Certain properties of \(\omega\)-\(Q\)-fuzzy subrings

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ABSTRACT

In this paper, we define the \(\omega\)-\(Q\)-fuzzy subring and discussed various fundamental aspects of \(\omega\)-\(Q\)-fuzzy subrings. We introduce the concept of \(\omega\)-\(Q\)-level subset of this new fuzzy set and prove that \(\omega\)-\(Q\)-level subset of \(\omega\)-\(Q\)-fuzzy subring form a ring. We define \(\omega\)-\(Q\)-fuzzy ideal and show that set of all \(\omega\)-\(Q\)-fuzzy cosets form a ring. Moreover, we investigate the properties of homomorphic image of \(\omega\)-\(Q\)-fuzzy subring.

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1. INTRODUCTION

In mathematics, ring theory is one of the most important part of abstract algebra. In algebra, ring theory studies the algebraic structures of rings. Rings algebraic structure is a framework in which addition and multiplication are well defined with some more properties.

in [24]. Shafei et al [25] studied the fuzzy logic control systems for demand management in airports and energy efficiency by using 3D simulator.

This paper is organized as the section 2 contains the elementary definition of \(Q\)-fuzzy subrings and related results which are thoroughly crucial to understand the novelty of this article. In section 3, we define the \(\omega-Q\)-fuzzy subring and prove that the level subset of \(\omega-Q\)-fuzzy subrings is a subring. We also define \(\omega-Q\)-fuzzy ideal and discuss its properties. In section 5, we use the classical ring homomorphism to investigate the behavior of homomorphic image (inverse-image) of \(\omega-Q\)-fuzzy subring.

2. PRELIMINARIES

We recall first the elementary notion of fuzzy sets which play a key role for our further analysis.

**Definition 2.1.** [1]: A fuzzy set \(A\) of a nonempty set \(M\) is a function,

\[
A : P \to [0, 1]
\]

**Definition 2.2.** [10]: Let \(A\) be fuzzy subset of a ring \(R\). Then \(A\) is said to be a fuzzy subring if

i. \(A(u - v) \geq \min\{A(u), A(v)\}\)

ii. \(A(uv) \geq \min\{A(u), A(v)\}\), for all \(u, v \in R\).

**Definition 2.3.** [14]: Let \(M\) and \(Q\) be two nonempty sets. A \(Q\)-fuzzy subset \(A\) of set \(M\) is a function \(A: X \times Q \to [0, 1]\) for all \(u, v \in M\) and \(q \in Q\).

**Definition 2.4.** [14]: A function \(A : R \times Q \to [0, 1]\) is a QFSR of a ring \(R\) if

i. \(A(u - v, q) \geq \min\{A(u, q), A(v, q)\}\)

ii. \(A(uv, q) \geq \min\{A(u, q), A(v, q)\}\), for all \(u, v \in R\) and \(q \in Q\).

**Definition 2.5.** Let the mapping \(f: R_1 \to R_2\) be a homomorphism. Let \(A\) and \(B\) be \(\omega\)-QFSRs of \(R_1\) and \(R_2\) respectively, then \(f(A)\) and \(f^{-1}(B)\) are image of \(A\) and the inverse image of \(B\) respectively, defined as

i. \(f(A)(v, q) = \{\sup(A(u, q) : u \in f^{-1}(v)), if f^{-1}(v) \neq \emptyset\}, for every v \in R_2\) and \(q \in Q\)

ii. \(f^{-1}(B)(u, q) = B(f(u), q), for every u \in R_1\) and \(q \in Q\)

**Definition 2.6.** [13]: Let \(t_p: [0, 1] \times [0, 1] \to [0, 1]\) be the algebraic product \(t\)-norm on \([0, 1]\) and is described as \(t_p(a, b) = ab, 0 \leq a \leq 1, 0 \leq b \leq 1\)

