TELKOMNIKA Indonesian Journal of Electrical Engineering
Vol. 14, No. 1, April 2015, pp. 140 ~ 146
DOI: 10.11591/telkomnika.v14i1.7461

Hopf Bifurcation in Numerical Approximation for the Generalized Lienard Equation with Finite Delay

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Abstract

The numerical approximation of the generalized Lienard equation is considered using delay as parameter. First, the delay difference equation obtained by using Euler method is written as a map.According to the theories of bifurcation for discrete dynamical systems,the conditions to guarantee the existence of Hopf bifurcation for numerical approximation are given. The relations of Hopf bifurcation between the continuous and the discrete are discussed. Then when the generalized Lienard equation has Hopf bifurcations at $r = r_0$*, the numerical approximation also has Hopf bifurcations at* $r_h = r_0 + o(h)$ *is proved. At last, the text listed an example of numerical simulation, the result shows that system (8) discretized by Euler keeps the dynamic characteristic of former system (1), and the theory is proved.*

Keywords: *the generalized Lienard equation, Euler method, Hopf bifurcation, numerical approximation*

1. Introduction

In recent years, the generalized Lienard equation:

$$\ddot{x}(t) - f(x(t))\dot{x}(t) + g(x(t-r)) = 0 \tag{1}$$

The behavior of its solution attracted attention of many scholars. Delays are the key to cause differences between delay differential equation and ordinary differential equation, so use delays as parameter to study Hopf bifurcation is meaningful. Many scholars have done in-depth research about the Hopf bifurcation of system (1) [1–3]. For example, in 1998, reference [1] uses delay r as parameter studied Hopf bifurcation of system (1), proved the existence of Hopf bifurcation and formula to count Hopf bifurcation was given. Reference [2] uses τ-D partitioning method of index polynomial to discuss the Hop bifurcation of system (1) using k as a parameter. Reference [3] discusses Hopf bifurcation of system (1) using b as a parameter, and gives the Hopf bifurcation diagram in the $r-b$ parameter plane.

This text discussed the Hopf bifurcation in numerical approximation of the system (1) by choosing r as the bifurcation parameter, using the Euler method. The reference 4 to 7 took the lead in studying the Hopf bifurcation in numerical approximation of the finite delay Logistic equation and got satisfied results. What is called the numerical approximation is to examining whether its numerical solution can maintain the dynamic characteristic of the system while using the numerical method to achieve the discretization of system.

2. The Existence of Hopf Bifurcation for the Generalized Lienard Equation

As to system (1), set delay $r > 0$ as constant, $f, g \in C^2$, and $g(x)$ satisfying $g(0) = 0, xg(x) > 0$. Set $f(0) = a, g'(0) = b,$ and $a > 0, b > 0.$

System (1) is equivalent to the following second-order-finite-delay system.

Received January 4, 2015; Revised March 11, 2015; Accepted March 25, 2015

$$\begin{cases} \dot{x}(t) = y(t), \\ \dot{y}(t) = -f(x(t))y(t) - g(x(t-r)), \end{cases} \tag{2}$$

Let $\dot{x} = y$, then do the time conversion $t = rs$, and still note $x(rs), y(rs)$ as $x(t), y(t)$, therefore Equation (2) can be transformed into its equivalent system.

$$\begin{cases} \dot{x}(t) = ry(t), \\ \dot{y}(t) = -rf(x(t))y(t)y(t) - rg(x(t-1)), \end{cases} \tag{3}$$

Its linear part is:

$$\begin{cases} \dot{x}(t) = ry(t), \\ \dot{y}(t) = -arx(t-1) - bry(t), \end{cases} \tag{4}$$

The characteristic equation of (4) Is:

$$\lambda^2 + ar\lambda + br^2 e^{-\lambda} = 0 \tag{5}$$

Lemma 1: Set r as a parameter, so when $r = r_0$, Equation (3) exists Hopf bifurcation, and r_0 satisfies following conditions:

$$\begin{cases} r_0 = \dfrac{1}{\omega_0} \sin^{-1}(\dfrac{a\omega_0}{b}), \\ \omega_0 = -\dfrac{1}{\sqrt{2}} \left[\sqrt{a^2 + 4b^2} - a^2 \right]^{\frac{1}{2}}, \end{cases} \tag{6}$$

a) Equation (5) has a pair of conjugate complex roots $\lambda_{1,2} = \alpha(r) \pm i\beta(r)$, and the α, β here are real numbers, while $\alpha(r_0) = 0, \beta(r_0) = \omega_0 > 0$.

