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Solutions of reaction-diffusion equations using similarity reduction and HSSOR iteration

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Article Info	ABSTRACT
	Similarity method is used in finding the solutions of partial differential equation (PDE) in reduction to the corresponding ordinary differential equation (ODE) which are not easily integrable in terms of elementary or tabulated functions. Then, the Half-Sweep Successive Over-Relaxation (HSSOR) iterative method is applied in solving the sparse linear system which is generated from the discretization process of the corresponding second order ODEs with Dirichlet boundary conditions. Basically, this ODEs has been constructed from one-dimensional reaction-diffusion equations by using wave variable transformation. Having a large-scale and sparse linear system, we conduct the performances analysis of three iterative methods such as Full-sweep Gauss-Seidel (FSGS), Full-sweep Successive Over-Relaxation (FSSOR) and HSSOR iterative methods to examine the effectiveness of their computational cost. Therefore, four examples of these problems were tested to observe the performance of the proposed iterative methods. Throughout implementation of numerical experiments, three parameters have been considered which are number of iterations, execution time and maximum absolute error. According to the numerical results, the HSSOR method is the most efficient iterative method in solving the proposed problem with the least number of iterations and execution time followed by FSSOR and FSGS iterative methods.

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1.   INTRODUCTION

The problem of parabolic PDEs are very important to be solved numerically since this problem will be applied in variety of application including in branches of science and engineering [1-3]. Therefore, it is clearly showed that development of an efficient and reliable method is very needed in tackling this issue. Actually, it is necessary to use the numerical approach in many application parts to get an approximate solution of one-dimensional reaction-diffusion equation. In the similar context, several numerical methods such as Hopf Bifurcation [4] and Method of Moments [5] have been proposed to solve particularly on mathematical models based on differential equations. Apart from the above numerical methods, the use of the discretization process via the two-level [6] or three-level scheme [7] of finite difference method particularly an implicit or Crank-Nicolson has been imposed to discretize the mathematical models and then get the corresponding approximation equation. It can be observed that this approximation equation leads a sequence of corresponding linear systems at each time level. It means that these linear systems need to be solved iteratively or directly. Directly, the computational time of getting an approximate solution will be increasing due to their high computational complexity. To avoid this matter, this study has proposed the similarity reduction method reduce the computational complexity during the discretization process in order to form a single similarity linear system.

Since having the similarity linear system as mentioned in the first paragraph, we focus on using Successive Over Relaxation (SOR) method combined with Half-Sweep (HS) concept whereas this iterative method is very useful in order to solve sparse linear systems [8]. Based on the previous studies of solving linear systems conducted by Young [9] as well as Hackbusch [10], they pointed out that SOR method is one of the effective point iterative methods in solving a large scale and sparse linear system. Due to its advantage of using SOR iteration, the simulation process will converge faster in which their execution time and number of iterations get smaller in solving the one-dimensional reaction-diffusion equation as compared to GS method.

In this paper, the main objective is to deal with effectiveness of HSSOR iterative method with the similarity reduction in solving one-dimensional reaction-diffusion equation. To examine the effectiveness of this combination, let us consider the one-dimensional reaction-diffusion equation as follows.

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial t^2} + p(t)u + \phi(x,t), \, 0 \le x \le n, \, 0 < t \le T$$

(1)

with the initial condition

$$u(x,0) = f(x), \, 0 \le x \le n$$

and the boundary conditions

$$u(0,t) = g_0(t), \quad 0 < t \le T,$$
$$u(n,t) = g_1(t), \quad 0 < t \le T$$

where f, g_0, g_1, ϕ, E and α are known, while the functions u need to be determined. In order to develop an approximation equation of problem (1), let length of $[0,n]$ for derivation of the n subinterval over the original internal $[0,n]$ finite difference scheme be given as follows

$$\Delta x = \frac{(n-0)}{n} = h,$$

(2)

Based on equation (2), the distribution of uniformly grid points over the solution domain $[0,n]$ is illustrated in Figure 1
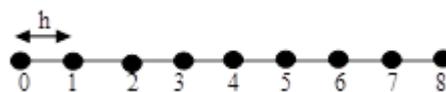

Figure 1. Finite grid network for solution domain at $n=8$

For the next discussion, this paper is organized as follows. In section 2, discretization of the similarity problem which is derived by problem (1) is carried out by using second-order central finite difference schemes to generate a linear system. In section 3, we show the formulation of HSSOR in order to solve the corresponding linear system. Then, four tested numerical results and their discussion are given in section 4. Finally, in section 5 we give some conclusions and future work for this problem.