3. PROPERTIES OF \(\omega-Q\)-FUZZY SUBRINGS

**Definition 3.1.** Let \(M\) and \(Q\) be any two nonempty sets and \(A\) be a \(Q\)-fuzzy subset of a set \(P\), any \(\omega \in [0, 1]\). Then fuzzy set \(A^\omega\) of \(M\) is said to be \(\omega-Q\)-fuzzy subset of \(M\) (w.r.t \(Q\)-fuzzy set \(A\)) and defined by:

\[
A^\omega(m, q) = t_p(A(m, q), \omega), for all m \in M and q \in Q
\]

**Remark 3.2.** Clearly, \(A^1(m, q) = A(m, q)\) and \(A^0(m, q) = 0\).

**Remark 3.3.** If \(A\) and \(B\) be two \(Q\)-fuzzy sets of \(M\). Then \((A \cap B)^\omega = A^\omega \cap B^\omega\).

**Definition 3.4.** A \(Q\)-fuzzy subset of a ring \(R\) is called \(\omega\)-QFSR, and \(\omega \in [0, 1]\), if

i. \(A^\omega(m - n, q) \geq \min\{A^\omega(m, q), A^\omega(n, q)\}\), for all \(m, n \in R\) and \(q \in Q\).

**Theorem 3.5.** If \(A\) is a \(\omega\)-QFSR of a ring \(R\), then \(A^\omega(m, q) \leq A^\omega(0, q)\), for all \(m \in R\) and \(q \in Q\) where 0 is the additive identity of \(R\).

**Proof:** Consider \(A^\omega(0, q) = A^\omega(m - m, q) \geq \min\{A^\omega(m, q), A^\omega(m, q)\} = \min\{A^\omega(m, q), A^\omega(m, q)\}\)

Hence, \(A^\omega(0, q) \geq A^\omega(m, q)\), for all \(m \in R\)

**Theorem 3.6.** If \(A\) is QFSR of a ring \(R\), then \(A\) is an \(\omega\)-QFSR of \(R\).

**Proof:** Assume that \(A\) is a QFSR of a ring \(R\) and \(\forall a, b \in R\) and \(q \in Q\).
Assume that, \( A^\omega(a - b, q) = t_p\{A(a - b, q), \mu\} \geq t_p\{\min\{A(a, q), A(b, q)\}, \omega\} \)
\[ \text{min}\{t_p\{A(a, q), \omega\}, t_p\{A(b, q), \omega\}\} = \text{min}\{A^\omega(a, q), A^\omega(b, q)\} \]
\[ A^\omega(a - b, q) \geq \text{min}\{A^\omega(a, q), A^\omega(b, q)\} \]
Further \( A^\omega(ab, q) = t_p\{A(ab, q), \mu\} \geq t_p\{\min\{A(a, q), A(b, q)\}, \omega\} \)
\[ \text{min}\{t_p\{A(a, q), \omega\}, t_p\{A(b, q), \omega\}\} = \text{min}\{A^\omega(a, q), A^\omega(b, q)\} \]
\[ A^\omega(ab, q) \geq \text{min}\{A^\omega(a, q), A^\omega(b, q)\} \]
Consequently, \( A \) is \( \omega \)-QFSR of \( R \). In general, the converse may not be true.

**Note 3.7.** We take \( A^\omega = \{A(a, q), \omega\} \) in all the examples.

**Example 3.8** Let \( R = \{0, 1, 2, 3\} \), be a ring and \( Q = \{q\} \). Let the \( Q \)-fuzzy set \( A \) of \( R \) described by:
\[
A(a, q) = \begin{cases} 
0.3, & \text{if } a = 0 \\
0.5, & \text{if } a = 1 \text{ or } 3 \\
0.4, & \text{if } a = 2 
\end{cases}
\]

Take \( \omega = 0 \) then,
\[
A^\omega(a, q) = t_p\{A(a, q), \omega\} = t_p\{A(a, q), 0\} = 0, \text{ for all } a \in R
\]
\[ \Rightarrow A^\omega(a - b, q) \geq \text{min}\{A^\omega(a, q), A^\omega(b, q)\} \]
Further, we have \( A^\omega(ab, q) \geq \text{min}\{A^\omega(a, q), A^\omega(b, q)\} \)
Consequently \( A \) is \( \omega \)-QFSR of \( R \) and \( A \) is not QFSR of \( R \).

