Investigation of Practical Representation and Parameterization of the Rational Cubic Conic Sections

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Abstract
This paper presents a practical representation containing a parameter of rational cubic conic sections and research's deeply the inner properties. Firstly, the parameter how to affect the control points, inner weights and shoulder point is discussed. Secondly, the inner relation between the parameter and the weights of the nonstandard-form quadratic rational conic sections is analyzed in detail. Change in the parameter value actually corresponds to a rational linear parameter transformation. Finally, we discuss the inverse calculation of the cubic rational conic sections and obtain the inverse calculation methods suitable for engineering applications.

1. Introduction

Non-uniform rational B-splines provide an exact representation for the family of conic or quadric curves (ellipse, parabol and hyperbola) [1]-[4]. As a form of NURBS methods, the rational Bezier methods play a very important role in practical applications. Designers often need to reparametrize the curves without changing the shape of the curves. The non-standard rational Bezier curves can generally be achieved by following ways[5,6]: to maintain the same control points and change the weights by projective transformation of the parameter space, to add the control points and change the weights by increasing the curves degree and re-parameterizing the curves, to add the control points and change the weights by improperly parameterizing the curve through a non-linear parameters transformation. However, for the standard rational Bezier curves, some basic problems (in particular the case of infinite representation methods of the standard rational cubic Bezier curves) have not yet been resolved. Guojin Wang [7],Kaihuai Qin [8] and Qiang Li[9] pointed out that a circular arcs has infinite representation methods in standard rational cubic Bezier form. Houjun Hang and Wanggen Li[10,11] discussed deeply the practical methods and inner properties of representing circular arcs with standard rational cubic Bezier curves.

On the basis of the paper[11], we research deeply the practical representation and inner properties of the conic section to present a series of important results and point out a conic section has infinite representation methods in(standard) rational cubic Bezier form. The different (standard) rational cubic Bezier representations of a conic section correspond actually to rational linear parameter transformation. Finally,we discuss several inverse calculation problems and obtain the inverse calculation methods suitable for engineering applications.

2. Practical representation of the rational cubic conic sections

Theorem 1 Let $\bar{b}_0, \bar{b}_1, \bar{b}_2$ are control points of standard rational quadratic Bezier representation of the conic section Γ.The inner weight is $\omega (> -1/2$ and $\neq 0)$.Then the conic section Γ can be represented by the following rational cubic Bezier equation with a parameter.

$$P(u) = \frac{\sum_{j=0}^{3} \omega_j \boldsymbol{b}_j \; B_{j,3}(u)}{\sum_{j=0}^{3} \omega_j B_{j,3}(u)}, \qquad 0 \le u \le 1 \tag{1}$$

where $\boldsymbol{b_0} = \bar{\boldsymbol{b}}_0, \boldsymbol{b_3} = \bar{\boldsymbol{b}}_2$, ω_i (i=0,1,2,3) are weights, and $\omega_0 > 0, \omega_3 > 0$.

$$\boldsymbol{b_1} = \frac{(1 + 2\lambda\omega(1 + 2\omega))\bar{\boldsymbol{b}}_0 + (2\omega - 2\lambda\omega(1 + 2\omega))\bar{\boldsymbol{b}}_1}{(1 + 2\omega)}$$

$$\boldsymbol{b_2} = \frac{(1 - \lambda(1 + 2\omega))\bar{\boldsymbol{b}}_2 + (2\omega + 4\lambda\omega^2(1 + 2\omega))\bar{\boldsymbol{b}}_1}{(1 + 2\omega)(1 - \lambda(1 - 4\omega^2))}$$

$$\omega_1 = \frac{1}{3}\sqrt[3]{\omega_3\omega_0^2} \; \frac{1 + 2\omega}{1 - \lambda(1 + 2\omega)} \cdot \sqrt[3]{\frac{1 - \lambda(1 + 2\omega)}{1 + 2\lambda\omega(1 + 2\omega)}}$$

$$\omega_2 = \frac{1}{3}\sqrt[3]{\omega_0\omega_3^2} \; \frac{(1 + 2\omega)(1 - \lambda(1 - 4\omega^2))}{1 + 2\lambda\omega(1 + 2\omega)} \cdot \sqrt[3]{\frac{1 + 2\lambda\omega(1 + 2\omega)}{1 - \lambda(1 + 2\omega)}} \tag{2}$$

$$\lambda \in \begin{cases} (-\dfrac{1}{2\omega(1 + 2\omega)}, \dfrac{1}{1 + 2\omega}) & \omega > 0 \\[3mm] (-\infty, \dfrac{1}{1 - 4\omega^2}) & -\dfrac{1}{2} < \omega < 0 \end{cases}$$