b) The roots of equation (5) in $r = r_0$ all have strictly negative real parts, except $\lambda(r_0), \bar{\lambda}(r_0)$.

c) $\left. \dfrac{d \operatorname{Re} \lambda(r)}{dr} \right|_{r=r_0} > 0$.

3. Hopf Bifurcation in Numerical Approximation for the Generalized Lienard Equation

Using the $EulerMethod$[4] $(h = \frac{1}{m}, m \in Z_+)$, we get the numerical solution of Equation (3).

$$\begin{cases} x_{n+1} = x_n + rhy_n \\ y_{n+1} = y_n - brhx_{n-m} - arhy_n \end{cases} \tag{7}$$

Introducing new vector $X_n = (x_n, y_n, x_{n-1}, y_{n-1}, \ldots x_{n-m}, y_{n-m})^T$, we can express (7) as:

$$X_{n+1} = F(X_n, r) \tag{8}$$

The $F(x)=(F_0,F_1,\dots,F_m)^T$ is a vector-valued function with $2(m+1)$ dimensions, i.e.

$$F_k=\begin{cases}\begin{cases}x_n+rhy_n\\ y_n-brhx_{n-m}-arhy_n\end{cases} & k=0\\ \begin{cases}x_n\\ y_n\end{cases} & 1\le k\le m\end{cases}$$

Expand the Equation (8) at $(0,0)$,

$$X_{n+1}=\widehat{A}X_n+\widehat{B}(X_n,X_n)+\widehat{C}(X_n,X_n,X_n)+\dots \tag{9}$$

Its linear part is :

$$X_{n+1}=\widehat{A}X_n \tag{10}$$

In which,

$$\widehat{A}=\begin{bmatrix}A & 0 & \cdots & 0 & B\\ I & 0 & \cdots & 0 & 0\\ 0 & I & \cdots & 0 & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots & I & 0\end{bmatrix}$$

I is a second order unit matrix, $A=\begin{pmatrix}1 & rh\\ 0 & 1-arh\end{pmatrix},\quad B=\begin{pmatrix}0 & 0\\ -brh & 0\end{pmatrix}$

The characteristic equation of $\widehat{A}$ is:

$$d_m(z,r,h)=z^{2m}(z-1)^2+arhz^{2m}(z-1)+br^2h^2z^m=0 \tag{11}$$

In order to facilitate the discussion about the bifurcation problem of the numerical solution in Equation (3), we introduce equation:

$$D(\mu,r,h)=\mu^2e^{2\mu}g^2(\mu h)+ar\mu e^{2\mu}g(\mu h)+br^2e^{\mu}=0 \tag{12}$$

In which $g(x)=\dfrac{e^x-1}{x}$, providing $g(0)=1$

Just like the lemma 4.1 in literature [8], we can get lemma 2.

Lemma 2: if characteristic (5) satisfies condition (6), then $D(\mu,r,h)=0$ satisfies:

a) $D(\mu,r,h)=0$ has a pair of conjugate complex roots $\mu_{1,2}=\sigma(r)\pm i\omega(r)$;

b) There exists $r_h=r_0+o(h)$, $\sigma(r_h)=0,\omega(r_h)\ne 0$;

c) $\left.\dfrac{d\sigma(r)}{dr}\right|_{r=r_h}>0$;

d) There exist $\varepsilon>0$ (nothing to do with r, h) to make for $h=\dfrac{1}{m}, m\in N$. There exists $(r,h)\in N(r_0,0)$ and $D(\mu,r,h)=0=\begin{cases}\mu=\sigma(r,h)\pm i\omega(r,h)\\ \operatorname{Re}\mu<-\varepsilon\end{cases}$.