**Definition 3.9.** Let \( A \) be \( \omega \)-Q-fuzzy set of universe set \( M \). For \( t, \omega \in [0, 1] \) the level subset \( A^\omega_t \) of \( \omega \)-Q-fuzzy set is given by:
\[ A^\omega_t = \{m \in M : A^\omega(m, q) \geq t\} \]

**Theorem 3.10.** Let \( A \) is \( \omega \)-Q-fuzzy subring of \( R \) then \( A^\omega_t \) is subring of \( R \) for all \( t \leq A(0, q) \).

**Proof:** It is quite obvious that \( A^\omega_t \) is non-empty. Since \( A \) be \( \omega \)-Q-fuzzy subring of a ring \( R \), which implies that \( A^\omega(m, q) \leq A^\omega(0, q) \), for all \( m \in R \) and \( q \in Q \). Let \( m, n \in A^\omega_t \) then \( A^\omega(m, q) \geq t \) \( A^\omega(n, q) \geq t \).

Now,
\[ A^\omega(m - n, q) \geq \text{min}\{A^\omega(m, q), A^\omega(n, q)\} \geq \text{min}\{t, t\} = t \]
\[ A^\omega(mn, q) \geq \text{min}\{A^\omega(m, q), A^\omega(n, q)\} \geq \text{min}\{t, t\} = t \]
This implies that \( -n, mn \in A^\omega_t \). Hence, \( A^\omega_t \) is subring of \( R \).

**Definition 3.11.** Let \( A \) be a \( Q \)-fuzzy subset of a ring \( R \) and \( \omega \in [0, 1] \). Then \( A^\omega \) is \( \omega \)-Q-fuzzy left ideal (\( \omega \)-QFLI) of \( R \) if,
\[ i. \quad A^\omega(m - n, q) \geq \text{min}\{A^\omega(m, q), A^\omega(n, q)\} \]
\[ ii. \quad A^\omega(mn, q) \geq A^\omega(n, q), \text{ for all } m, n \in R \text{ and } q \in Q \]

**Definition 3.12.** Let \( A \) be a \( Q \)-fuzzy subset of a ring \( R \) and \( \omega \in [0, 1] \). Then \( A^\omega \) is \( \omega \)-Q-fuzzy right ideal (\( \omega \)-QFRI) of \( R \) if,
\[ i. \quad A^\omega(m - n, q) \geq \text{min}\{A^\omega(m, q), A^\omega(n, q)\} \]
\[ ii. \quad A^\omega(mn, q) \geq A^\omega(m, q), \text{ for all } m, n \in R \text{ and } q \in Q \]

**Definition 3.13.** Let \( A \) be a \( Q \)-fuzzy subset of a ring \( R \) and \( \omega \in [0, 1] \). Then \( A^\omega \) is \( \omega \)-QFI of \( R \) if,
\[ i. \quad A^\omega(m - n, q) \geq \text{min}\{A^\omega(m, q), A^\omega(n, q)\} \]
\[ ii. \quad A^\omega(mn, q) \geq \text{max}\{A^\omega(m, q), A^\omega(n, q)\}, \text{ for all } m, n \in R \text{ and } q \in Q \]

**Definition 3.14.** Let \( A \) be a \( \omega \)-QFSR of a ring \( R \) and \( \omega \in [0, 1] \). For any \( m \in R \) and \( q \in Q \), the \( \omega \)-Q-fuzzy coset of \( A \) in \( R \) is represented by \( m + A^\omega \) as defined as,
Theorem 3.15. Let $A$ be an $\omega$-QFI of ring $R$. Then the set $A_0^\omega = \{ m \in R : A_0^\omega(m, q) = A_0^\omega(0, q) \}$ is an ideal of ring $R$.