2. SIMILARITY REDUCTION VIA HALF-SWEEP FINITE DIFFERENCE SCHEME

In this section, a linear system is generated using a finite difference approximate equation. Before having this linear system, the similarities reduction method will be imposed into problem (1) which means that PDE (1) will be reduced to an ODE in order to form only a linear system. In addition to do that, there are various techniques for PDE to be reduced to an ODE include various integral transform and eigenfunction expansions [11]. However, this study use the similarities reduction for PDE namely wave variable transformation technique. The essential idea of this procedure is to seek solutions of the form $u(x,t)$

becomes $u(\xi)$ where $\xi = (x - ct)$ is called as wave variable [12, 13]. To apply this reduction, the general linear PDE is considered and given as follows

$$J(u, u_x, u_t, u_{xx}, u_{tt}, \ldots) = 0 \tag{3}$$

The following equation (4) shows the general linear ODE after applying wave variable transform

$$K(u, u_\xi, u_{\xi\xi}, \ldots) = 0 \tag{4}$$

Before discretizing the proposed problem in equation (1), let us consider the following terms of wave variable transformation to reduce the variables in equation (1) as follows

$$\left. \begin{array}{l} \dfrac{\partial}{\partial t} = -c \dfrac{\partial}{\partial \xi}, \dfrac{\partial^2}{\partial t^2} = c^2 \dfrac{\partial^2}{\partial \xi^2}, \\[3mm] \dfrac{\partial}{\partial x} = \dfrac{\partial}{\partial \xi}, \dfrac{\partial^2}{\partial x^2} = \dfrac{\partial^2}{\partial \xi^2}. \end{array} \right\} \tag{5}$$

By using the corresponding terms (5), problem (1) can be easily transformed into the linear ODE and state as follows

$$\alpha \frac{\partial^2 u}{\partial \xi^2} + c \frac{\partial u}{\partial \xi} + p(\xi)u(\xi) = -\phi(\xi) \tag{6}$$

Figure 2 shows the implementation of the Full-sweep (FS) and Half-sweep (HS) iterations that will compute the approximate values of interior node points of type ● only until the convergence criterion is reached. Meanwhile, the node points type ○ shows the other approximate solutions at remaining points which solving by using direct methods [14].

(a)

(b)
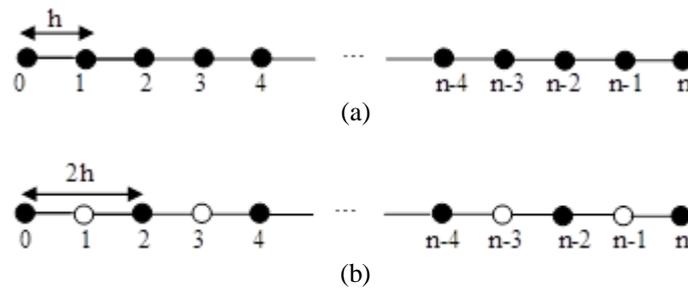
Figure 2. (a) and (b) represent the illustration of finite grid network for FS and HS iteration respectively

By considering the distribution of node points in Figure 1, Figure 2(a) and Figure 2(b) represent the illustration of finite grid network for FS and HS iteration respectively. According to Figure 2, it seems that the implementation of the HS iteration just involves half of whole inner node points as shown in Figure 2(b) compared with the FS iteration as shown in Figure 2(a). Then, in order to construct the generated linear system, let the corresponding ODE (6) be discretized by applying the second-order HS central finite difference scheme to form the corresponding approximation equation. Before discretizing equation (6), let us consider the following second-order HS central finite difference schemes [14].

$$\left. \frac{du}{d\xi}\right|_i = \frac{u_{i+2} - u_{i-2}}{4h}$$

$$\left. \frac{d^2u}{d\xi^2}\right|_i = \frac{u_{i+2} - 2u_i + u_{i-2}}{4h^2}$$

(7)

Thus, by imposing the second-order HS central difference scheme in equation (7), the second-order HS finite difference approximation equation for problem (6) can be formed as follows

$$\alpha\left(\frac{u_{i+2} - 2u_i + u_{i-2}}{4h^2}\right) + c\left(\frac{u_{i+2} - u_{i-2}}{4h}\right) + p_i(u_i) = -\phi_i$$