Proof: (a-c)Because $D(\mu,r,0)=d(\mu,r)$, so $D(i\omega_0,r,0)=d(i\omega_0,r)$ .In $(i\omega_0,r_0,0)$, $\sigma'(r_0)=-\frac{d_r(\mu(r_0),r_0)}{d_\mu(\mu(r_0),r_0)}$, therefore $d_\mu(i\omega_0,r_0)\neq 0$. By the implicit function theorem, in the neighborhood of $(r_0,0)$, there exists only one function $\sigma(r,h),\omega(r,h)$ making $\mu_{1,2}=\sigma(r)\pm i\omega(r)$. Because $\sigma(r_0,0)=0,\sigma'(r_0,0)\neq 0$, there exists $r=r_h$ making $\sigma(r_h)=0,\ r_h=r_0+o(h),\omega(r_h)\neq 0$. By the implicit function theorem again, in the neighborhood of $(r_0,0)$, $\left.\frac{d\sigma(r)}{dr}\right|_{r=r_h}>0$. If $D(\mu,r,h)=0$,then $D(\bar{\mu},r,h)=0$, so there exists a neighborhood of r_0, making $d(\mu,r)=0$ has only one root $\mu_1(r)$, satisfying to $r>0$,there is $\mathrm{Re}(\mu_1(r))>-\varepsilon,\quad \mathrm{Im}(\mu_1(r))>0,$ and $D(\bar{\mu},r,h)=0$ also has similar character.

Set $\{\mu_m,r_m,h_m\}$ to make $D(\mu_m,r_m,h_m)=0,\ (r_m,h_m)\in N(r_0,0),\ \lim_{m\to\infty}h_m=0$, so $|\mu_m|$ is uniformly bounded. So there exists m_j, to make $\mu_{m_j}\to\mu_0,\ r_{m_j}\to r_0,\ h_{m_j}\to 0$. By the continuity of $D(\mu_0,r_0,0)=0$, there exists $\mu_0=i\omega_0, r_h=r_0$. So:

$$D(\mu,r,h)=0=\begin{cases}\mu=\sigma(r,h)\pm i\omega(r,h)\\ \mathrm{Re}\,\mu<-\varepsilon\end{cases}$$

Lemma 3: When $h=\frac{1}{m}$, the necessary and sufficient condition of $D(\mu,r,h)=0$ has the root μ is (11) has the root $Z=e^{\frac{\mu}{m}}$.

Proof: Substitute $e^{\frac{\mu}{m}}$ for Z in (11).

$$\mu^2e^{2\mu}g^2(\mu h)+ar\mu e^{2\mu}g(\mu h)+br^2e^{\mu}=0$$

So the lemma 3 is proved.

Lemma 4: $\left.\frac{d|z|}{dr}\right|_{r=r_h}\neq 0$

Proof: $Z=e^{\frac{\mu}{m}}$, $h=\frac{1}{m}$, $|z|^2=z\bar{z}$, so there exists:

$$\frac{d|z|^2}{dr}=z\frac{d\bar{z}}{dr}+\bar{z}\frac{dz}{dr}=he^{\mu h}e^{\bar{\mu}h}\frac{d\bar{\mu}}{dr}+he^{\mu h}e^{\bar{\mu}h}\frac{d\mu}{dr}=2he^{(\mu+\bar{\mu})h}\frac{d\sigma(r,h)}{dr},$$

Because $\left.\frac{d\sigma(r,h)}{dr}\right|_{r=r_h}>0$, so $\left.\frac{d|z|}{dr}\right|_{r=r_h}>0$.

Theorem 1: If differential Equation (3) has Hopf bifurcation in $r=r_0$, so when step size h is sufficiently small, differential Equation (8)will produce Hopf bifurcation in $r_h=r_0+o(h)$.

Proof: We can learn by lemma 3 and 4 that to the step size $h=\frac{1}{m}$ $(m \geq m_0)$, in the neighborhood of r_0, if characteristic equation (5) has root, $Z = e^{\frac{\mu}{m}}$ is the root of (11). If (5) have a pair of simple conjugate complex roots $\mu = \pm i\omega_0$, while other roots have strictly real parts. So the differential Equation (8) have a pair of conjugate complex roots $e^{\pm\frac{i\omega_h}{m}}$ in $r_h = r_0 + o(h)$ $(h=\frac{1}{m})$, and $\left|e^{\pm\frac{i\omega_h}{m}}\right| = 1$, while other roots' modules less than 1, and $\left.\frac{d|z|}{dr}\right|_{r=r_h} > 0$.