Proof. Let $\dfrac{\sum_{j=0}^{2} \bar{\omega}_j \bar{\boldsymbol{b}}_j B_{j,2}(u)}{\sum_{j=0}^{2} \bar{\omega}_j B_{j,2}(u)} = \dfrac{\sum_{j=0}^{3} \omega_j \boldsymbol{b}_j B_{j,3}(u)}{\sum_{j=0}^{3} \omega_j B_{j,3}(u)}$ .Then

$$\sum_{j=0}^{2} \bar{\omega}_j \bar{\boldsymbol{b}}_j B_{j,2}(u) \cdot \sum_{j=0}^{3} \omega_j B_{j,3}(u) = \sum_{j=0}^{3} \omega_j \boldsymbol{b}_j B_{j,3}(u) \cdot \sum_{j=0}^{2} \bar{\omega}_j B_{j,2}(u),$$

Comparing the coefficient of $u^i(1-u)^{5-i}$,we have

$$(2\bar{\omega}_1\omega_0 - 3\frac{2\omega - 2\lambda\omega(1 + 2\omega)}{1 + 2\omega}\bar{\omega}_0\omega_1)(\bar{\boldsymbol{b}}_1 - \bar{\boldsymbol{b}}_0) = 0$$

$$(2\bar{\omega}_1\omega_3 - 3\frac{2\omega + 4\lambda\omega^2(1 + 2\omega)}{(1 + 2\omega)(1 - \lambda(1 - 4\omega^2))}\bar{\omega}_2\omega_2)(\bar{\boldsymbol{b}}_1 - \bar{\boldsymbol{b}}_2) = 0$$

So

$$\overline{\omega}_1\omega_0 - 3\frac{\omega - \lambda\omega(1+2\omega)}{1+2\omega}\overline{\omega}_0\omega_1 = 0$$

$$\overline{\omega}_1\omega_3 - 3\frac{\omega + 2\lambda\omega^2(1+2\omega)}{(1+2\omega)(1-\lambda(1-4\omega^2))}\overline{\omega}_2\omega_2 = 0$$

We obtain

$$\frac{\overline{\omega}_1}{\overline{\omega}_0} = \omega \cdot \sqrt[3]{\frac{\omega_3(1-\lambda(1+2\omega))}{\omega_0(1+2\lambda\omega(1+2\omega))}} \tag{3}$$

$$\frac{\overline{\omega}_1}{\overline{\omega}_2} = \omega \cdot \sqrt[3]{\frac{\omega_0(1+2\lambda\omega(1+2\omega))}{\omega_3(1-\lambda(1+2\omega))}} \tag{4}$$

It is easily checked that $\dfrac{\overline{\omega}_0\overline{\omega}_2}{\overline{\omega}_1^2} = \dfrac{1}{\omega^2}$ .So equation (1) represents the conic section Γ .

Theorem 2 Let $\overline{b}_0, \overline{b}_2$ are control points of standard rational quadratic Bezier representation of the semi- ellipse Γ .Vector $\overrightarrow{b}$ is parallel to the conjugate diameter of $\overline{b}_0\overline{b}_2$ and has the half length of the conjugate diameter. Then the semi-ellipse Γ can be represented by the following rational cubic Bezier equation with a parameter.

$$\boldsymbol{P}(u) = \frac{\sum\limits_{j=0}^{3}\omega_j\boldsymbol{b}_j B_{j,3}(u)}{\sum\limits_{j=0}^{3}\omega_j B_{j,3}(u)}, \quad 0 \le u \le 1 \tag{5}$$

where $\boldsymbol{b}_0 = \overline{b}_0, \boldsymbol{b}_3 = \overline{b}_2$, ω_i (i=0,1,2,3) are weights, and $\omega_0 > 0, \omega_3 > 0$.

