1] that S_n is an $(n-1)$ -regular, $(n-1)$ -connected vertex-transitive graph with $n!/(n-k)!$ vertices.

Lemma 2.2 For any $\alpha \in P(n, k-1)$ ($k \geq 2$), let $V_\alpha = \{p\alpha : p \in J_n \setminus \alpha\}$. Then the subgraph of $S_{n,k}$ induced by V_α is a complete graph K_{n-k+1}^α , denoted by K_{n-k+1}^α .

Proof. For any two vertices $p\alpha$ and $q\alpha$ in V_α with $p \neq q$, by the condition (2) of definition 2.1, $p\alpha$ and $q\alpha$ are linked in $S_{n,k}$ by an unswap-edge. Thus, the subgraph of $S_{n,k}$ induced by V_α is a complete graph K_{n-k+1} . □

Let $|P(n, k-1)| = p(n, k-1)$. The vertex-set $P(n, k)$ of $S_{n,k}$ can be decomposed into $p(n, k-1)$ subsets, each of which induces a complete graph by Lemma 2.2. It is clear that, for any two distinct elements x and y in $P(n, k)$, if they are in different complete subgraphs K_{n-k+1}^α and K_{n-k+1}^β ($\alpha \neq \beta$), then there is at most one swap-edge between x and y in $S_{n,k}$, which is an i -edge if and only if α and β differ only in the i -th bit. Thus, we have the following conclusion.

Lemma 2.3 *The vertex-set of $S_{n,k}$ can be partitioned into $|P(n, k-1)|$ subsets, each of which induces a complete graph of order $(n-k+1)$. Furthermore, there is at most one swap-edge between any two complete graphs.*

Let $S_{n,k}^i$ denote a subgraph of $S_{n,k}$ induced by all vertices with the last symbol i for some $i \in J_n$.

Lemma 2.4 (Chiang and Chen [1]) *$S_{n,k}$ can be decomposed into n subgraphs $S_{n-1,k-1}^i$, which is isomorphic to $S_{n-1,k-1}$, for each $i \in J_n$. Moreover, there are $\frac{(n-2)!}{(n-k)!}$ k -edges between $S_{n-1,k-1}^i$ and $S_{n-1,k-1}^j$, which forms a matching between them, for any $i, j \in J_n$ with $i \neq j$.*

Corollary 2.5 *An $S_{n,2}$ can be decomposed into n subgraphs $S_{n-1,1}^i$, which is isomorphic to complete K_{n-1} , for each $i \in J_n$.*

3. Independent number of $S_{n,k}$

In this section, we mainly determine the independent number of $S_{n,k}$. Since $S_{n,1} \cong K_n$, we only consider the case $k \geq 2$, in the following discussion.

Lemma 3.1 $\alpha(S_{n,k}) \leq \frac{n!}{(n-k+1)!}$ for $2 \leq k \leq n-1$.

Proof. Assume that $I_k (\subset V(S_{n,k}))$ is a maximum independent set of $S_{n,k}$, then $|I_k| = \alpha(S_{n,k})$ by definition 1.1. If $\alpha(S_{n,k}) > \frac{n!}{(n-k+1)!}$, then there are at least two nodes of I_k , which are from one K_{n-k+1}^α ($\alpha \in P(n, k-1)$) since $S_{n,k}$ can be decomposed into different $p(n, k-1) (= \frac{n!}{(n-k+1)!})$ complete graphs K_{n-k+1}^α ($\alpha \in P(n, k-1)$) by Lemma 2.3, so that the two nodes is adjacent. Thus, it is contrary to definition 1.1. □

Lemma 3.2 $\alpha(S_{n,2}) = n$ for $n \geq 3$.

Proof. Let $I_2 (\subset V(S_{n,2}))$ be a maximum independent set of $S_{n,2}$, then $|I_2| = \alpha(S_{n,2})$ by definition 1.1. By Lemma 3.1, we get $|I_2| = \alpha(S_{n,2})$. Thus, Lemma 3.2 can be proved if we can construct an I_2 , so that $|I_2| = n$.