4. Numerical Simulation

This section gives an example of numerical simulation of system (1). The result shows that system (8) discretized by Euler keeps the dynamic characteristic of former system (1), and the theory is proved.

Set $f'(0) = a_1 = 0.8, g'(0) = b = 1.$ and the system turned into:

$$\begin{cases} \dot{x}(t) = y(t), \\ \dot{y}(t) = -0.8y(t) - x(t-r), \end{cases} \tag{13}$$

System (13) exists only equilibrium point $E^* = (0,0)$.

According to the theorem 4.1 of reference [3], it's easy to get:

$$r_0 \approx 0.\,378\,316\,029\,857\,13,$$

So system (13) generates Hopf bifurcation at $r = r_0$.

Diagram 1 to 3 express waveforms and trajectory diagram of solution system (13) before discretized. Diagram 4 to 6 express waveforms and trajectory diagram of system (8) discretized by Euler. The diagram 1 shows that when $r < r_0$, zero solution of system is asymptotically stabled. The diagram 2 shows that when $r = r_0$, system experiences Hopf bifurcation at origin, and stable bifurcating periodic solution was produced around equilibrium point. The diagram 3 shows that when $r > r_0$, zero solution of system is unstable. The diagram 4 to 6 shows that when $r < r_0$, zero solution of system (8) is asymptotically stabled, and stable periodic solution was produced around $r = r_0$. When $r > r_0$, zero solution of system (8) is unstable, which means system (8) discretized by Euler keeps the dynamic characteristic of former system (1).