$$\boldsymbol{b_1} = \overline{b}_0 + 2(1-\lambda)\overrightarrow{b}, \boldsymbol{b_2} = \overline{b}_2 + \frac{2}{1-\lambda}\overrightarrow{b}, \omega_1 = \frac{1}{3}\sqrt[3]{\frac{\omega_3\omega_0^2}{(1-\lambda)^2}}, \omega_2 = \frac{1}{3}\sqrt[3]{\omega_0\omega_3^2(1-\lambda)^2}, -\infty < \lambda < 1$$

Proof.Let $\dfrac{(1-u)^2\overline{\omega}_0\overline{b}_0 + u^2\overline{\omega}_2\overline{b}_2}{(1-u)^2\overline{\omega}_0 + u^2\overline{\omega}_2} + \dfrac{2u(1-u)}{(1-u)^2\overline{\omega}_0 + u^2\overline{\omega}_2}\overrightarrow{b} = \dfrac{\sum\limits_{j=0}^{3}\omega_j\boldsymbol{b}_j B_{j,3}(u)}{\sum\limits_{j=0}^{3}\omega_j B_{j,3}(u)}$

Comparing the coefficient of $u^i(1-u)^{5-i}$,we have

$$\overline{\omega}_0 = \sqrt[3]{\frac{\omega_0}{\omega_3(1-\lambda)}}, \overline{\omega}_2 = \sqrt[3]{\frac{\omega_3}{\omega_0}(1-\lambda)} \tag{6}$$

here $\overline{\omega}_1 = 0$ .So equation (5) represents the semi-ellipse Γ .

For $\omega_0 = \omega_3 = 1$, (1) is standard rational cubic Bezier representation with a parameter of the conic section.

Because

$$1 - \lambda(1 - 4\omega^2) > 0, 1 + 2\lambda\omega(1 + 2\omega) > 0,$$

$$1 + 2\omega > 0, 1 - \lambda(1 + 2\omega) > 0,$$

So it is clear that $\omega_1 > 0, \omega_2 > 0$.

Figure 1 shows the inner weights ω_1, ω_2 curves of λ. Where (a) for $\omega = 1, \omega_0 = \omega_3 = 1$ (b) for $\omega_0 = \omega_3 = 1, \omega = 0$ (c) for $\omega_0 = \omega_3 = 1, \omega = -1/4$
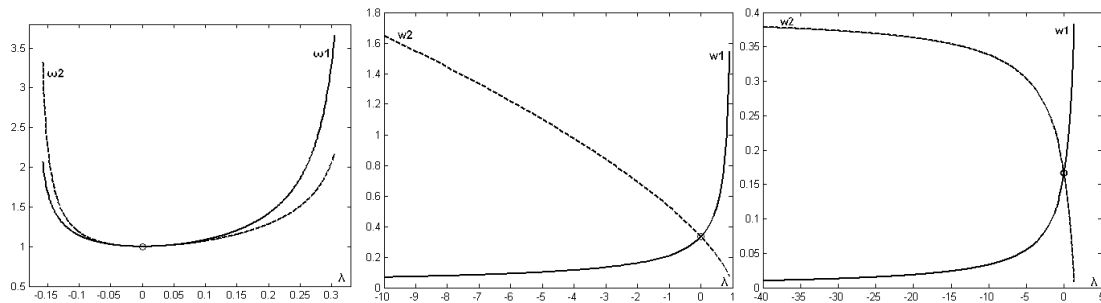

Figure 1 Inner weights curve of λ

(a)for $\omega = 1, \omega_0 = \omega_3 = 1$ (b)for $\omega_0 = \omega_3 = 1, \omega = 0$ (c)for $\omega_0 = \omega_3 = 1, \omega = -1/4$

3. Parametrization Analysing

From (3),(4) and (6), if a rational cubic Bezier representation of the conic section is given, we can get the rational quadratic Bezie equation that has same parametrization effect.