(8)

For convenience, we simplify equation (8) and rewrite it in the following equation

$$Au_{i-2} + B_i u_i + Cu_{i+2} = -\phi_i, \quad i = 2,4,6,..., n-2$$

(9)

where

$$A = \frac{\alpha}{4h^2} - \frac{c}{4h}, B_i = p_i - \frac{\alpha}{2h^2}, C = \frac{\alpha}{4h^2} + \frac{c}{4h}.$$

The following linear system generated using finite difference method can be easily show n in matrix form as follows

$$W\underline{U} = \underline{\phi}$$

(10)

where

$$W = \begin{bmatrix} B_2 & C & & & & \\ A & B_4 & C & & & \\ & A & B_6 & C & & \\ & & \ddots & \ddots & \ddots & \\ & & & A & B_{n-4} & C \\ & & & & A & B_{n-2} \end{bmatrix} \quad \underline{U} = \begin{bmatrix} U_2 \\ U_4 \\ U_6 \\ \vdots \\ U_{n-4} \\ U_{n-2} \end{bmatrix} \quad \underline{\phi} = \begin{bmatrix} -AU_0 - \phi_2 \\ \phi_4 \\ \phi_6 \\ \vdots \\ \phi_{n-4} \\ -CU_n - \phi_{n-2} \end{bmatrix}.$$

According to linear system (10), it can be revealed that the coefficient matrix A is categorized as large-scale and sparse matrix. Due to get the similarity solution of linear system (10), the iterative method is the key to the solution of the linear system. Therefore, we consider the iterative methods such as HSSOR, FSSOR and FSGS in solving the linear system (10) as explained in the next section.

3. FORMULATION OF HALF-SWEEP SOR ITERATION

As mentioned in section 2, the iterative method namely SOR is applied with combination of HS concept to solve the sparse linear system (10). The HS iteration concept is known as the complexity reduction approach [15] which is introduced by Abdullah [16]. Due to its advantage, many authors have been attracted to use the applications of the HS iteration concept in solving the numerical issues in their research [17-20]. Therefore, the corresponding sparse linear system (10) also has been solved by using the HS iteration concept via the SOR method since the SOR method is the common option for efficient solution. Based on Hadjidimos [21] and Young [22], they have mentioned that the SOR method is one of the most known and widely used iterative method in solving any system of linear equations.

In order to implement the iterative process, let the formulation of the HSSOR iterative methods be constructed from the approximation equation (9) by considering the following splitting of its coefficient matrix as:

$$W = D - L - V \tag{11}$$

where each matrix of D, L and V can be defined as

$$D = \begin{bmatrix} B_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & B_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & B_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & B_{n-4} & 0 \\ 0 & 0 & 0 & 0 & 0 & B_{n-2} \end{bmatrix}, L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -A & 0 & 0 & 0 & 0 & 0 \\ 0 & -A & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & -A & 0 & 0 \\ 0 & 0 & 0 & 0 & -A & 0 \end{bmatrix}, V = \begin{bmatrix} 0 & -C & 0 & 0 & 0 & 0 \\ 0 & 0 & -C & 0 & 0 & 0 \\ 0 & 0 & 0 & -C & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & -C \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

By manipulating equation (11), it can be showed that the general form of HSSOR iterative methods can be stated as [18, 19]

$$\underline{U}^{(k+1)} = (1-\omega)\underline{U}^{(k)} + \omega(D+L)^{-1}\left(-V\underline{U}^{(k)} + \underline{\phi}\right) \tag{12}$$

Actually, SOR method is generalization and improvement of the Gauss-Seidel (GS) method together with the weighted parameter, ω in the range of $1 \leq \omega < 2$. For $\omega = 1$, the SOR becomes the GS method. Weighted parameter can be determined essentially by sequentially selecting a value with some accuracy until the optimal value can be obtained. Based on equation (12), the general algorithm for the implementation of HSSOR iterative method in solving the linear system (10) may be demonstrated in Algorithm 1 [23].