**Proof:** Obviously $A_0^\omega \neq \emptyset$ because $0 \in R$. Let $m, n \in A_0^\omega$ be any elements. Consider

\[ A_0^\omega(m - n, q) \leq \min\{A_0^\omega(m, q), A_0^\omega(n, q)\} = \min\{A_0^\omega(0, q), A_0^\omega(0, q)\} \]

Implying that $A_0^\omega(m - n, q) = A_0^\omega(0, q)$. But $A_0^\omega(m - n, q) \leq A_0^\omega(0, q)$

Therefore, $A_0^\omega(m - n, q) = A_0^\omega(0)$

Implying that $m - n \in A_0^\omega$.

Further, let $m \in A_0^\omega$ and $n \in R$. Consider

$A_0^\omega(mn, q) \geq \max\{A_0^\omega(m, q), A_0^\omega(n, q)\} = \max\{A_0^\omega(0, q), A_0^\omega(n, q)\}$.

Implying that $A_0^\omega(mn, q) \geq A_0^\omega(0, q)$. But $A_0^\omega(mn, q) \leq A_0^\omega(0, q)$

Therefore, $A_0^\omega(mn, q) = A_0^\omega(0, q)$.

Similarly, $A_0^\omega(nm, q) = A_0^\omega(0, q)$

Implying that $mn, nm \in A_0^\omega$.

Implying that $A_0^\omega$ is an ideal.

Theorem 3.16. Let $A_0^\omega$ be an $\omega$-QFI of ring $R$, $m, n \in R$ and $q \in Q$. Then,

\[ m + A_0^\omega = n + A_0^\omega \]

if and only if $m - n \in A_0^\omega$.

**Proof:** For any $m, n \in S$, we have $m + A_0^\omega = n + A_0^\omega$.

Consider,

\[ A_0^\omega(m - n, q) = (n + A_0^\omega)(m, q) = (m + A_0^\omega)(m, q) = A_0^\omega(0, q) \]

Therefore, $m - n \in A_0^\omega$.

Conversely, let $m - n \in A_0^\omega$

Implying that $A_0^\omega(m - n, q) = A_0^\omega(0, q)$

Consider,

\[ (m + A_0^\omega)(h, q) = A_0^\omega(h - m, q) = A_0^\omega((h - n) - (m - n), q) \]

\[ \geq \min\{A_0^\omega((h - n), q), A_0^\omega((m - n), q)\} \]

\[ = \min\{A_0^\omega((h - n), q), A_0^\omega(0, q)\} = A_0^\omega((h - n), q) = (n + A_0^\omega)(h, q) \]

Interchange the role of $p$ and $q$ we get

\[ (n + A_0^\omega)(h, q) \geq (m + A_0^\omega)(h, q) \]

Therefore, $(m + A_0^\omega)(h, q) = (n + A_0^\omega)(h, q)$, for all $h \in R$.

Definition 3.17. Let $A$ be a $\omega$-QFI of a ring $R$. The set of all $\omega$-Q-fuzzy cosets of $A$ denoted by $R/A_0^\omega$ form a ring with respect to binary operation $*$ defined by

\[ (m + A_0^\omega) * (n + A_0^\omega) = (m + n) + A_0^\omega, \text{where } m + A_0^\omega, n + A_0^\omega \in R/A_0^\omega, m, n \in R. \]

\[ (m + A_0^\omega) * (n + A_0^\omega) = (m * n) + A_0^\omega, \text{where } m + A_0^\omega, n + A_0^\omega \in R/A_0^\omega, m, n \in R. \]

The ring $R/A_0^\omega$ is called the factor ring of $R$ with respect to $\omega$-QFI $A_0^\omega$.