Consider two rational cubic Bezier representation of the conic section $\lambda = \lambda_1$ and $\lambda = \lambda_2$, the correspondent weights separately are $\omega_0^1, \omega_1^1, \omega_2^1, \omega_3^1$ and $\omega_0^2, \omega_1^2, \omega_2^2, \omega_3^2$. Using (3), (4) and (6), we get separately weights $\overline{\omega}_0^1, \overline{\omega}_1^1, \overline{\omega}_2^1$ and $\overline{\omega}_0^2, \overline{\omega}_1^2, \overline{\omega}_2^2$ under non-standard rational quadratic Bezier representation. Then the rational linear parameter transformation that has same parametrization effect as follows

$$u = \frac{\overline{u}}{\sqrt{\dfrac{\overline{\omega}_2^1 \overline{\omega}_0^2}{\overline{\omega}_0^1 \overline{\omega}_2^2}}(1 - \overline{u}) + \overline{u}} , \tag{7}$$

From(3),(4),we can represent（7）with λ_1, λ_2

$$u = \frac{\overline{u}}{\Delta(1 - \overline{u}) + \overline{u}} \tag{8}$$

Where

$$\Delta = \sqrt[3]{\frac{\omega_3^1 \omega_0^2 (1 - \lambda_1(1 + 2\omega))(1 + 2\lambda_2\omega(1 + 2\omega))}{\omega_0^1 \omega_3^2 (1 + 2\lambda_1\omega(1 + 2\omega))(1 - \lambda_2(1 + 2\omega))}}$$

u is the parameter for $\lambda = \lambda_1$ and $\overline{u}$ is the parameter for $\lambda = \lambda_2$ .Moreover,(8) is right when $\omega = 0$.

Now, we associate λ with the rational linear parameter transformation. By adjusting the value of the λ ,we can not only change the inner control points and weights,but also the location of the shoulder point.Actually the corresponding relations between the points on the curve and the points in parameter field are changed.That is to say, the rational cubic conic sections is re-parametrized.

4. Te inverse calculation methods suitable for engineering applications
4.1 Calculating inversely parameter value of a point on rational cubic conic sections

Given a rational cubic Bezier representation of the conic section ($\lambda = \lambda^*$)

$$P(u) = \frac{\sum_{j=0}^{3} \omega_j b_j^{\;*} B_{j,3}(u)}{\sum_{j=0}^{3} \omega_j B_{j,3}(u)}, \quad 0 \leq u \leq 1$$

where $b_j^{\;*}$ ($i = 0,1,2,3$)are control points and ω_j ($i = 0,1,2,3$)are weights.We now Calculate inversely parameter value $\overline{u}$ of P on the rational cubic conic section.(see Fig.2).

With $\lambda = \lambda_1 = 0$,using (3),(4) and (6),we get the weights of the rational quadratic Bezier repre -sentation of the conic section

$$\frac{\overline{\omega_1}}{\overline{\omega_0}} = \omega \cdot \sqrt[3]{\frac{\omega_3^1}{\omega_0^1}} \quad , \quad \frac{\overline{\omega_1}}{\overline{\omega_2}} = \omega \cdot \sqrt[3]{\frac{\omega_0^1}{\omega_3^1}} \;\; \text{(for } \omega \neq 0 \text{)}$$

Or

$$\overline{\omega}_0 = \sqrt[3]{\frac{\omega_0^1}{\omega_3^1}}, \overline{\omega}_1 = 0, \overline{\omega}_2 = \sqrt[3]{\frac{\omega_3^1}{\omega_0^1}} \;\; \text{(for } \omega = 0 \text{)}$$

Parameter value of p is $u = \dfrac{\sqrt{\mu \dfrac{\overline{\omega_0}}{\overline{\omega_2}}}}{1 + \sqrt{\mu \dfrac{\overline{\omega_0}}{\overline{\omega_2}}}}$,So $u = \dfrac{\sqrt[3]{\mu \sqrt{\mu} \dfrac{\omega_0^1}{\omega_3^1}}}{1 + \sqrt[3]{\mu \sqrt{\mu} \dfrac{\omega_0^1}{\omega_3^1}}}$,where $\mu = \dfrac{\overline{b_0 m}}{\overline{m b_2}}$

(see Figure 3).