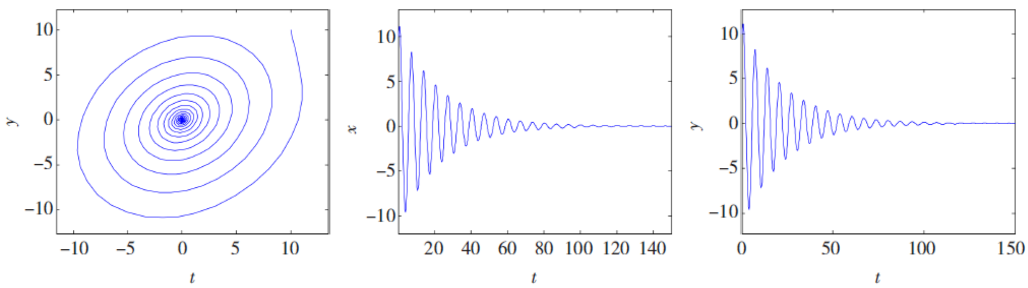

Figure 1. Waveform and phase orbit of system（13）when $r = 0.2 < r_0$

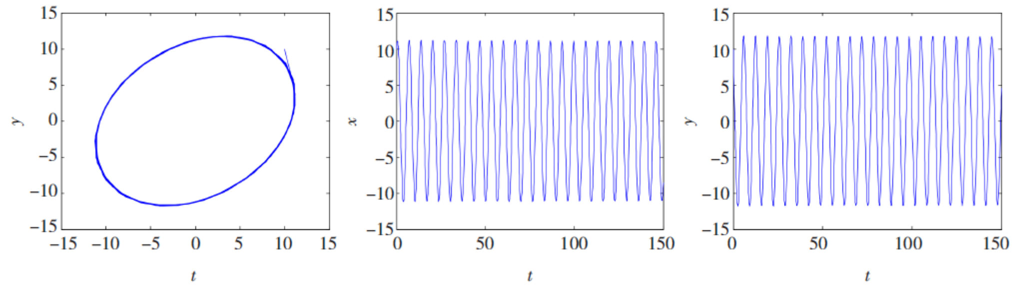

Figure 2. Waveform and phase orbit of system（13） when $r = r_0$

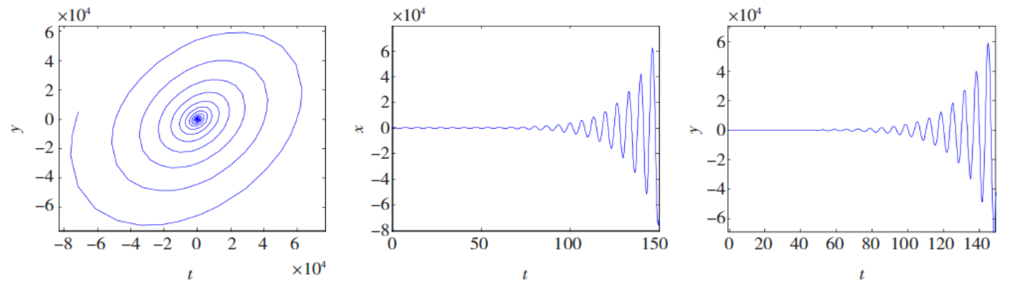

Figure 3. Waveform and phase orbit of system (13) when $r = 0.55 > r_0$

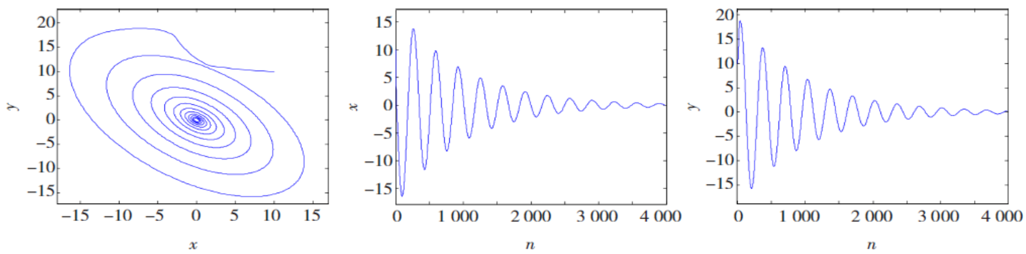

Figure 4. Waveform and phase orbit of discrete system (8) when $r = 0.2 < r_0, h = 0.02$

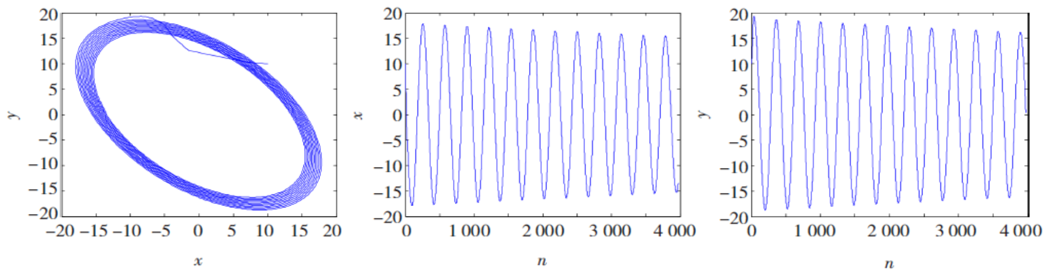

Figure 5. Waveform and phase orbit of discrete system (8) when $r = r_0, h = 0.02$

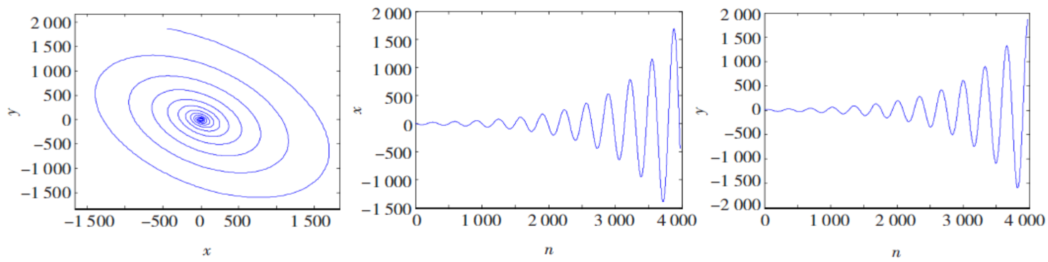

Figure 6. Waveform and phase orbit of discrete system (8) when $r = 0.55 > r_0, h = 0.02$

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