By Lemma 2.5, let $I_2 = \{12, 23, \dots, (n-1)n, n1\}$, then each vertex of I_2 is from different complete $K_{n-1}^\alpha (\alpha \in P(n,1))$, clearly, any two nodes of I_2 is not adjacent in $S_{n,2}$ by Definition 2.1, and $|I_2| = n$. \square

Theorem 3.3 $\alpha(S_{n,k}) = \frac{n!}{(n-k+1)!}$ for $2 \leq k \leq n-1$.

Proof. By Lemma 3.1, we get $|I_k| = \alpha(S_{n,k}) \leq \frac{n!}{(n-k+1)!}$. Thus, Theorem 3.3 can be shown if we can construct an I_k , so that $|I_k| = \frac{n!}{(n-k+1)!}$.

Let $I_k^{(i,j)}$ be a vertex-set, which includes the vertices of I_k if the vertices don't include element i , and $I_k^{(i,j)}$ includes the vertices of I_k if the vertices include element i but swap i with j .

Step 1: In $S_{n-k+2,2}$, by Lemma 3.2, let $I_2 = \{12, 23, \dots, (n-k+1)(n-k+2), (n-k+2)1\}$. Clearly, $|I_2| = n-k+2$;

Step2: In $S_{n-k+3,3}$, let $I_{3(n-k+3)} = \{\beta(n-k+3) | \beta \in I_2\}$ and $I_{3x} = \{\beta x | x \in J_{n-k+2}, \beta \in I_2^{(x,n-k+3)}\}$. Now, we let $I_3 = \sum_{x \in J_{n-k+3}} I_{3x}$, clearly, $|I_{3x}| = (n-k+3)(n-k+2); \dots$

Step k: In $S_{n-k+k,k}$, let $I_{k(n-k+k)} = \{(n-k+k) | \beta \in I_{k-1}\}$ and $|I_{kx}| = \{\beta x | x \in J_{n-k+(k-1)}, \beta \in I_{k-1}^{(x,n-k+k)}\}$. Now, we let $I_k = \sum_{x \in J_{n-k+k}} I_{kx}$, clearly, $|I_k| = n(n-1) \dots (n-k+3)(n-k+2) = \frac{n!}{(n-k+1)!}$.

In step $i (i \in J_k)$, it is easy to verify that any two vertices of I_i are not adjacent in $S_{n-k+i,i}$ by the rules of our construction. \square

Corollary 3.4 In n-star graph S_n , $\alpha(S_n) = \frac{n!}{2}$ for $n \geq 2$.

4. Dominating number of $S_{n,k}$

In this section, we mainly determine the dominating number of $S_{n,k}$. Similarly, since $S_{n,1} \cong K_n$, we only consider the case $k \geq 2$ in the following discussion.

Lemma 4.1 $\gamma(S_{n,k}) \geq \frac{(n-1)!}{(n-k)!}$ for $2 \leq k \leq n-1$.

Proof. Let $S (\subset V(S_{n,k}))$ be a minimum dominating set of $S_{n,k}$, then $|S| = \gamma(S_{n,k})$ by definition 1.2. By definition 2.1, we have known that $S_{n,k}$ is a $(n-1)$ -regular graph, so each vertex of S can at most dominate $(n-1)$ vertices in $S_{n,k} - S$. If $\gamma(S_{n,k}) \leq \frac{(n-1)!}{(n-k)!} - 1$ then

S can at most dominate $\left[\frac{(n-1)!}{(n-k)!} - 1 \right] (n-1)$ vertices in $S_{n,k} - S$. Thus,

$$\begin{aligned} |S| + |V(S_{n,k} - S)| &\leq \left[\frac{(n-1)!}{(n-k)!} - 1 \right] (n-1) + \frac{(n-1)!}{(n-k)!} - 1 \\ &= \frac{n!}{(n-k)!} - n < \frac{n!}{(n-k)!} = |V(S_{n,k})| \end{aligned}$$

It is contrary to the definition of dominating number. \square

Theorem 4.2 $\gamma(S_{n,k}) = \frac{(n-1)!}{(n-k)!}$ for $2 \leq k \leq n-1$.