Algorithm 1: HSSOR iteration

i.   Set all the parameters.
ii. Assign the optimal value of ω.
iii. Solve the system by calculating the value of   using

$$\underline{U}^{(k+1)} = (1-\omega)\underline{U}^{(k)} + \omega(D+L)^{-1}\left(-V\underline{U}^{(k)} + \underline{\phi}\right).$$

iv. Perform the convergence test, $\left|U_i^{(k+1)} - U_i^{(k)}\right| \leq \varepsilon = 10^{-10}$. If yes, move to step (v). Otherwise go to step (iii).
v.   Stop.

4. NUMERICAL RESULTS AND DISCUSSION

In this section, four numerical tests have been conducted in order to verify the effectiveness of the HSSOR method as compared with the FSSOR and FSGS methods. As a comparison among of these three iterative methods, three criteria including number of iterations, execution time which measured in second and maximum absolute error are considered. Through the numerical simulations, we have set up the tolerance error, $\varepsilon = 10^{-10}$ as a convergence test at different sizes grid which are *m = 512, 1024, 2048, 4096* and *8192*. We have used maximum absolute error as stopping error in which it is very helpful when the convergence is very slow to converge. We have tested the proposed methods on the following four examples:

4.1. Example 1 [24]

Let us consider the one-dimensional problem as follows

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial t^2} + \left(1+t^2\right) + \left[\pi^2 - (t+1)^2\right]e^{-t^2}\left[\cos(\pi x) + \sin(\pi x)\right], \tag{13}$$

with the initial condition

$$u(x,0) = f(x), 0 \leq x \leq 1$$

and boundary conditions

$$u(0,t) = g_0(t), \quad 0 < t \le T,$$
$$u(1,t) = g_1(t), \quad 0 < t \le T$$

The analytical solution of problem (13) is

$$u(x,t) = \exp\left(-t^2\right)\left(\cos(\pi x) + \sin(\pi x)\right) \tag{14}$$

4.2. Example 2 [25]

Consider the following PDE which is found in many transport phenomena

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 4 - 6x, \quad t > 0, 0 < x < 1 \tag{15}$$

with the initial condition

$$u(x,0) = \sin \pi x + x(1-x)^2 \quad 0 \le x \le 1$$

and boundary conditions

$$u(0,t) = u(1,t) = 0$$

The analytical solution of problem (15) is

$$u(x,t) = e^{-\pi^2 t} \sin(\pi x) + x(1-x)^2 \tag{16}$$

4.3. Example 3 [26]

Consider the following equation

$$\frac{\partial u}{\partial t} = [2 + \cos(t)]\frac{\partial^2 u}{\partial x^2} + [3 + \cos(t)]e^t \cos(x), \quad (x,t) \in (0,1) \times (0,T] \tag{17}$$

with the initial condition

$$u(x,0) = f(x), \ x \in [0,1]$$

and boundary conditions

$$u(0,t) = g_0(t) \ t \in [0,T]$$
$$u(1,t) = g_1(t) \ t \in [0,T]$$

The analytical solution of problem (17) is

$$u(x,t) = e^t \cos(x) \tag{18}$$

4.4. Example 4 [27]

Consider the following diffusion problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u, \tag{19}$$

with the initial condition

$$u(x,0) = x + e^{-x}$$

While, the analytical solution of problem (19) is

$$u(x,t) = e^{-x} + xe^{-t} \tag{20}$$

Table 1 shows the comparison of the three criteria by using three purposed iterative methods in order to examine their performance. Meanwhile, Table 2 represents the percentage of reduction for the HSSOR and FSSOR method as compared to FSGS iterative method.

Table 1. Comparison of the three criteria using HSSOR, FSSOR and FSGS iterative methods