Theorem 3.18. The set $R/A_0^\omega$ forms a ring with respect to the above stated binary operation.

**Proof:** Let $m_1 + A_0^\omega = m_2 + A_0^\omega$ and $n_1 + A_0^\omega = n_2 + A_0^\omega$ for some $m_1, m_2, n_1, n_2 \in R$. Let $g \in R$ be any element of $R$ and $q \in Q$.

\[ (m_2 + n_2 + A_0^\omega)(g, q) = A_0^\omega(g - (m_2 + n_2), q) \]

\[ = A_0^\omega(g - m_2 - n_2, q) = n_2 + A_0^\omega((g - m_2), q) = n_1 + A_0^\omega((g - m_2), q) \]

\[ = A_0^\omega((g - m_2 - n_1), q) = m_2 + A_0^\omega((g - n_1), q) = m_1 + A_0^\omega((g - n_1), q) \]

\[ = A_0^\omega((g - m_1 - n_1), q) = A_0^\omega(g - (m_1 + n_1), q) = (m_1 + n_1 + A_0^\omega)(g, q) \]

Moreover,

\[ (m_2 n_2 + A_0^\omega)(g, q) = A_0^\omega(g - m_1 n_1 - (m_2 n_2 - m_1 n_1), q) \]
\[
\geq \min\{A^\omega(g - m_1n_1), A^\omega((m_2n_2 - m_1n_1), q)\}
\]

But we have, \(A^\omega((m_2n_2 - m_1n_1), q) = A^\omega((m_1n_1 - m_2n_1 + m_2n_1 - m_2n_2), q)\)

\[
= A^\omega((m_1 - m_2)n_1 + m_2(n_1 - n_2), q) \geq \min\{A^\omega((m_1 - m_2)n_1, q), A^\omega(m_2(n_1 - n_2), q)\}
\]

\[
= \min\{A^\omega(0, 0), A^\omega(0, q)\}
\]

\(A^\omega((m_2n_2 - m_1n_1), q) \geq A^\omega(0, q)\)

\(m_2n_2 + A^\omega(g, q) \geq A^\omega(g - m_1n_1, q)\)

\[
(0 + A^\omega) + (0 + A^\omega) = (n + A^\omega)
\]

Similarly, we can prove that \((m_2n_2 + A^\omega)(g, q) \leq (m_1n_1 + A^\omega)(g, q)\)

Consequently, \((m_2n_2 + A^\omega)(g, q) = (m_1n_1 + A^\omega)(g, q)\).

Therefore \(\ast\) is well defined. Now we prove that the following axioms of ring, for any \(m, n \in R\).

1) \((m + A^\omega) + (n + A^\omega) = m + n + A^\omega\)
2) \((m + A^\omega) + ((n + A^\omega) + (r + A^\omega)) = m + A^\omega + [n + r + A^\omega] = (m + n) + r + A^\omega = [m + n + A^\omega] + r + A^\omega = [(m + A^\omega) + (n + A^\omega)] + (r + A^\omega)\)
3) \((m + A^\omega) + (n + A^\omega) = m + n + A^\omega = n + m + A^\omega = (n + A^\omega) + (m + A^\omega)\)
4) \((0 + A^\omega) + (n + A^\omega) = (n + A^\omega)\)
5) \((m + A^\omega) + (-m + A^\omega) = A^\omega\)
6) \((m + A^\omega)(n + A^\omega) = mn + A^\omega\)
7) \((m + A^\omega)[(n + A^\omega)(r + A^\omega)] = m + A^\omega + [nr + A^\omega] = mnr + A^\omega = [mn + A^\omega] + r + A^\omega = [(m + A^\omega)(n + A^\omega)](r + A^\omega)\)
8) \([(n + A^\omega) + (r + A^\omega)](m + A^\omega) = (n + A^\omega)[(n + r) + A^\omega] = m(n + r) + A^\omega = (mn + mr) + A^\omega = (mn + A^\omega) + (mr + A^\omega) = [(m + A^\omega)(n + A^\omega)] + (m + A^\omega)(r + A^\omega)\)
9) \([(n + A^\omega) + (r + A^\omega)](m + A^\omega) = [(n + r) + A^\omega][m + A^\omega] = (n + r)m + A^\omega = (nm + rm) + A^\omega = (mn + A^\omega) + (rm + A^\omega) = [(n + A^\omega)(m + A^\omega)] + (r + A^\omega)(m + A^\omega)\)

Consequently, \((R/A^\omega, +, \ast)\) is a ring.