Substitution of $u = \dfrac{\sqrt[3]{\mu \sqrt{\mu} \dfrac{\omega_0^1}{\omega_3^1}}}{1 + \sqrt[3]{\mu \sqrt{\mu} \dfrac{\omega_0^1}{\omega_3^1}}}$, $\lambda_1 = 0$ and $\lambda_2 = \lambda^*$ into (8) gives
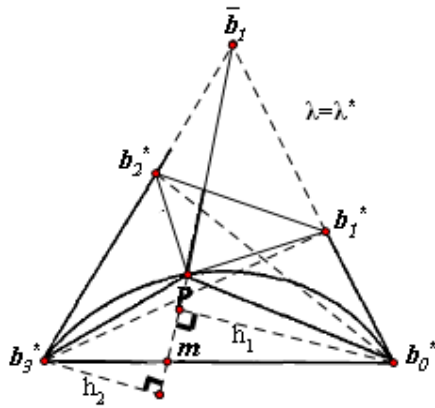
Figure 2. Calculating inversely parameter value of P on rational cubic conic sections $(\lambda = \lambda^*)$
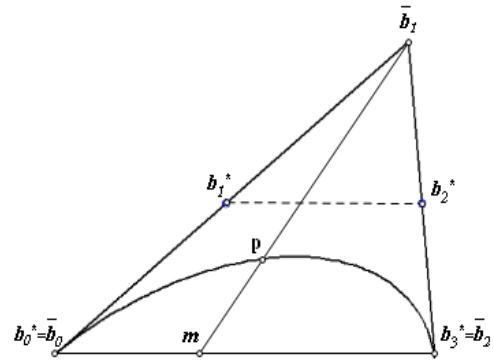

Figure 3. Calculating inversely parameter value of P on quadratic rational Bezier curve

$$\bar{u} = \cfrac{1}{1 + \sqrt[3]{\cfrac{\omega_3(1 - \lambda^*(1 + 2\omega))}{\mu\sqrt{\mu}\,\omega_0(1 + 2\lambda^*\omega(1 + 2\omega))}}} \tag{9}$$

Particularly, when $\omega = 0$

$$\bar{u} = \cfrac{1}{\sqrt[3]{\cfrac{\omega_3(1 - \lambda^*)}{\omega_0\mu\sqrt{\mu}}} + 1} \tag{10}$$

Refer to Fig.2. Let S_3, S_0, Δ_2 and Δ_0 separately be the area of triangle $b_0^*b_1^*b_2^*$, $b_1^*b_2^*b_3^*$, $b_0^*\bar{b}_1p$ and $p\bar{b}_1b_3^*$.

So $\mu = \dfrac{\overline{b_0^*m}}{mb_3^*} = \dfrac{h_1}{h_2} = \dfrac{\Delta_2}{\Delta_0}$. Using $b_1^* = \dfrac{(1 + 2\lambda\omega(1 + 2\omega))b_0^* + (2\omega - 2\lambda\omega(1 + 2\omega))\bar{b}_1}{(1 + 2\omega)}$

we have $b_1^* - b_0^* = \dfrac{2\omega - 2\lambda\omega(1 + 2\omega)}{1 + 2\lambda\omega(1 + 2\omega)}(\bar{b}_1 - b_1^*)$.

Using $b_2^* = \dfrac{(1 - \lambda(1 + 2\omega))b_3^* + (2\omega + 4\lambda\omega^2(1 + 2\omega))\bar{b}_1}{(1 + 2\omega)(1 - \lambda(1 - 4\omega^2))}$

we have $b_2^* - b_3^* = \dfrac{2\omega + 4\lambda\omega^2(1 + 2\omega)}{1 - \lambda(1 + 2\omega)}(\bar{b}_1 - b_2^*)$.

$$S_3 = \dfrac{1}{2}\left|(b_1^* - b_0^*) \times (b_2^* - b_1^*)\right| = \dfrac{2\omega - 2\lambda\omega(1 + 2\omega)}{1 + 2\lambda\omega(1 + 2\omega)} \cdot \dfrac{1}{2}\left|(\bar{b}_1 - b_1^*) \times (b_2^* - b_1^*)\right|$$

$$S_0 = \frac{1}{2}\left|(\boldsymbol{b_2}^* - \boldsymbol{b_3}^*) \times (\boldsymbol{b_1}^* - \boldsymbol{b_2}^*)\right| = \frac{2\omega + 4\lambda\omega^2(1+2\omega)}{1-\lambda(1+2\omega)} \cdot \frac{1}{2}\left|(\bar{\boldsymbol{b}}_1 - \boldsymbol{b_2}^*) \times (\boldsymbol{b_1}^* - \boldsymbol{b_2}^*)\right|$$

So

$$\frac{1-\lambda(1+2\omega)}{1+2\lambda\omega(1+2\omega)} = \sqrt{\frac{S_3}{S_0}}$$