Proof. By Lemma 4.1, we have shown $|S| = \gamma(S_{n,k}) \geq \frac{(n-1)!}{(n-k)!}$. Thus, by definition 1.2,

Theorem 4.2 can be proved if we can construct a dominating set S , so that $|S| = \frac{(n-1)!}{(n-k)!}$.

We now split $V(S_{n,k})$ into 3 vertex-subsets: $V_n = \{n\alpha \mid \alpha \in P(n-1, k-1)\}$, $V'_n = \{\alpha \mid \alpha \in P(n-1, k)\}$ and $V''_n = \{p_1 p_2 \dots p_{a-1} n p_{a+1} \dots p_k \mid p_i \in J_{n-1}, a \geq 2\}$. It is easy to verify that V_n, V'_n and V''_n have no intersection, and $|V_n| + |V'_n| + |V''_n| = |V(S_{n,k})| = \frac{n!}{(n-k)!}$ since

$$|V_n| = \frac{(n-1)!}{(n-k)!}, |V'_n| = \frac{(n-1)!}{(n-k-1)!} \text{ and } |V''_n| = \frac{(n-1)!}{(n-k)!} (k-1).$$

Let $p_1 p_2 \dots p_k$ be any one vertex of V_n , then all neighboring-edges of $p_1 p_2 \dots p_k$ must have one unswap-edge connected to $n p_2 \dots p_k$ of V_n

Let $p_1 p_2 \dots p_{a-1} n p_{a+1} \dots p_k$ be any one vertex of V''_n , then all neighboring-edges of $p_1 p_2 \dots p_{a-1} n p_{a+1} \dots p_k$ must have one swap-edge connected to $n p_2 p_{a-1} p_1 p_{a+1} \dots p_k$ of V_n .

Thus, we can let $V_n = S$, and $|V_n| = \frac{(n-1)!}{(n-k)!}$. \square

Corollary 4.3 In n -star graph S_n , $\gamma(S_n) = (n-1)!$.

Corollary 4.4 If let $V_x = \{x p_1 p_2 \dots p_{k-1} \mid p_j \in J_n \setminus x\} (x \in J_n)$, then each V_x is a minimum dominating set of $S_{n,k}$ for $x = 1, 2, \dots, n$.

Corollary 4.5 If S is a minimum dominating set of $S_{n,k}$, then any two vertices of S aren't adjacent in $S_{n,k}$, and any two neighboring-vertices of S aren't common.

Proof. Let v_1 and v_2 be any two vertices of S , if v_1 and v_2 are adjacent in $S_{n,k}$, then v_1 and v_2 can at most dominate $2n-4$ vertices of $S_{n,k} - S$ since either v_1 or v_2 only dominate $n-2$ vertices of $S_{n,k} - S$. Thus, we can get that S can at most dominate

$(|S|-2)(n-1)+2n-4$ vertices of $S_{n,k}-S$, and can get $|S|+(|S|-2)(n-1)+2n-4=n|S|-2=|V(S_{n,k})|-2 < |V(S_{n,k})|$, a contradiction.

If there exist two neighboring-vertices of S who are common, then S can at most dominate $|S|(n-1)-1$ vertices of $S_{n,k}-S$. Therefore, we have $|S|+|S|(n-1)-1=|V(S_{n,k})|-1 < |V(S_{n,k})|$, a contradiction. \square

In any case, in Graph Theory, it is rather difficult to compute independent number and dominating number of the graphs. Up to now, the conclusions in this respect are confined only to a few specific graphs such as cube, hypercube and so on. Thus, the paper is very valuable since it solves independent number and dominating number of (n,k) -star graphs and n -star graphs.

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