EXAMPLE	GRID SIZES	ITERATION NUMBER			EXECUTION TIME (SECOND)			MAXIMUM ABSOLUTE ERROR		
		FSGS	FSSOR	HSSOR	FSGS	FSSOR	HSSOR	FSGS	FSSOR	HSSOR
1	512	193056	1617	792	3.78	0.06	0.02	2.2249e-02	2.2252e-02	2.2252e-02
	1024	627163	3271	1617	24.48	0.16	0.07	2.2242e-02	2.2252e-02	2.2252e-02
	2048	1931198	6580	3271	150.44	0.59	0.20	2.2211e-02	2.2252e-02	2.2252e-02
	4096	5420290	17064	6580	847.91	3.02	0.63	2.2087e-02	2.2252e-02	2.2252e-02
	8192	12408135	32231	17064	3881.56	11.34	3.01	2.1592e-02	2.2252e-02	2.2252e-02
2	512	275982	1522	765	5.40	0.08	0.02	1.9675e-02	1.9672e-02	1.9672e-02
	1024	960413	2815	1522	37.38	0.15	0.06	1.9682e-02	1.9672e-02	1.9672e-02
	2048	3267333	5388	2815	254.54	0.51	0.25	1.9709e-02	1.9672e-02	1.9672e-02
	4096	10771529	14687	5388	7076.77	2.59	0.52	1.9819e-02	1.9673e-02	1.9672e-02
	8192	33893854	34560	14687	10665.73	12.10	2.61	2.0259e-02	1.9673e-02	1.9673e-02
3	512	368920	2049	1025	8.58	0.08	0.03	5.5183e-02	5.5185e-02	5.5186e-02
	1024	1329011	4097	2049	61.65	0.25	0.10	5.5175e-02	5.5185e-02	5.5185e-02
	2048	4729285	8193	4097	439.85	0.86	0.25	5.5145e-02	5.5185e-02	5.5185e-02
	4096	16569953	24155	8193	3089.34	4.95	0.88	5.5026e-02	5.5185e-02	5.5185e-02
	8192	56890787	46350	24155	21221.07	19.00	5.01	5.4550e-02	5.5185e-02	5.5185e-02
4	512	335335	2049	1025	6.49	0.08	0.02	4.5357e-02	4.5360e-02	4.5360e-02
	1024	1197647	4097	2049	46.24	0.20	0.07	4.5350e-02	4.5360e-02	4.5360e-02
	2048	4215906	8193	4097	326.38	0.71	0.21	4.5319e-02	4.5360e-02	4.5360e-02
	4096	14565094	22540	8193	2263.54	3.95	0.76	4.5194e-02	4.5360e-02	4.5360e-02
	8192	49066665	43174	22540	15239.63	15.11	3.94	4.4697e-02	4.5360e-02	4.5360e-02

According to Table 1, the numerical results of four examples show that the performance of proposed iterative methods consists of number of iterations, execution time in seconds and maximum absolute error. These three parameters have been required to compare the effectiveness of three iterative methods which are HSSOR, FSSOR and FSGS iterative methods in solving the reaction-diffusion equations. As we can see, the combination between SOR method and HS approach has required less number of iterations followed by FSSOR and FSGS methods. Also, we found that the HSSOR method is the fastest iterations process in term of execution time at every grid sizes compared other methods.

Table 2. Percentage reduction for the HSSOR and FSSOR compared to FSGS iterative method

Example	Number of Iterations		Time (second)	
	FSSOR	HSSOR	FSSOR	HSSOR
1	99.16 - 99.74	99.59-99.88	98.41 - 99.71	99.47 - 99.93
2	99.45 - 99.90	99.72 - 99.96	98.52 - 99.96	99.63 - 99.99
3	99.44 - 99.92	99.72 - 99.96	99.07 - 99.91	99.65 - 99.98
4	99.39 - 99.91	99.69 - 99.95	98.77 - 99.90	99.69 - 99.97

Based on the performance analysis in Table 2, the percentage of reduction for number of iterations by using HSSOR iterative method is a little bit higher than FSSOR by approximately 99.59–99.96% and 99.16–99.92% respectively compared to FSGS iterative method. While in terms of execution time, HSSOR and FSSOR proved to be very faster by approximately 99.47–99.99% and 98.41-99.96% respectively compared to FSGS iterative method. It can be concluded that the most efficient iterative method in this study is HSSOR method and followed by FSSOR method.

5. CONCLUSION

This paper presents the similarities solution technique in advantages of forming only a linear system that can be reduced the computational complexity and time during the iteration process. Then, the applications of the HS iteration concept combined with SOR method can be carried out in solving the sparse linear systems that has been generated from corresponding second-order HS central finite difference approximate equation via similarities solution technique. On the whole, the numerical results have showed that HSSOR iterative method is drastically better than FSSOR and FSGS iterative methods due to its smaller number of iterations and execution time. For the future work, we will apply the family of quarter-sweep iteration [28] as linear solvers to solve this problem.

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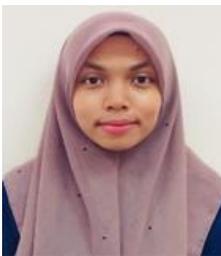
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