Substitution of $\mu = \dfrac{\Delta_2}{\Delta_0}, \dfrac{1-\lambda(1+2\omega)}{1+2\lambda\omega(1+2\omega)} = \sqrt{\dfrac{S_3}{S_0}}$ into (9) gives another formula for calculating the parameter of p :

$$\bar{u} = \frac{1}{1 + \sqrt[6]{(\dfrac{\omega_3}{\omega_0})^2 \cdot \dfrac{S_3}{S_0}} \cdot \sqrt{\dfrac{\Delta_0}{\Delta_2}}} \tag{11}$$

4.2 Given the parameter value of a point p on conic section, Calculating inversely the correspondence rational cubic Bezier representation

Supposing the parameter value of p is $\bar{u}^*$ .Substitution of

$u = \dfrac{\sqrt[3]{\mu\sqrt{\mu}\dfrac{\omega_0^1}{\omega_3^1}}}{1 + \sqrt[3]{\mu\sqrt{\mu}\dfrac{\omega_0^1}{\omega_3^1}}}$, $\lambda_1 = 0$ and $\bar{u} = \bar{u}^*$ into (7) gives

$$\lambda^* = \frac{\omega_3^2 - \mu\sqrt{\mu}\omega_0^2(\dfrac{1}{\bar{u}^*}-1)^3}{(1+2\omega)(\omega_3^2 + 2\omega\omega_0^2\mu\sqrt{\mu}(\dfrac{1}{\bar{u}^*}-1)^3)} \tag{12}$$

1) When $\omega = 0$

$$\lambda^* = \omega_3^2 - \mu\sqrt{\mu}\omega_0^2(\dfrac{1}{\bar{u}^*}-1)^3 \tag{13}$$

2) When $\bar{u}^* = \dfrac{1}{2}$ (p is the shoulder point of the rational cubic conic section）

$$\lambda^* = \frac{\omega_3^2 - \mu\sqrt{\mu}\omega_0^2}{(1+2\omega)(\omega_3^2 + 2\omega\omega_0^2\mu\sqrt{\mu})} \tag{14}$$

5. Examples Analyzing

Example 1 Given a quadratic elliptical arc defined by control points $b_0(0,0), b_1(1+\sqrt{3}/2, \sqrt{3}), b_2(2,0)$ and weights $\overline{\omega}_0 = \overline{\omega}_2 = 1, \overline{\omega}_1 = \omega = -\dfrac{1}{4}$.

Using theorem 1,We can get the control points and weights of the rational cubic elliptical arc.

$$b_0^* = b_0, b_3^* = b_2, \omega_0 > 0, \omega_3 > 0$$

$$b_1^* = (2-\lambda/2)b_0 + (-1+\lambda/2)b_1, b_2^* = \frac{(2-\lambda)b_2 + (-1+\lambda/4)b_1}{1 - \dfrac{3}{4}\lambda}$$

$$\omega_1 = \frac{1}{3(2-\lambda)}\sqrt[3]{\omega_3\omega_0^2 \frac{4-2\lambda}{4-\lambda}}, \omega_2 = \frac{2(1-\dfrac{3}{4}\lambda)}{3(4-\lambda)}\sqrt[3]{\omega_0\omega_3^2 \frac{4-\lambda}{4-2\lambda}}$$

$$-\infty < \lambda < \frac{4}{3}\,\circ$$

Figure 4 shows the correspondence control points and weights with our choice λ =-3,0,0.7,1.1 for $\omega_0 = \omega_3 = 1$.Table 1 gives the coordinates of correspondence shoulder point and the value of inner weights.
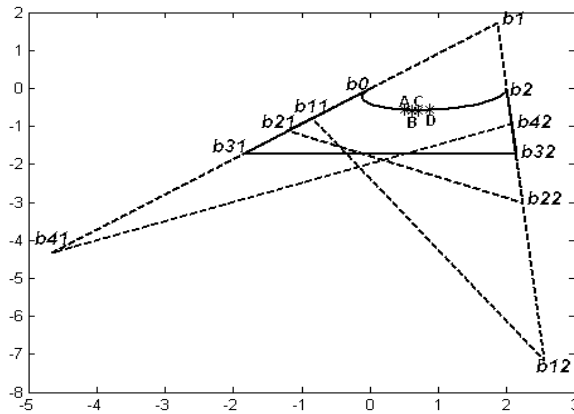