4. HOMOMORPHISM OF \(\omega\)-Q-FUZZY SUBRINGS

In this section, we investigate the behavior of homomorphic image and inverse image of \(\omega\)-QFSR.

Lemma 4.1. Let \(f: M \rightarrow N\) be a mapping and \(A\) and \(B\) be two fuzzy subsets of \(M\) and \(N\) respectively, then

(i) \(f^{-1}(B^\omega)(m, q) = (f^{-1}(B))^\omega(m, q)\), for all \(m \in M\) and \(q \in Q\)
(ii) \(f(A^\omega)(n, q) = (f(A))^\omega(n, q)\), for all \(n \in N\) and \(q \in Q\)

Proof:
(i) \(f^{-1}(B^\omega)(m) = B^\omega(f(m)) = t^\omega[p(B(f(m)), \omega)] = t^\omega[f^{-1}(B)(m), \omega]\)

\[
f^{-1}(B^\omega)(m) = (f^{-1}(B))^\omega(m), \text{ for all } m \in M
\]

(ii) \(f(A^\omega)(n, q) = \sup\{A^\omega(m, q): f(m) = y\} = \sup\{t^\omega[A(m, q), \omega]: f(m) = n\}\)

\[
= t^\omega[\{A(m, q): f(m) = n\}, \omega] = t^\omega[f(A)(n, q), \omega] = (f(A))^\omega(n, q), \text{ for all } n \in N
\]

Hence, \(f(A^\omega)(n, q) = (f(A))^\omega(n, q)\)

Theorem 4.2. Let \(f: R_1 \rightarrow R_2\) be a homomorphism from a ring \(R_1\) to a ring \(R_2\) and \(A\) be a \(\omega\)-QFSR of ring \(R_1\). Then \(f(A)\) is a \(\omega\)-QFSR of ring \(R_2\).

Proof: Let \(A\) be a \(\omega\)-QFSR of ring \(R_1\). Let \(n_1, n_2 \in R_2\) be any element. Then there exists unique elements \(m_1, m_2 \in R_1\) such that \(f(m_1) = n_1\) and \(f(m_2) = n_2\) and for \(q \in Q\).

Consider,
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\( (f(A))_1^{\omega} (n_1 - n_2, q) = t_p[f(A) (n_1 - n_2, q), \omega] = t_p[f(A) (f(m_1) - f(m_2), q), \omega] \)
\( = t_p[f(A) (f(m_1 - m_2, q), \omega) = t_p[A (m_1 - m_2, q), \omega] = A^{\omega} (m_1 - m_2, q) \)
\( \geq \min(A^{\omega} (m_1, q), A^{\omega} (m_2, q)) \), for all \( m_1, m_2 \in H \), such that \( f(m_1) = n_1 \) and \( f(m_2) = n_2 \)
\( \geq \min(\sup(A^{\omega} (m_1, q) : f(m_1) = n_1), \sup(A^{\omega} (m_2, q) : f(m_2) = n_2) ) \)
\( = \min(f(A^{\omega} (n_1, q), f(A^{\omega} (n_2, q)) = \min\{(f(A))^{\omega} (n_1, q), (f(A))^{\omega} (n_2, q) \}\)
Thus, \( (f(A))^{\omega} (n_1, n_2, q) \geq \min\{(f(A))^{\omega} (n_1, q), (f(A))^{\omega} (n_2, q) \}\)