Figure 4. λ affecting the control points and shoulder point

Table 1 Correspondence inner weights and shoulder point for $\lambda = 1.1, 0.7, 0, -3$

λ	Control points	weights	Shoulder point
1.1	b0,b11,b12,b2	ω_1=0.3159 ω_2=0.0472	A(0.5068, -0.5678)
0.7	b0,b21,b22,b2	ω_1=0.2368 ω_2=0.1039	B(0.6069, -0.5749)
0	b0,b31,b32,b2	ω_1=0.1667 ω_2=0.1667	C(0.7113, -0.5774)
-3	b0,b41,b42,b2	ω_1=0.0751 ω_2=0.2748	D(0.8714, -0.5720)

Example 2 Given a semi-ellipse defined by control points $\bar{b}_0(0,0), \bar{b}_2(2,0)$ and direction vector $\vec{b} = \{\frac{1}{2}, 1\}$. (1)Given the parameter of $p(\frac{3}{2}, 1)$ on the semi-ellipse $\bar{u} = 0.5296880$,find the correspondence standard rational cubic Bezier representation of the semi- ellipse .(2) Find the parameter value of $p^*(1.4244, 0.9972)$.

Suppose the intersection point of $\overline{\bar{b}_0\bar{b}_2}$ and the line passing p and being parallel to $\vec{b} = \{\frac{1}{2}, 1\}$ is m. Substitution of $\bar{u} = 0.5296880$, $\mu = 1$ and $\omega_0 = \omega_3 = 1$ into(13) gives $\lambda^* = 0.3$.Refer to theorem 2,we obtain the standard rational cubic Bezier representation of the semi-ellipse.The control points and weights are

$$b_0^* = \bar{b}_0, b_3^* = \bar{b}_2, b_1^* = \bar{b}_0 + 1.4\vec{b},$$

$$b_2^* = \bar{b}_2 + \frac{20}{7}\vec{b}$$

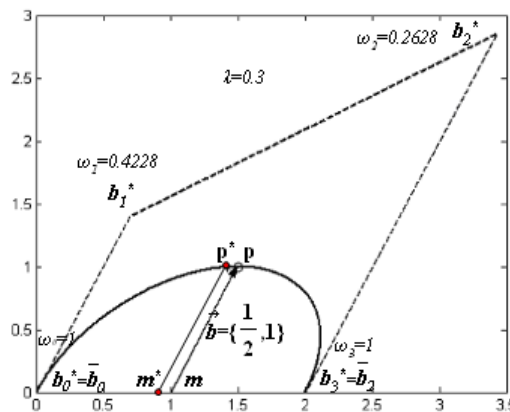$$\omega_0 = \omega_3 = 1, \omega_1 = 0.4228, \omega_2 = 0.2628$$

(see Figure 5).

Figure 5. Inverse calculation of semi-ellipse represented
in standard cubic rational Bezier form

Suppose the intersection point of $\overline{\bar{b}_0\bar{b}_2}$ and the line passing $p^*(1.4244, 0.9972)$ and being parallel to $\vec{b} = \{\frac{1}{2}, 1\}$ is m^*.We can get coordinates $(0.9258, 0)$ of m^*.So

$$\mu = \frac{\overline{\bar{b}_0 m^*}}{\overline{m^* \bar{b}_2}} = \frac{0.9258}{2 - 0.9258} = 0.8618507,$$

Substitution of $\mu = 0.8618507$, $\lambda^* = 0.3$ and $\omega_0 = \omega_3 = 1$ into (10) gives the parameter value of $p^*(1.4244, 0.9972)$ $\bar{u} = 0.5111429$.

6. Conclusion

A conic section has infinite representation methods in(standard) rational cubic Bezier form. The (standard) rational cubic Bezier representation with a parameter of conic sections shows the inner relation between the parameter and the inner control points and the weights. By adjusting the parameter value,we can not only change the inner control points and weights,but also the location of the shoulder point.Actually the corresponding relations between the points on the curve and the points in parameter field are changed. That is to say, the rational cubic conic sections are reparametrized. We obtain the inverse calculation methods suitable for engineering applications.It is clear that the rational high degree Bezier representation of the conic section has been solved by the ascending degree algorithm of Bezie curves. Generally, do the standard rational nth-degree Bezier curves have the similar conclusions ?etc.All these will be our next major work.

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