Further, \( (f(A))^{\omega} (n_1, n_2, q) = t_p[f(A)(n_1, n_2, q), \omega] = t_p[f(A)(f(m_1)f(m_2), q), \omega] \)
\( = t_p[f(A)(f(m_1m_2), q), \omega] = t_p[A(m_1m_2, q), \omega] = A^{\omega} (m_1m_2, q) \)
\( \geq \min(A^{\omega} (m_1, q), A^{\omega} (m_2, q)) \), for all \( m_1, m_2 \in H \), such that \( f(m_1) = n_1 \) and \( f(m_2) = n_2 \)
\( \geq \min(\sup(A^{\omega} (m_1, q) : f(m_1) = n_1), \sup(A^{\omega} (m_2, q) : f(m_2) = n_2) ) \)
\( = \min(f(A^{\omega} (n_1, q), f(A^{\omega} (n_2, q)) = \min\{(f(A))^{\omega} (n_1, q), (f(A))^{\omega} (n_2, q) \}\)
Thus, \( (f(A))^{\omega} (n_1, n_2, q) \geq \min\{(f(A))^{\omega} (n_1, q), (f(A))^{\omega} (n_2, q) \}\)

Consequently, \( f(A) \) is \( \omega \)-QFSR of \( R_2 \).

**Theorem 4.3.** Let \( f : R_1 \rightarrow R_2 \) be a homomorphism from ring \( R_1 \) into a ring \( R_2 \) and \( B \) be a \( \omega \)-QFSR of ring \( R_2 \). Then \( f^{-1}(B) \) is \( \omega \)-QFSR of ring \( R_1 \).

**Proof:** Let \( B \) be \( \omega \)-QFSR of ring \( R_2 \). Let \( m_1, m_2 \in R_1 \) be any elements, then \( (f^{-1}(B))^{\omega} (m_1 - m_2, q) = f^{-1}(B^{\omega})(m_1 - m_2, q) = B^{\omega}(f(m_1 - m_2, q) \)
\( = B^{\omega}(f(m_1) - f(m_2), q) \)
\( \geq \min(B^{\omega}(f(m_1), q), B^{\omega}(f(m_2), q)) = \min(f^{-1}(B^{\omega})(m_1, q), f^{-1}(B^{\omega})(m_2, q) \)
\( = \min\{(f^{-1}(B))^{\omega} (m_1, q), (f^{-1}(B))^{\omega} (m_2, q) \}\)
Thus, \( (f^{-1}(B))^{\omega} (m_1, m_2, q) \geq \min\{(f^{-1}(B))^{\omega} (m_1, q), (f^{-1}(B))^{\omega} (m_2, q) \}\)

Further, \( (f^{-1}(B))^{\omega} (m_1m_2, q) = f^{-1}(B^{\omega})(m_1m_2, q) = B^{\omega}(f(m_1)m_2, q) \)
\( \geq \min(B^{\omega}(f(m_1), q), B^{\omega}(f(m_2), q)) = \min(f^{-1}(B^{\omega})(m_1, q), f^{-1}(B^{\omega})(m_2, q) \)
\( = \min\{(f^{-1}(B))^{\omega} (m_1, q), (f^{-1}(B))^{\omega} (m_2, q) \}\)
Thus, \( (f^{-1}(B))^{\omega} (m_1m_2, q) \geq \min\{(f^{-1}(B))^{\omega} (m_1, q), (f^{-1}(B))^{\omega} (m_2, q) \}\)

Consequently, \( f^{-1}(B) \) is \( \omega \)-QFSR of a ring \( R_1 \).

5. CONCLUSION

In paper, we have proved the level subset of two \( \omega \)-Q-fuzzy subrings is a subring. In addition, we have extended the study of this ideology to investigate the effect of image and inverse image of \( \omega \)-QFSR under ring homomorphism.

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Conflict of interest
All authors declare no conflict of interest in this paper

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